Business Mathematics

Lecture 6

Probability & Decision-making

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Introduction to Lecture 6

This lecture introduces you to probability theory and its application to decision-making in economics and business-related problems. Probability theory is a branch of mathematics that deals with the analysis of random phenomena. It offers rules for assigning probabilities to outcomes of random experiments or events. In probability theory, the outcome of a random event cannot be determined before it occurs, but it may be any one of several possible outcomes. The actual outcome is determined by chance.

Further Readings

The resources below are recommended for further reading to gain more insights on the application of probability and decision-making to economics, and other areas. The resources offer a detailed background introduction to input-output analysis that may not be covered in this lecture. These are (Kahenya, 2021; Lay et al., 2016; Upton & Cook, 2001; Werner & Sotskov, 2006).

Intended Learning Outcomes

At the end of this lecture, you will be able to;

(i) Define basic terms used in probability theory.
(ii) Solve problems involving basic probability theory.

Definition of key terms in probability theory

To help us understand the basic terms of probability theory, consider tossing a coin with two sides, the head, and the tail. The appearance of getting a head or tail is called an **event**, while the two appearances are called the **outcomes.**

The probability or the chance or likelihood of an event happening is always between 0 and 1. *We always assume that the process is fair or unbiased.* The probability of an event occurring with certainty is 1 e.g. the probability that Sunday follows Saturday. On the other hand, the probability of an event not occurring with certainty i.e. it is impossible is 0 e.g. the probability that you will grow a tail at age 4.

Probability Space: it is the set of all possible outcomes of an experiment. Any specified subset of the sample space is called an **Event** F. We have two types of probability space, **discrete** and **continuous** probability space. A **probability function** P assigns to each event in the space a probability which is between 0 and 1, inclusive.

Random Variables: These are variables whose possible values are outcomes of a random phenomenon.

Probability Distributions: These are mathematical functions that provide the probabilities of occurrence of different possible outcomes for an experiment or trial.

Stochastic Processes: These are mathematical models used to describe phenomena that evolve over time in a probabilistic manner.

a) Discrete Probability Space

A sample space Ω is referred to as a discrete space if it is finite or countable. For example,

i) The total outcomes of tossing a coin twice i.e. {HH, HT, TT, TH}

ii) Rolling a dice once or several times i.e. {1,2,3,4,5,6}

Consider a sample space Ω that has $n(\Omega)$ possible outcomes and that each has an equal chance of occurring i.e.it is fair or unbiased. Supposed the number of outcomes for each event is $n(E)$, then the probability that the event happens is

$$p(E) = \frac{n(E)}{n(\Omega)}$$

Example 1: Find the probability of getting an even number when a dice is rolled once.

Solution: The set of all even numbers is E = {2,4,6}. Then the probability of getting an even number P(E) is the probability of getting a 2 or the probability of getting a 4 or the probability of getting a 6 i.e.

$$p(E) = p(2) \text{ or } p(4) \text{ or } p(6) = p(2) + p(4) + p(6) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2} = 0.5$$

Alternatively;

$$p(E) = \frac{n(E)}{n(\Omega)} = \frac{3}{6} = \frac{1}{2} = 0.5$$

Example 2: Find the probability of picking a prime number between 1 and 10 inclusive.

Solution: The set of all prime numbers between 1 and 10 inclusive is $V = \{2,3,5,7\}$.

$$\Rightarrow p(V) = \frac{n(E)}{n(\Omega)} = \frac{4}{10} = 0.4$$

Alternatively

$$p(V) = p(2) \text{ or } p(3) \text{or } p(5) \text{or } p(7) = p(2) + p(3) + p(5) + p(7)$$

$$= \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{4}{10} = 0.4$$

b) Continuous Probability Space

A space is said to be continuous if it has uncountable number of members or objects. For example;

(i) The real numbers between 0 and 1 inclusive.

(ii) Space Ω of a random point in the unit circle, that is, $\Omega = \{(x, y) : x^2 + y^2 \leq 1\}$.

A continuous probability space uses a probability density function pdf (f(x)) to describe the likelihood of outcomes. This function gives the probability per unit sample space. The pdf is a non-negative function that integrates to 1 i.e.

$$f(x) = \begin{cases} f(x) \geq 0, & x \in \mathbb{R} \\ \int f(x)dx = 1 \end{cases}$$

In general the probability of X taking a value in the interval (a, b) is given by the area

$$p(a < x < b) = \int_a^b f(x)dx$$

Example 1: The continuous random variable X has probability density function given by

$$f(x) = \begin{cases} \frac{1}{8}x, & 0 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

Determine $P(X > 3)$

Solution: The area of interest is the shaded region in the diagram below.

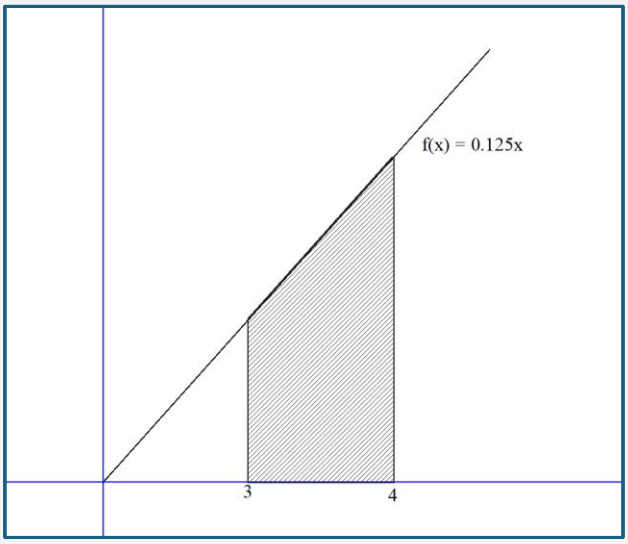


Figure 1

$$P(X > 3) = \int_3^4 \frac{1}{8}x\,dx = \left[\frac{x^2}{16}\right]_3^4 = \frac{16}{16} - \frac{9}{16} = \frac{7}{16}$$

Example 2: Given the function below, determine the value of k, $P(X < 1)$ and $P(X > 2)$

$$f(x) = \begin{cases} kx^2 + \frac{2}{5}, & 0 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

Solution:

$$\int_0^3 \left(kx^2 + \frac{2}{5}\right) dx = \left[\frac{kx^3}{3} + \frac{2}{5}x\right]_0^3 = \frac{27k}{3} + \frac{6}{5} = 1$$

$$\Rightarrow \frac{27k}{3} = 1 - \frac{6}{5} = -\frac{1}{5}$$

$$27k = -\frac{3}{5} \therefore k = -\frac{1}{45}$$

Next we find $P(X < 1)$ i.e.

$$\int_0^1 \left(kx^2 + \frac{2}{5}\right) dx = \left[\frac{kx^3}{3} + \frac{2}{5}x\right]_0^1 = \frac{k}{3} + \frac{2}{5}$$

But $k = -\frac{1}{45} \Rightarrow \frac{k}{3} + \frac{2}{5} = -\frac{1}{135} + \frac{2}{5} = \frac{53}{135}$

Lastly we find $P(x > 2)$ i.e.

$$\int_2^3 \left(kx^2 + \frac{2}{5}\right) dx = \left[\frac{kx^3}{3} + \frac{2}{5}x\right]_2^3 = \left(\frac{27k}{3} + \frac{6}{5}\right) - \left(\frac{8k}{3} + \frac{4}{5}\right) = \frac{19k}{3} + \frac{2}{5}$$

But $k = -\frac{1}{45} \Rightarrow \frac{19k}{3} + \frac{2}{5} = -\frac{19}{135} + \frac{2}{5} = \frac{7}{27}$

Rules of Probability

a) Suppose X is an event, then the probability of it occurring is $0 \le P(X) \le 1$

b) If the event X is impossible then, $P(X) = 0$ but if event X is certain to occur then $P(X) = 1$. $P(X) = 0.0001$ implies event X is very unlikely to occur while $P(X) = 0.9999$ implies that event X is highly likely to occur.

c) **Mutually Exclusive Events (Addition Rule):** Two events say A and B or even more, are said to be mutually exclusive if the occurrence of one implies that the other cannot occur. We can also say that $P(A \text{ and } B) = 0$ or $P(A \cap B) = 0$. This implies that, if event A and event B are mutually exclusive, then $p(A \cup B) = p(A \text{ or } B) = p(A) + p(B)$

Example 1: A fair dice is rolled once. Find the probability of a getting a number divisible by 3.

Solution: The numbers divisible by 3 are either 3 or 6. We can only get either a 3 or 6 showing up but not both. Hence we have; $p(3 \text{ or } 6) = p(3) + p(6) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$

Example 2: The prices of 32 candy-sticks in KES were recorded as below. Determine the probability of selecting a candy-stick that is either 10 KES or 13 KES.

KES	9	10	11	12	13	14	15
f	5	7	10	4	3	2	1

Solution: $p(10 \text{ or } 13) = p(10) + p(13) = \frac{7}{32} + \frac{3}{32} = \frac{10}{32} = 0.3125$

d) Complementation Rule

An event X either occurs or it does not. Such that for any event X, $\mathrm{P(X)} = 1 - \mathrm{P(not\ X)}$ i.e., the probability that an event X occurs equals to one minus the probability that its complement occurs. It can also be written as;

$$\mathrm{P(X)} = 1 - \mathrm{P}(\overline{\mathrm{X}})$$

Example 1: The price of 70 mangoes of different size in a market stall were recorded as below. Find the probability of selecting a mango whose price is greater than 10 KES.

KES	10	11	12	13	14	15	16
f	3	8	12	19	7	8	13

Solution: Let X be the event that the price of a mango is greater than 10.

Then by complementation rule $\mathrm{P(X)} = 1 - \mathrm{P}(\overline{\mathrm{X}})$ where $\overline{\mathrm{X}}$ is the event that the price of the mango is 10 KES or less.

$$\mathrm{P(X)} = 1 - \mathrm{P}(\overline{\mathrm{X}}) = 1 - \frac{3}{70} = \frac{67}{70}$$

Example 2: Two fair dice are rolled at once. Find the probability $\mathrm{P(A)}$of getting the sum of two numbers showing to be greater than 4.

Solution: We can represent the outcomes graphically as shown below;

+	**Dice 1**						
Dice 2		*1*	*2*	*3*	*4*	*5*	*6*
	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

The probability $\mathrm{P}(\overline{\mathrm{A}})$ of getting a sum less than or equals to 4,

$$\{\text{Dice 1, Dice 2}\} = \{(1,1), (1,2), (1,3), (2,1), (2,2), (3,1)\}$$

$$\Rightarrow n(\overline{A}) = 6$$

We have $n(S) = 36$ equally likely outcomes. Then

$$P(\overline{A}) = \frac{n(\overline{A})}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

Hence $P(A) = 1 - P(\overline{A}) = 1 - \frac{1}{6} = \frac{5}{6}$

e) Joint Probability

Consider two independent events A and B. That is the occurrence of A does not influence the occurrence of B and vice versa, then

$$p(A \cap B) = p(A \text{ and } B) = p(A) \times p(B)$$

Example 1: A fair coin is tossed twice. Find the probability of getting a head in the first toss and in the second toss.

Solution: when you toss a fair coin, the outcome of the second toss is independent of the outcome of the first toss. We write this as;

$$p(H \text{ and } H) = p(H) \times p(H) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Alternatively, we can list all the possible outcomes in a contingency table as shown below;

		1st Toss	
2nd Toss		**H**	**T**
	H	*HH*	*HT*
	T	*TH*	*TT*

The probability of getting heads in both tosses; $\frac{n(HH)}{n(S)} = \frac{1}{4}$

Example 2: Consider tossing a fair coin C and a fair dice D, at once. The possible outcomes can be represented in a contingency table as shown

		Dice D					
		1	**2**	**3**	**4**	**5**	**6**
Coin C	**H**	*H1*	*H2*	*H3*	*H4*	*H5*	*H6*
	T	*T1*	*T2*	*T3*	*T4*	*T5*	*T6*

Find the probability of getting a 5 and a Tail.

Solution: $p(5 \cap T) = p(5 \text{ and } T) = p(5) \times p(T) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$.

Note that the probability of getting a 5 is independent of a head. Again note that $n(T5) = 1$ and $n(S) = 12$, therefore $p(5T) = \frac{n(T5)}{n(S)} = \frac{1}{12}$

f) Conditional probability

Consider two events A and B are events. The probability that event B occurs given that event A has occurred is called a conditional probability, denoted $P(B|A)$, read as, '*Probability of B given A*'. Similarly the probability that event A occurs given that event B has occurred is denoted P(A|B), read as, '*probability A given B*'.

Example 1: A die is tossed. What is the probability of getting a 4 given that the die comes up even?

Solution: Let A = getting a 4; Let B = getting an even number, then

$$P(A|B) = \frac{1}{3}$$

Note that $p(A|B) = \frac{n(A \cap B)}{n(B)} = \frac{1}{3}$

Remark 1:

Note that $p(A|B) = \frac{n(A \cap B)}{n(B)} \cdots (*)$ However, we have seen from previous examples that,

$p(A \cap B) = \frac{n(A \cap B)}{n(S)} \Rightarrow n(A \cap B) = n(S) \cdot p(A \cap B)$

Again $\mathrm{p(B)} = \frac{\mathrm{n(B)}}{\mathrm{n(S)}} \Rightarrow \mathrm{n(B)} = \mathrm{n(S)} \cdot \mathrm{p(B)}$

Therefore our equation (*) becomes $\mathrm{p(A|B)} = \frac{\mathrm{n(A \cap B)}}{\mathrm{n(B)}} = \frac{\mathrm{n(S) \cdot p(A \cap B)}}{\mathrm{n(S) \cdot p(B)}} = \frac{\mathrm{p(A \cap B)}}{\mathrm{p(B)}}$

That is, $\mathrm{p(A|B)} = \frac{\mathrm{p(A \cap B)}}{\mathrm{p(B)}} \cdots (**)$ similarly we can show that $\mathrm{p(B|A)} = \frac{\mathrm{p(A \cap B)}}{\mathrm{p(A)}} \cdots (***)$

From equation (**) we have $\mathrm{p(B) \times p(A|B) = p(A \cap B)}$

From equation (***) we have $\mathrm{p(A) \times p(B|A) = p(A \cap B)}$

$$\Rightarrow \mathrm{p(A \cap B) = p(B) \times p(A|B) = p(A) \times p(B|A)}$$

Bayes' Theorem

It states that given two events A and B then;

$$\mathrm{p(A|B)} = \frac{\mathrm{p(A \cap B)}}{\mathrm{p(B)}} = \frac{\mathrm{p(A) \times p(B|A)}}{\mathrm{p(B)}}$$

Alternatively

$$\mathrm{p(B|A)} = \frac{\mathrm{p(A \cap B)}}{\mathrm{p(A)}} = \frac{\mathrm{p(B) \times p(A|B)}}{\mathrm{p(A)}}$$

Example 2: Consider two events A and B. The $p(A) = 0.6$ and $p(B) = 0.2$ and that $p(A|B) = 0.1$. Determine the probability that neither A nor B occurs.

Solution: *Alternative I (using set theory):*

From the precious Remark, we have $\mathrm{p(A \cap B)} = \mathrm{p}(B) \times \mathrm{p(A|B)} = 0.2 \times 0.1 = 0.02$

From set theory we have, $\mathrm{p(A \cup B) = p(A) + p(B) - p(A \cap B)} = 0.6 + 0.2 - 0.02 = 0.78$

Again applying set theory, we have;

$$\mathrm{p(neither\ A\ nor\ B)} = \mathrm{p}\overline{\mathrm{(A \cup B)}} = 1 - \mathrm{p(A \cup B)} = 1 - 0.78 = 0.22$$

Alternative II (Using tables of probabilities)

We create tables for the joint probabilities as shown below. Let *neither A nor B* is $\overline{A} \cap \overline{B} = x$;

$\cap$	B	$\overline{B}$	Total
A	a_{11}	a_{12}	0.6
$\overline{A}$	a_{21}	x	0.4
Total	0.2	0.8	1

We start by filling cell a_{21} i.e. $a_{21} = 0.4 - x$; Next we fill $a_{12} = 0.8 - x$

Then cell $a_{11} = 0.2 - a_{21} = 0.2 - (0.4 - x) = x - 0.2$

Our table will now look this way;

$\cap$	B	$\overline{B}$	Total
A	$x - 0.2$	$0.8 - x$	0.6
$\overline{A}$	$0.4 - x$	x	0.4
Total	0.2	0.8	1

Next we have; $0.1 = p(A|B) = \frac{p(A \cap B)}{p(B)} = \frac{x - 0.2}{0.2} \Rightarrow 0.02 = x - 0.2 \therefore x = 0.02 + 0.2 = 0.22$

Note that;

$$p(A \cap \overline{B}) = 0.8 - x = 0.8 - 0.22 = 0.58$$

$$p(A \cap B) = x - 0.2 = 0.22 - 0.2 = 0.02$$

$$p(\overline{A} \cap B) = 0.4 - x = 0.4 - 0.22 = 0.18$$

Example 3: A survey was carried out at a certain firm. It was observed that of those workers with smartphones, 70% uses the smartphone to transact business for the firm. Of the remaining workers only 40% uses smartphones to transact business. It has been established that 75% of the workers have smartphones. Find;

(i) The overall proportion of workers who uses smartphones to transact business for the firm.

(ii) The probability that a worker who transact business has a smartphone

Solution: Let A: transact business with a smartphone; B1: Have a smartphone B2: doesn't have a smartphone.

We can use a table to get;

	Have smartphones B1	No smartphones B2	Total
Transact business with smartphone A	52.5% (70% of 75)	10% (40% of 25)	62.5%
Doesn't transact business with smartphone $\overline{\mathrm{A}}$	22.5%	15%	37.5%
Total	75%	25	100

(i) The overall proportion of workers who uses smartphones to transact business is 62.5%

(ii) The probability of a worker who transact business has a smartphone is

$$\mathrm{p(B1|A)} = \frac{52.5}{62.5} = 0.84$$

Probability and Tree Diagrams

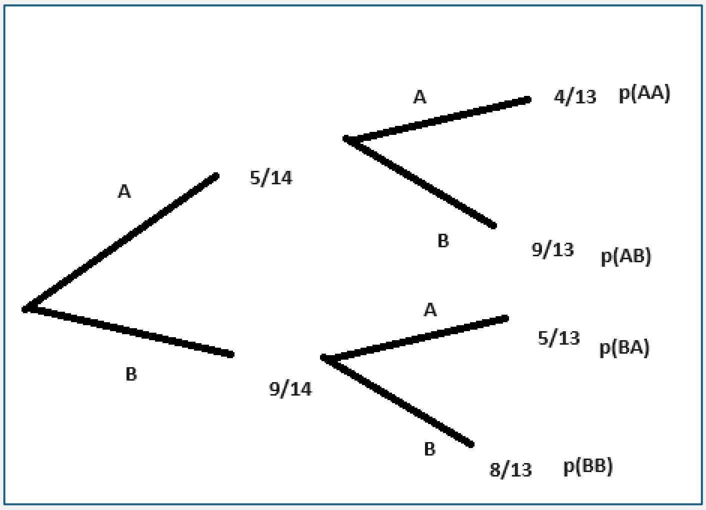
One can use of tree diagram to calculate the probabilities especially when dealing with the multiplication rule i.e., if A and B are any two events, then

$$\mathrm{P(A \quad and \quad B) = P(A) \times P(B|A)}$$

That is the probability that both events A and B occur equals the probability that event A occurs times the conditional probability that event B occurs given that event A occurs.

Example 1: Two spare parts of a car are selected from a batch of 5 defective spare parts and 9 non-defective spart parts. What is the probability of picking a defective spare part and a non-defective spare part without order and without replacement.

Solution: Let defective spare parts be A and non-defective spare parts be B then we can draw a tree diagram to represent this scenario.



The probability of picking ***a defective spare part and a non-defective spare part*** **is**

$$\mathbf{P(AB) or\ p(BA)} = \left(\frac{5}{14} \times \frac{9}{13}\right) + \left(\frac{9}{14} + \frac{5}{13}\right) = \frac{45}{182} + \frac{45}{182} = \frac{90}{182} = \frac{45}{91}$$

Example 2: A business entity is supplied with goods from three sources A, B, and C. 40% comes from source A, 50% comes from source B, and 10% from source C. The quality control department note that 10% of the goods supplied by source A have some defects, 2% of the goods supplied by source B had defects, while 3% of goods supplied by source C have defects. Suppose a good is picked at random find;

(i) The probability of picking a good that has a defect

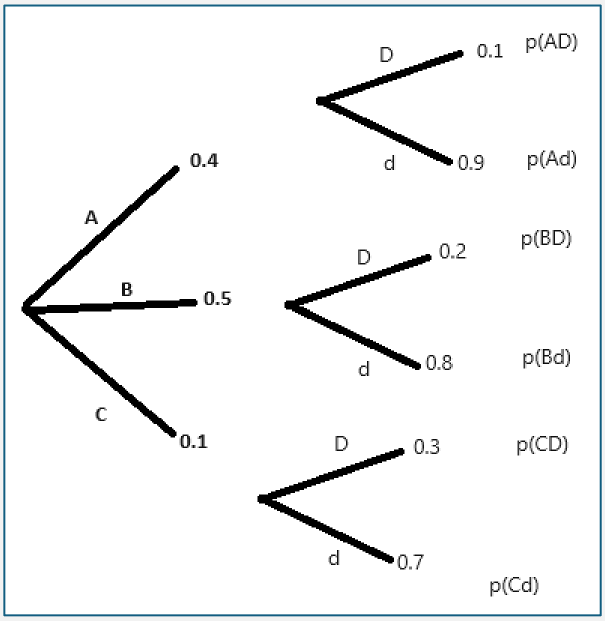
(ii) Suppose a good is picked at random and examined and is found to have defects, what is the probability that it came from source A?

Solution*: (by using a tree diagram)*

The probability of picking a defect good that has a defect is;

$$p(AD) + p(BD) + p(CD)$$

$$= (0.4 \times 0.1) + (0.5 \times 0.2) + (0.1 \times 0.3) = 0.04 + 0.10 + 0.03 = 0.17$$



(iii) The probability of picking a good from source A and it has defects is by Bayes' theorem;

$$\mathrm{p(A\ given\ it\ has\ defects)} = \frac{\mathrm{p(A\ AND\ has\ defects)}}{\mathrm{p(it\ has\ defects)}} = \frac{0.4 \times 0.1}{0.17} \approx 0.2353$$

Exercise

1) A fair coin and a dice are tossed at the same time. Find the probability of getting a tail and an odd number.
2) What is the probability that a positive integer picked at random from the first 30 natural numbers is a multiple of 7.
3) Two batches A and B of drinking water contain sparkling water and non-sparkling water. Batch A contains 12 bottles of sparkling water and 17 bottles of non-sparkling water, while batch B contains 23 bottles of sparkling water and 30 bottles of non-sparkling water. A batch is randomly picked, and two bottles randomly chosen, one at a time and without replacement. Find the probability of;
    a. Picking two bottles of sparkling water.
    b. Picking a bottle of sparkling water and a bottle of non-sparkling water.
    c. Picking two bottles of non-sparkling water.

4) A bowl contains 7 black marbles and 13 white marbles. If two marbles are randomly drawing from the bag one at a time, find the chance of picking a black marble and a red marble without replacement, with replacement and without order both instances.
5) A candy is chosen at random from a stall in the supermarket. Let A be the event that the candy is red and let B be the event that the candy is green. Suppose $p(A) = 0.4$, $p(B) = 0.1$ and $p(A|B) = 0.3$. Find $A \cap \overline{B}$, $p(A|\overline{B})$ and describe $A \cap \overline{B}$.
6) A biscuits processing factory produces two brands of biscuits X and Y. Samples of the biscuits are taken to the quality control department for checking. It is noted that 70% of the samples pass the quality mark with 2% inconsistences. The remaining 30% does not pass the quality mark and contain 7% inconsistences. Find;
    a. The probability the next batch to the quality control department will not pass the quality mark
    b. If a second sample of biscuit is taken and is inconsistent, what is the probability that the sample does not mark.

References

Kahenya, P. N. (2021). *Basic Mathematics*. HUFOCW. https://www.hufocw.org/Course/854

Lay, D. C., Lay, S. R., & McDonald, J. J. (2016). *Linear Algebra and its Application* (5th ed.). Pearson.

Upton, G., & Cook, I. (2001). *Introduction to Statistics* (2nd ed.). Oxford University Press.

Werner, F., & Sotskov, Y. N. (2006). *Mathematics of Economics and Business*. Routledge: Taylor & Francis Group.