BUSINESS MATHEMATICS

Lecture 6

Probability & Decision-making

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Introduction to Lecture 6

This lecture introduces you to probability theory and its application to decision-making in economics and business-related problems.

Probability theory is a branch of mathematics that deals with the analysis of random phenomena.

It offers rules for assigning probabilities to outcomes of random experiments or events.

In probability theory, the outcome of a random event cannot be determined before it occurs, but it may be any one of several possible outcomes.

The actual outcome is determined by chance.

Further Readings

- These notes have been derived from diverse resources.
- These resources are recommended for further reading to gain more insights on the application of matrix algebra and input-output analysis to business, and other areas.
- The resources offer a detailed background to matrix algebra and introduction to input-output analysis that may not be covered in this lecture.
- These are (Kahenya, 2021; Lay et al., 2016; Upton & Cook, 2001; Werner & Sotskov, 2006).

Intended Learning Outcomes

Define terms used in probability theory

Solve problems involving basic probability theory

Definition of key terms

- To help us understand the basic terms of probability theory, consider tossing a coin with two sides, the head, and the tail. T
- The appearance of getting a head or tail is called an **event**, while the two appearances are called the **outcomes.**
- The probability or the chance or likelihood of an event happening is always between 0 and 1. *We always assume that the process is fair or unbiased.*
- The probability of an event occurring with certainty is 1 e.g. the probability that Sunday follows Saturday.
- The probability of an event not occurring with certainty i.e. it is impossible is 0 e.g. the probability that you will grow a tail at age 4.

Definitions…contd.

- **Probability Space:** it is the set of all possible outcomes of an experiment
- We have two types of probability space, **discrete** and **continuous** probability space.
- A **probability function** P assigns to each event in the space a probability which is between 0 and 1, inclusive.
- **Random Variables:** These are variables whose possible values are outcomes of a random phenomenon.
- **Probability Distributions:** These are mathematical functions that provide the probabilities of occurrence of different possible outcomes for an experiment or trial.

Discrete Probability space

A sample space Ω is referred to as a discrete space if it is finite or countable. For example,

i) The total outcomes of tossing a coin twice i.e. {HH, HT, TT, TH}

ii) Rolling a dice once or several times i.e. {1,2,3,4,5,6}

Consider a sample space Ω that has $\mathrm{n}(\Omega)$ possible outcomes and that each has an equal chance of occurring i.e.it is fair or unbiased. Supposed the number of outcomes for each event is $\mathrm{n}(\mathrm{E})$, then the probability that the event happens is

$$\mathrm{p}(\mathrm{E}) = \frac{\mathrm{n}(\mathrm{E})}{\mathrm{n}(\Omega)}$$

Example 1

Find the probability of getting an even number when a dice is rolled once.

Solution: The set of all even numbers is E = {2,4,6}.

Then the probability of getting an even number P(E) is the probability of getting a 2 or the probability of getting a 4 or the probability of getting a 6 i.e.

$$\mathrm{p(E) = p(2)\ or\ p(4)\ or\ p(6) = p(2) + p(4) + p(6)} = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2} = 0.5$$

Alternatively;

$$\mathrm{p(E)} = \frac{\mathrm{n(E)}}{\mathrm{n(\Omega)}} = \frac{3}{6} = \frac{1}{2} = 0.5$$

Example 2

Find the probability of picking a prime number between 1 and 10 inclusive.

Solution: The set of all prime numbers between 1 and 10 inclusive is $V = \{2,3,5,7\}$.

$$\Rightarrow p(V) = \frac{n(E)}{n(\Omega)} = \frac{4}{10} = 0.4$$

Alternatively

$$p(V) = p(2) \text{ or } p(3) \text{or } p(5) \text{or } p(7) = p(2) + p(3) + p(5) + p(7)$$

$$= \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{4}{10} = 0.4$$

Continuous Probability Space

A space is said to be continuous if it has uncountable number of members or objects. For example;

- *The real numbers between 0 and 1 inclusive.*
- *Space* Ω *of a random point in the unit circle, that is,* $\Omega = \{(x, y)\colon\ x^2 + y^2 \leq 1\}$

A continuous probability space uses a probability density function pdf (f(x)) to describe the likelihood of outcomes.

This function gives the probability per unit sample space.

The pdf is a non-negative function that integrates to 1 i.e.

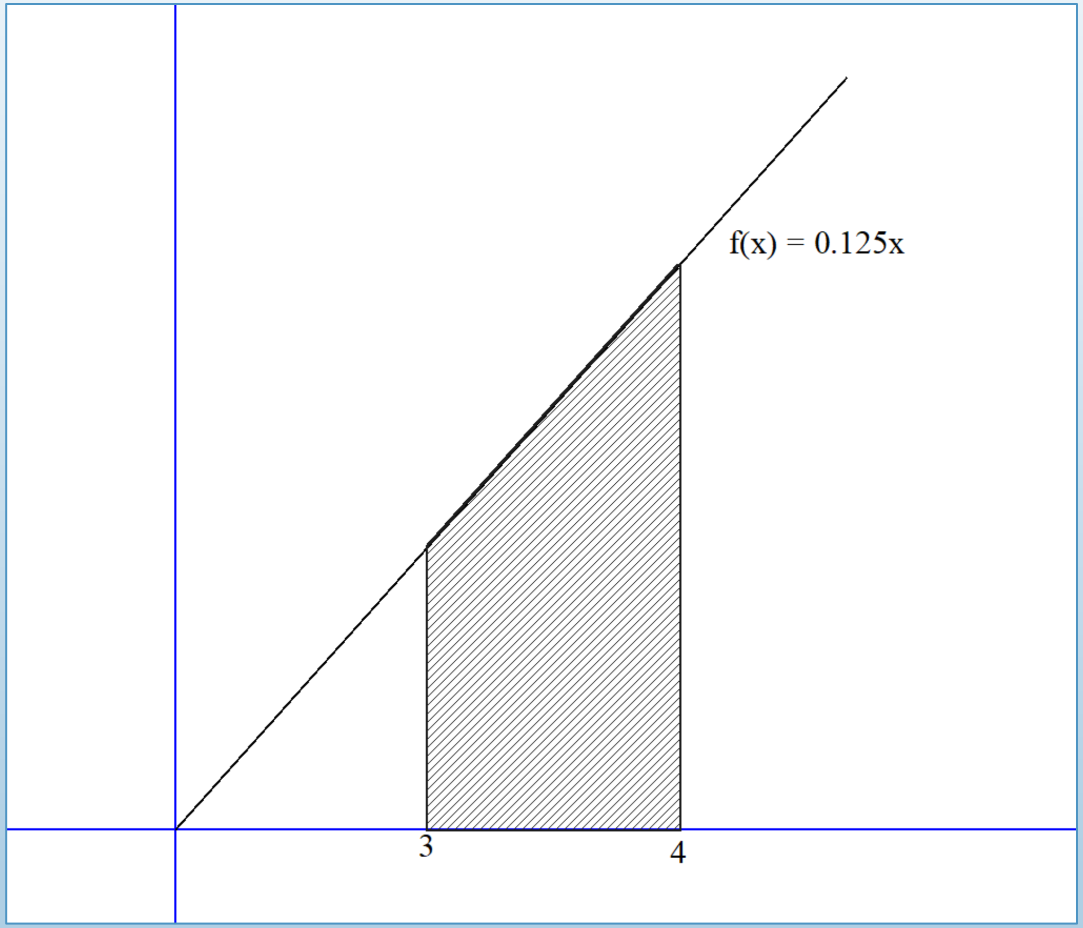
$$\mathrm{f(x)} = \begin{cases} \mathrm{f(x)} \geq 0, & \mathrm{x} \in \mathbb{R} \\ \quad \int \mathrm{f(x)dx} = 1 & \end{cases}$$

Example 1

The continuous random variable X has probability density function given by the function below. Determine $P(X > 3)$

$$f(x) = \begin{cases} \frac{1}{8}x, & 0 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

Solution: The area of interest is the shaded region in the diagram below.



$$\int_3^4 \frac{1}{8}x\,dx = \left[\frac{x^2}{16}\right]_3^4 = \frac{16}{16} - \frac{9}{16} = \frac{7}{16}$$

Example 2

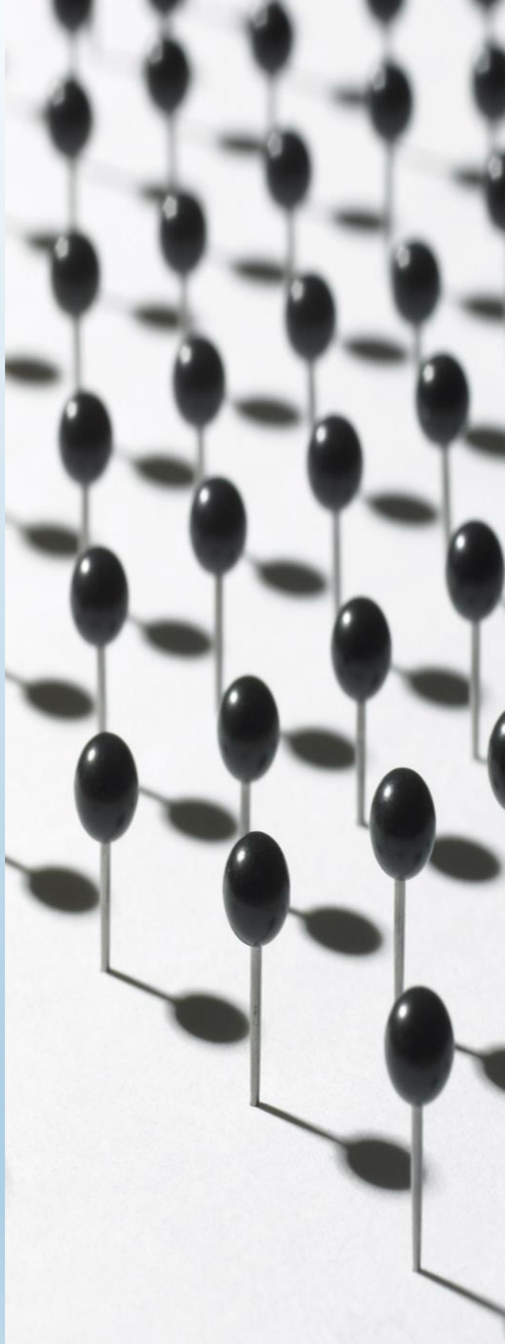
Given the function below, determine the value of k, $P(X < 1)$ and $P(X > 2)$

$$\mathrm{f(x)} = \begin{cases} \mathrm{kx^2} + \frac{2}{5}, & 0 < \mathrm{x} < 3 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_0^3 \left(\mathrm{kx^2} + \frac{2}{5}\right) \mathrm{dx} = \left[\frac{\mathrm{kx^3}}{3} + \frac{2}{5}\mathrm{x}\right]_0^3 = \frac{27\mathrm{k}}{3} + \frac{6}{5} = 1$$

$$\Rightarrow \frac{27\mathrm{k}}{3} = 1 - \frac{6}{5} = -\frac{1}{5}$$

$$27\mathrm{k} = -\frac{3}{5} \therefore \mathrm{k} = -\frac{1}{45}$$

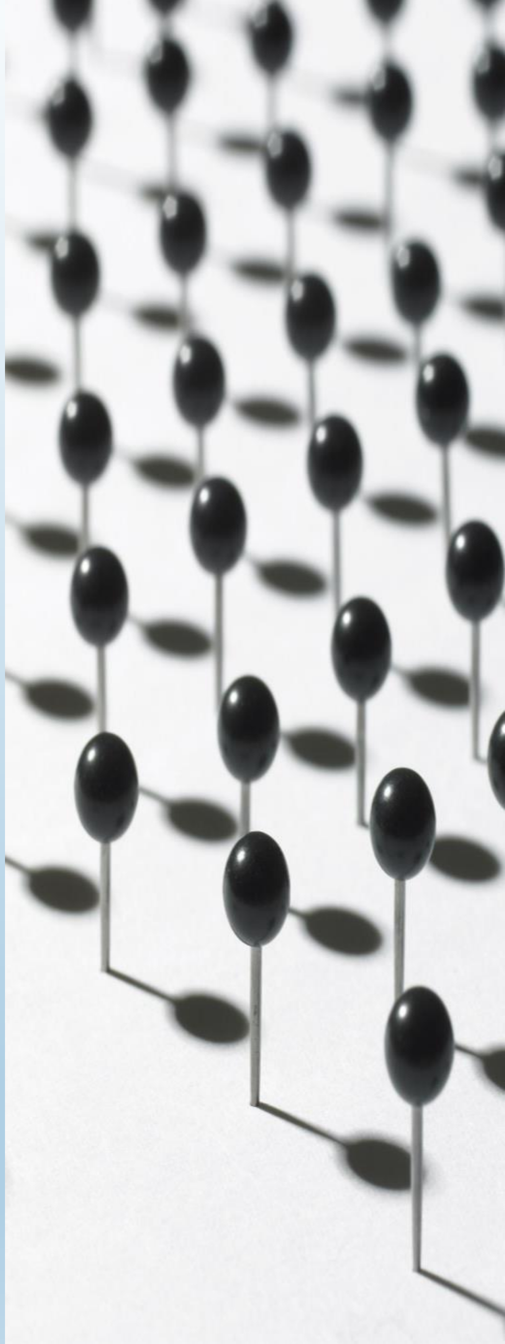


Example 2…contd.

Next, we find $P(x > 2)$

$$\int_2^3 \left(kx^2+\frac{2}{5}\right)dx = \left[\frac{kx^3}{3}+\frac{2}{5}x\right]_2^3 = \left(\frac{27k}{3}+\frac{6}{5}\right)-\left(\frac{8k}{3}+\frac{4}{5}\right)=\frac{19k}{3}+\frac{2}{5}$$

But $k=-\frac{1}{45} \Rightarrow \frac{19k}{3}+\frac{2}{5} = -\frac{19}{135}+\frac{2}{5}=\frac{7}{27}$

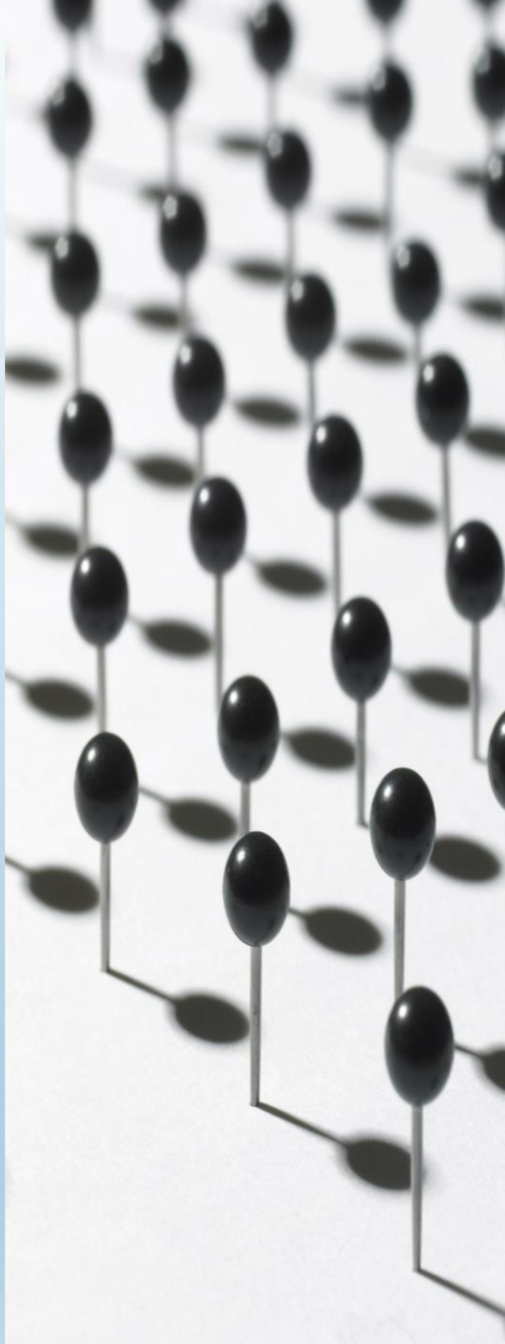


Example 2…contd.

Next, we find $P(X < 1)$ i.e.

$$\int_0^1 \left(kx^2 + \frac{2}{5}\right) dx = \left[\frac{kx^3}{3} + \frac{2}{5}x\right]_0^1 = \frac{k}{3} + \frac{2}{5}$$

But $k = -\frac{1}{45} \Rightarrow \frac{k}{3} + \frac{2}{5} = -\frac{1}{135} + \frac{2}{5} = \frac{53}{135}$

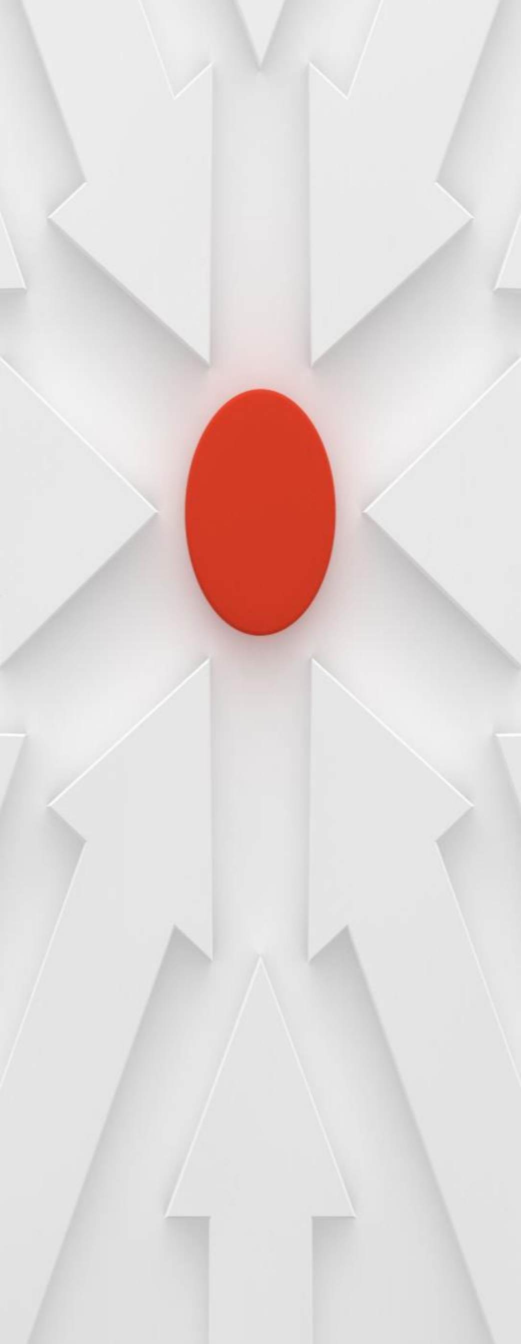


Rules of Probability

a) Suppose X is an event, then the probability of it occurring is $0 \leq P(X) \leq 1$

b) If the event X is impossible then, $P(X) = 0$ but if event X is certain to occur then $P(X) = 1$.

- $P(X) = 0.0001$ implies event X is very unlikely to occur
- while $P(X) = 0.9999$ implies that event X is highly likely to occur.



Rules of probability…contd.

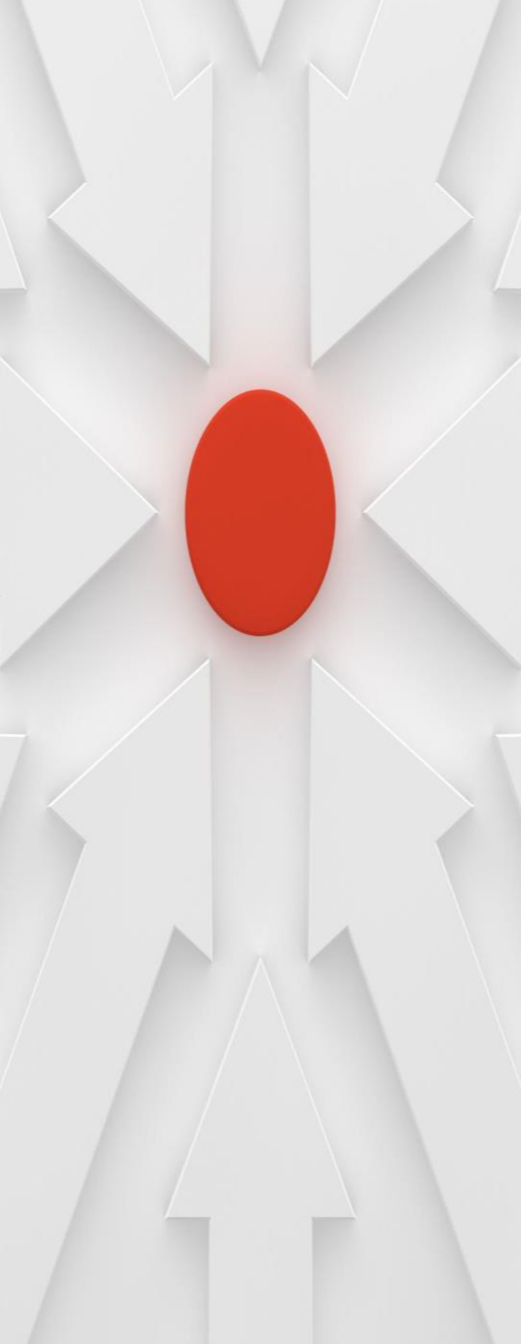
Mutually Exclusive Events (Addition Rule)

- ❑Two events say A and B or even more, are said to be mutually exclusive if the occurrence of one implies that the other cannot occur.
- ❑We can also say that $\mathrm{P(A\ and\ B)} = 0$ or $\mathrm{P}(A \cap B) = 0$.
- ❑This implies that, if event A and event B are mutually exclusive, then

$$p(A \cup B) = p(A \text{ or } B) = p(A) + p(B)$$

Example 1

- A fair dice is rolled once. Find the probability of a getting a number divisible by 3.
- **Solution:** The numbers divisible by 3 are either 3 or 6.
- We can only get either a 3 or 6 showing up but not both.
- Hence, we have; $p(3 \text{ or } 6) = p(3) + p(6) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$

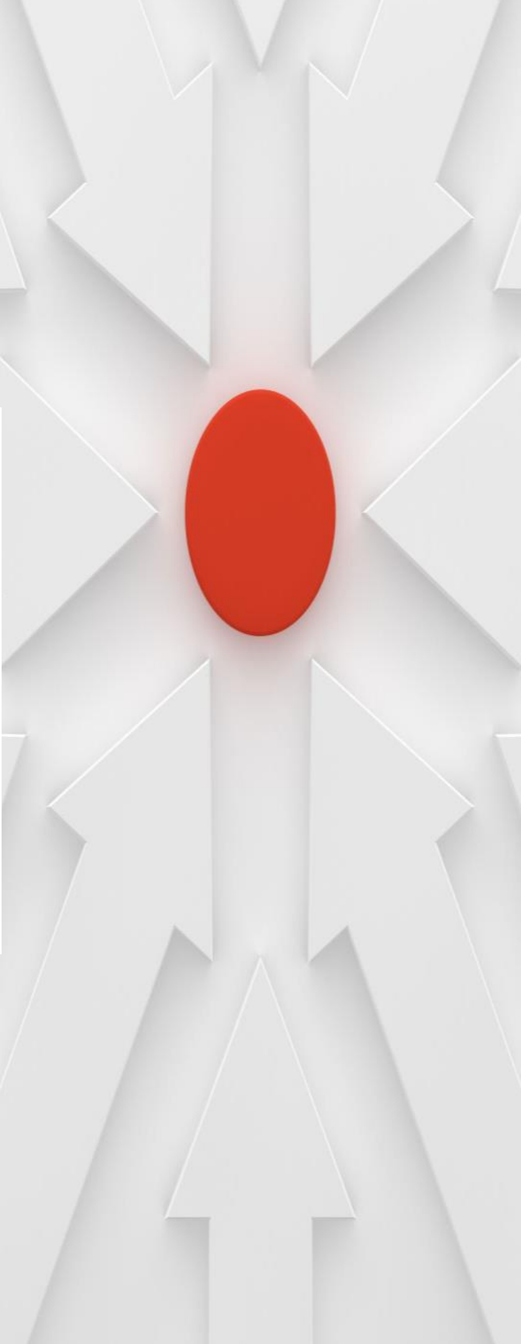


Example 2

The prices of 32 candy-sticks in KES were recorded as below. Determine the probability of selecting a candy-stick that is either 10 KES or 13 KES.

KES	**9**	**10**	**11**	**12**	**13**	**14**	**15**
f	**5**	**7**	**10**	**4**	**3**	**2**	**1**

Solution: $p(10 \text{ or } 13) = p(10) + p(13) = \frac{7}{32} + \frac{3}{32} = \frac{10}{32} = 0.3125$



Complementation Rule

An event X either occurs or it does not. Such that for any event X, $\mathrm{P(X) = 1 - P(not\,X)}$ i.e.,

the probability that an event X occurs equals to one minus the probability that its complement occurs.

It can also be written as;

$$\mathrm{P(X) = 1 - P(\bar{X})}$$

Example 1

The price of 70 mangoes of different size in a market stall were recorded as below. Find the probability of selecting a mango whose price is greater than 10 KES.

KES	10	11	12	13	14	15	16
f	3	8	12	19	7	8	13

Solution: Let X be the event that *the price of a mango is greater than 10*.

Then by complementation rule $\mathrm{P}(\mathrm{X}) = 1 - \mathrm{P}(\overline{\mathrm{X}})$ where $\overline{X}$ is the event that the price of the mango is 10 KES or less.

$$\mathrm{P}(\mathrm{X}) = 1 - \mathrm{P}(\overline{\mathrm{X}}) = 1 - \frac{3}{70} = \frac{67}{70}$$

Example 2

Two fair dice are rolled at once. Find the probability P(A)of getting the sum of two numbers showing to be greater than 4.

Solution: We can represent the outcomes graphically as shown below;

+	Dice 1						
Dice 2		1	2	3	4	5	6
	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

Joint Probability

- Consider two independent events A and B. That is the occurrence of A does not influence the occurrence of B and vice versa, then

$$\mathrm{p}(\mathrm{A} \cap \mathrm{B}) = \mathrm{p}(\mathrm{A\ and\ B}) = \mathrm{p}(\mathrm{A}) \times \mathrm{p}(\mathrm{B})$$

- **Example 1:** A fair coin is tossed twice. Find the probability of getting a head in the first toss and in the second toss.

- **Solution:** when you toss a fair coin, the outcome of the second toss is independent of the outcome of the first toss. We write this as;

$$\mathrm{p}(\mathrm{H\ and\ H}) = \mathrm{p}(\mathrm{H}) \times \mathrm{p}(\mathrm{H}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Example 2

Consider tossing a fair coin C and a fair dice D, at once.

The possible outcomes can be represented in a contingency table as shown below.

Find the probability of getting a 5 and a Tail.

		Dice D					
		1	2	3	4	5	6
Coin C	H	H1	H2	H3	H4	H5	H6
	T	T1	T2	T3	T4	T5	T6

Solution: $p(5 \cap T) = p(5 \text{ and } T) = p(5) \times p(T) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$.

Conditional probability

Consider two events A and B are events. The probability that event B occurs given that event A has occurred is called a conditional probability, denoted $P(B|A)$, read as, *'Probability of B given A'*.

Similarly, the probability that event A occurs given that event B has occurred is denoted P(A|B), read as, *'probability A given B'*.

Example 1: A die is tossed. What is the probability of getting a 4 given that the die comes up even?

Solution: Let A = getting a 4; Let B = getting an even number, then

$$\mathrm{P(A|B)} = \frac{1}{3}$$

Note that $\mathrm{p(A|B)} = \frac{\mathrm{n(A \cap B)}}{\mathrm{n(B)}} = \frac{1}{3}$

Remark 1

Note that $p(A|B) = \frac{n(A\cap B)}{n(B)} \cdots (*)$ However, we have seen from previous examples that, $p(A \cap B) = \frac{n(A\cap B)}{n(S)}$

$\Rightarrow n(A \cap B) = n(S) \cdot p(A \cap B)$

Again, $p(B) = \frac{n(B)}{n(S)} \Rightarrow n(B) = n(S) \cdot p(B)$

Therefore, our equation (*) becomes $p(A|B) = \frac{n(A\cap B)}{n(B)} = \frac{n(S)\cdot p(A\cap B)}{n(S)\cdot p(B)} = \frac{p(A\cap B)}{p(B)}$

That is, $p(A|B) = \frac{p(A\cap B)}{p(B)} \cdots (**)$ similarly we can show that $p(B|A) = \frac{p(A\cap B)}{p(A)} \cdots (***)$

From equation (**) we have $p(B) \times p(A|B) = p(A \cap B)$

From equation (***) we have $p(A) \times p(B|A) = p(A \cap B)$

$$\Rightarrow p(A \cap B) = p(B) \times p(A|B) = p(A) \times p(B|A)$$

Bayes' Theorem

It states that given two events A and B then;

$$\mathrm{p}(\mathrm{A}|\mathrm{B}) = \frac{\mathrm{p}(\mathrm{A} \cap \mathrm{B})}{\mathrm{p}(\mathrm{B})} = \frac{\mathrm{p}(\mathrm{A}) \times \mathrm{p}(\mathrm{B}|\mathrm{A})}{\mathrm{p}(\mathrm{B})}$$

Alternatively

$$\mathrm{p}(\mathrm{B}|\mathrm{A}) = \frac{\mathrm{p}(\mathrm{A} \cap \mathrm{B})}{\mathrm{p}(\mathrm{A})} = \frac{\mathrm{p}(\mathrm{B}) \times \mathrm{p}(\mathrm{A}|\mathrm{B})}{\mathrm{p}(\mathrm{A})}$$

Example 2(Using Set Theory)

Consider two events A and B. The $p(A) = 0.6$ and $p(B) = 0.2$ and that $p(A|B) = 0.1$. Determine the probability that neither A nor B occurs.

Solution: *Alternative I (using set theory):*

From the precious Remark, we have $\mathrm{p}(\mathrm{A} \cap \mathrm{B}) = \mathrm{p}(B) \times \mathrm{p}(\mathrm{A}|\mathrm{B}) = 0.2 \times 0.1 = 0.02$

From set theory we have, $\mathrm{p}(\mathrm{A} \cup \mathrm{B}) = \mathrm{p}(\mathrm{A}) + \mathrm{p}(\mathrm{B}) - \mathrm{p}(\mathrm{A} \cap \mathrm{B}) = 0.6 + 0.2 - 0.02 = 0.78$

Again, applying set theory, we have;

$$\mathrm{p}(\text{neither A nor B}) = \mathrm{p}\overline{(\mathrm{A} \cup \mathrm{B})} = 1 - \mathrm{p}(\mathrm{A} \cup \mathrm{B}) = 1 - 0.78 = 0.22$$

Example 2...contd. (Using tabled of probabilities)

We create tables for the joint probabilities as shown below. Let *neither A nor B* is $\bar{A} \cap \bar{B} = x$;

$\cap$	**B**	$\bar{B}$	**Total**
A	a_{11}	a_{12}	**0.6**
$\bar{A}$	a_{21}	**x**	**0.4**
Total	**0.2**	**0.8**	**1**

We start by filling cell a_{21} i.e. $a_{21} = 0.4 - x$; Next we fill $a_{12} = 0.8 - x$

Then cell $a_{11} = 0.2 - a_{21} = 0.2 - (0.4 - x) = x - 0.2$

Our table will now look this way;

Our table will now look this way;

$\cap$	**B**	$\overline{\mathbf{B}}$	**Total**
A	$\boldsymbol{x-0.2}$	$\boldsymbol{0.8-x}$	**0.6**
$\overline{\mathbf{A}}$	$\boldsymbol{0.4-x}$	**x**	**0.4**
Total	**0.2**	**0.8**	**1**

Next, we have; $0.1 = p(A|B) = \frac{p(A \cap B)}{p(B)} = \frac{x-0.2}{0.2} \Rightarrow 0.02 = x - 0.2 \therefore x = 0.02 + 0.2 = 0.22$

Note that;

$$p\left(A \cap \overline{B}\right) = 0.8 - x = 0.8 - 0.22 = 0.58$$

$$p(A \cap B) = x - 0.2 = 0.22 - 0.2 = 0.02$$

$$p\left(\overline{A} \cap B\right) = 0.4 - x = 0.4 - 0.22 = 0.18$$

Example 3

A survey was carried out at a certain firm.

It was observed that of those workers with smartphones, 70% uses the smartphone to transact business for the firm. Of the remaining workers only 40% uses smartphones to transact business.

It has been established that 75% of the workers have smartphones. Find;

(i) The overall proportion of workers who uses smartphones to transact business for the firm.

(ii) The probability that a worker who transact business has a smartphone

Example 3…contd.

Solution: Let A: transact business with a smartphone; B1: Have a smartphone B2: doesn't have a smartphone.

We can use a table to get;

	Have smartphones B1	**No smartphones B2**	**Total**
Transact business with smartphone A	52.5% (70% of 75)	10% (40% of 25)	62.5%
Doesn't transact business with smartphone $\overline{A}$	22.5%	15%	37.5%
Total	75%	25	100

(i) The overall proportion of workers who uses smartphones to transact business is 62.5%

(ii) The probability of a worker who transact business has a smartphone is

$$p(B1|A) = \frac{52.5}{62.5} = 0.84$$

Probability and Tree Diagrams

One can use of tree diagram to calculate the probabilities especially when dealing with the multiplication rule i.e., if A and B are any two events, then

$$P(A \quad \text{and} \quad B)=P(A) \times P(B|A)$$

That is the probability that both events A and B occur equals the probability that event A occurs times the conditional probability that event B occurs given that event A occurs.

Example 1

Two spare parts of a car are selected from a batch of 5 defective spare parts and 9 non-defective spart parts.

What is the probability of picking a defective spare part and a non-defective spare part without order and without replacement.

Example 2

- A business entity is supplied with goods from three sources A, B, and C. 40% comes from source A, 50% comes from source B, and 10% from source C.
- The quality control department note that 10% of the goods supplied by source A have some defects, 2% of the goods supplied by source B had defects, while 3% of goods supplied by source C have defects.
- Suppose a good is picked at random find;

  (i) The probability of picking a good that has a defect

  (ii) Suppose a good is picked at random and examined and is found to have defects, what is the probability that it came from source A?

Example 2…contd.

The probability of picking a good that has a defect

Example 2…contd.

Suppose a good is picked at random and examined and is found to have defects, what is the probability that it came from source A?

Example 2…contd.

The probability of picking a good from source A and it has defects is by Bayes' theorem

$$\text{p}(\text{A given it has defects}) = \frac{\text{p(A AND has defects)}}{\text{p(it has defects)}}$$

$$= \frac{0.4 \times 0.1}{0.17}$$

$$\approx 0.2353$$

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