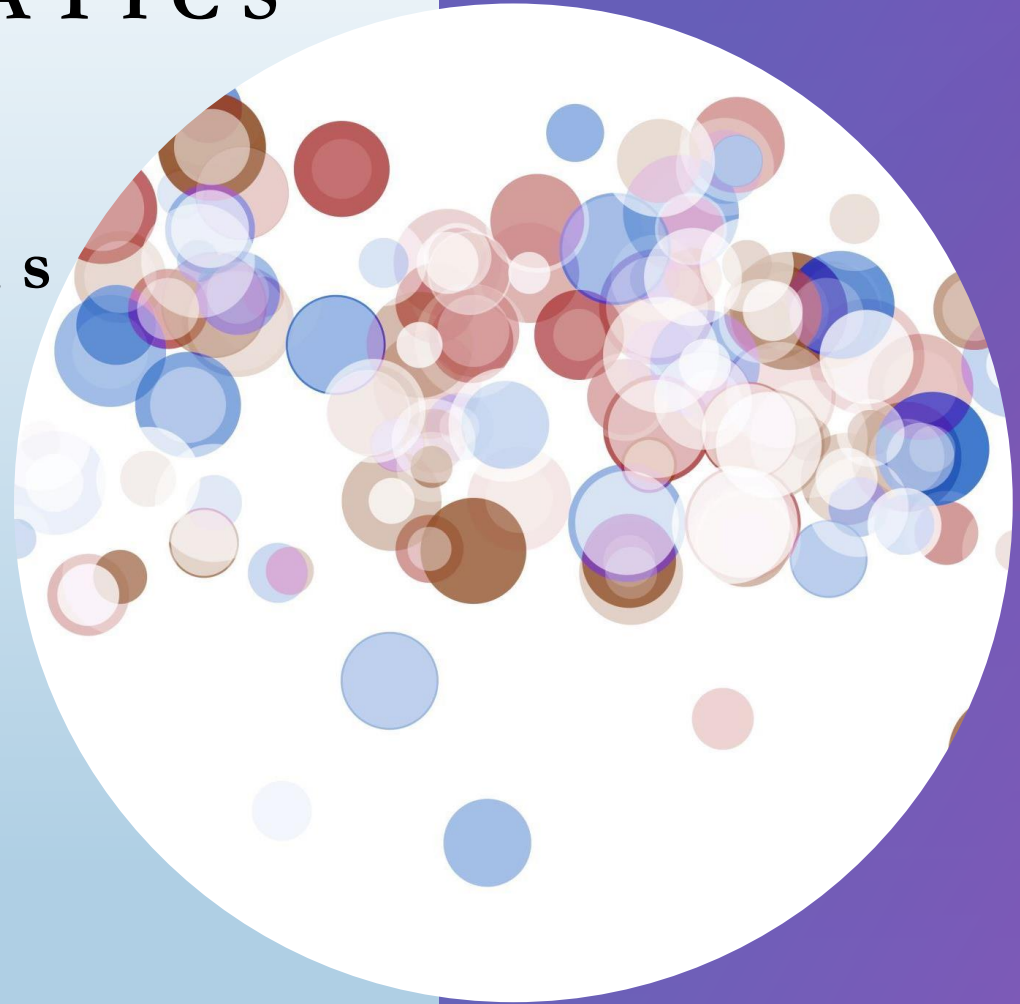


BUSINESS MATHEMATICS

Lecture 7

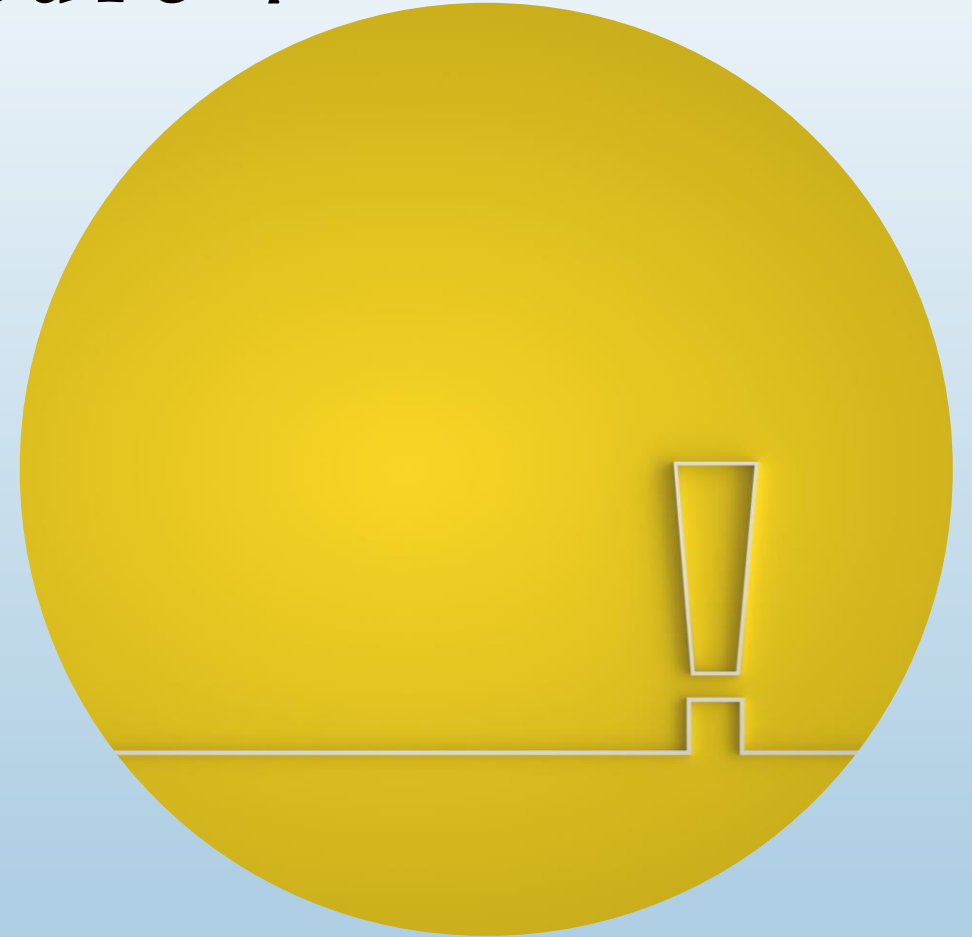
Decision Analysis

Kahenya, N.P



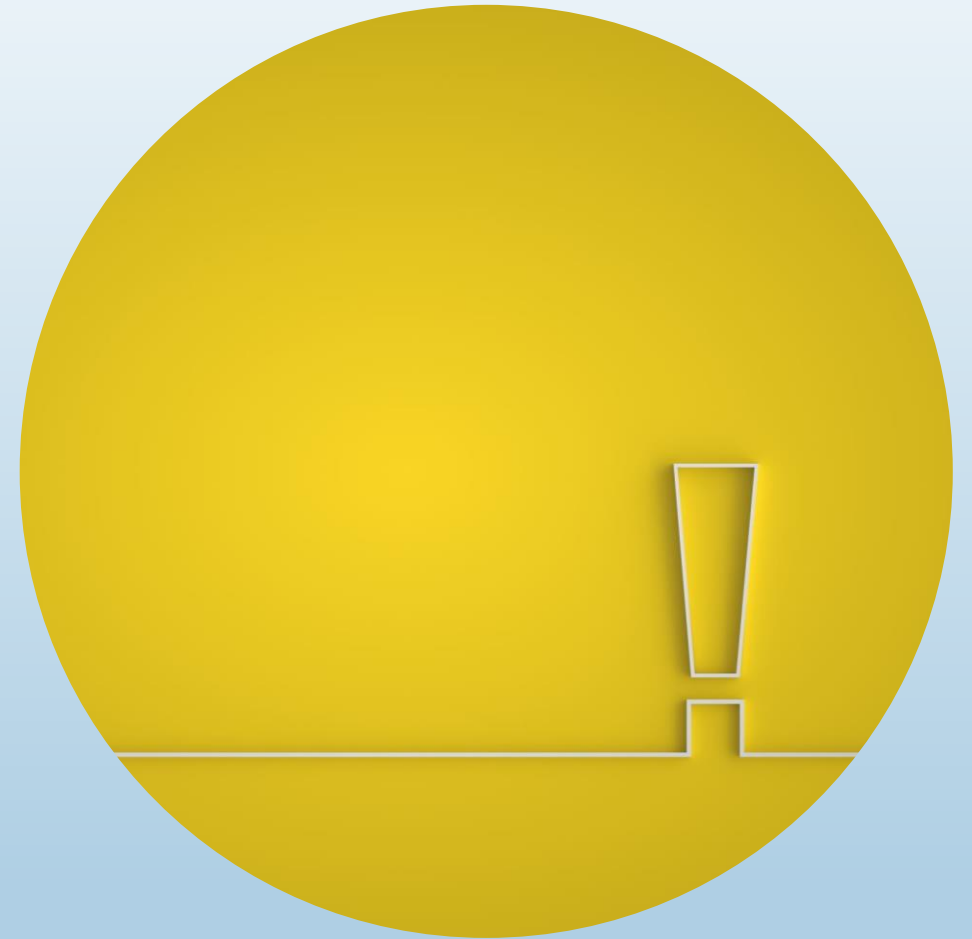
Introduction to Lecture 7

- This lecture is continuation of lecture 6 on probability and decision-making.
- It introduces the Markov chain and Decision Analysis.



Further Readings

You can get more insight into Markov chain and Decision analysis in the following resources; (Lay et al., 2016).



Intended Learning Outcomes

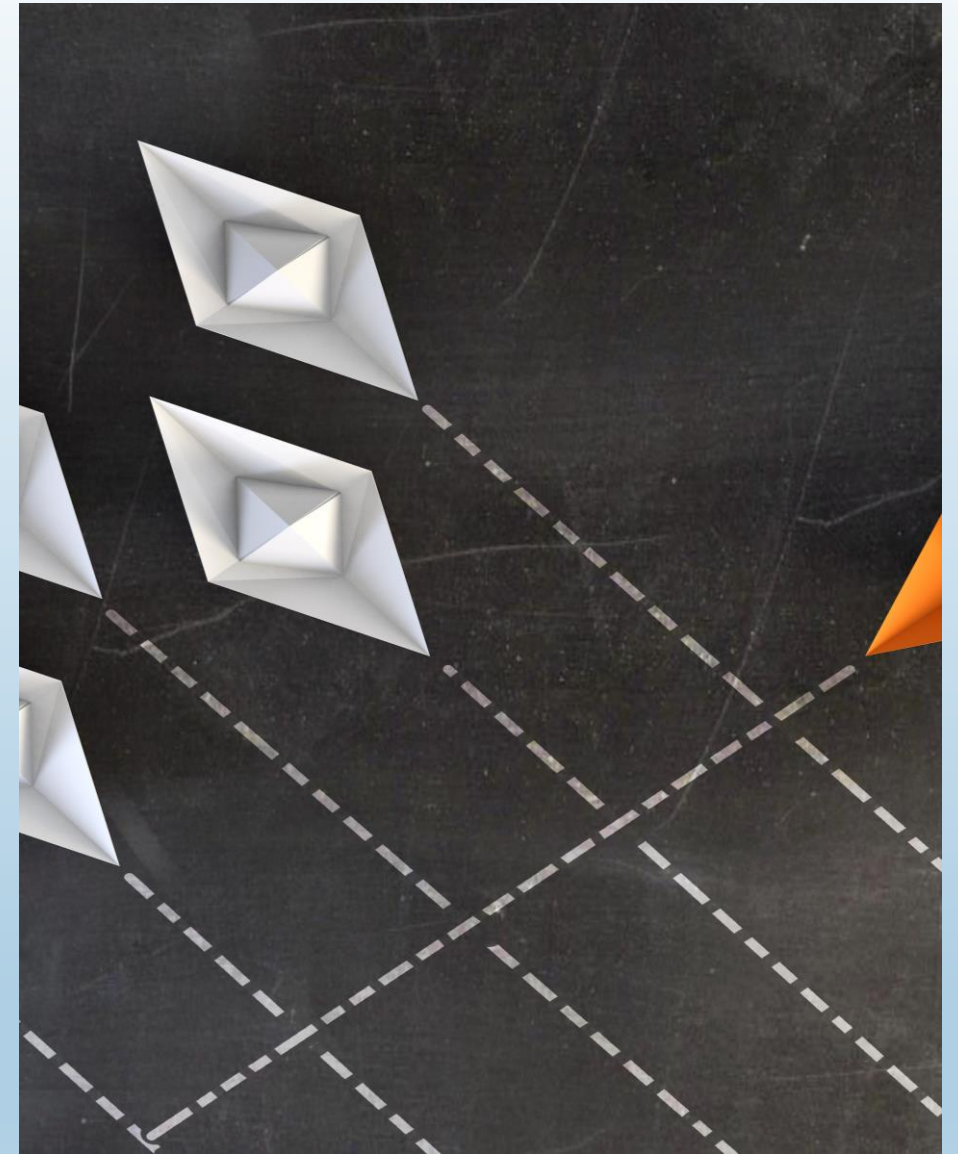
Explain Markov
Chain

Describe basic
decision analysis

Determine the
maximin minimax
and minimax
approach

Introduction

- Decision analysis is a systematic, quantitative, and visual approach that is used to evaluate complex decisions.
- This involves breaking down a decision problem into its components, such as alternatives, uncertainties, and objectives, and then applying mathematical and analytical techniques to identify the best course of action.
- It helps determine optimal strategies when one is faced with several options or alternatives and an uncertain risk-filled pattern of future happenings.



Components of Decision Analysis

Identifying the
decision

Identifying the
alternatives

Components of Decision Analysis... contd.

Identifying
uncertainties

Defining
Consequences

Components of Decision Analysis... contd.

Quantitative
Analysis

Visual
Representation

Markov Chain

A Markov chain is a mathematical model used to describe a sequence of events where the probability of transitioning from one state to another depends only on the current state and not on the sequence of events that preceded it

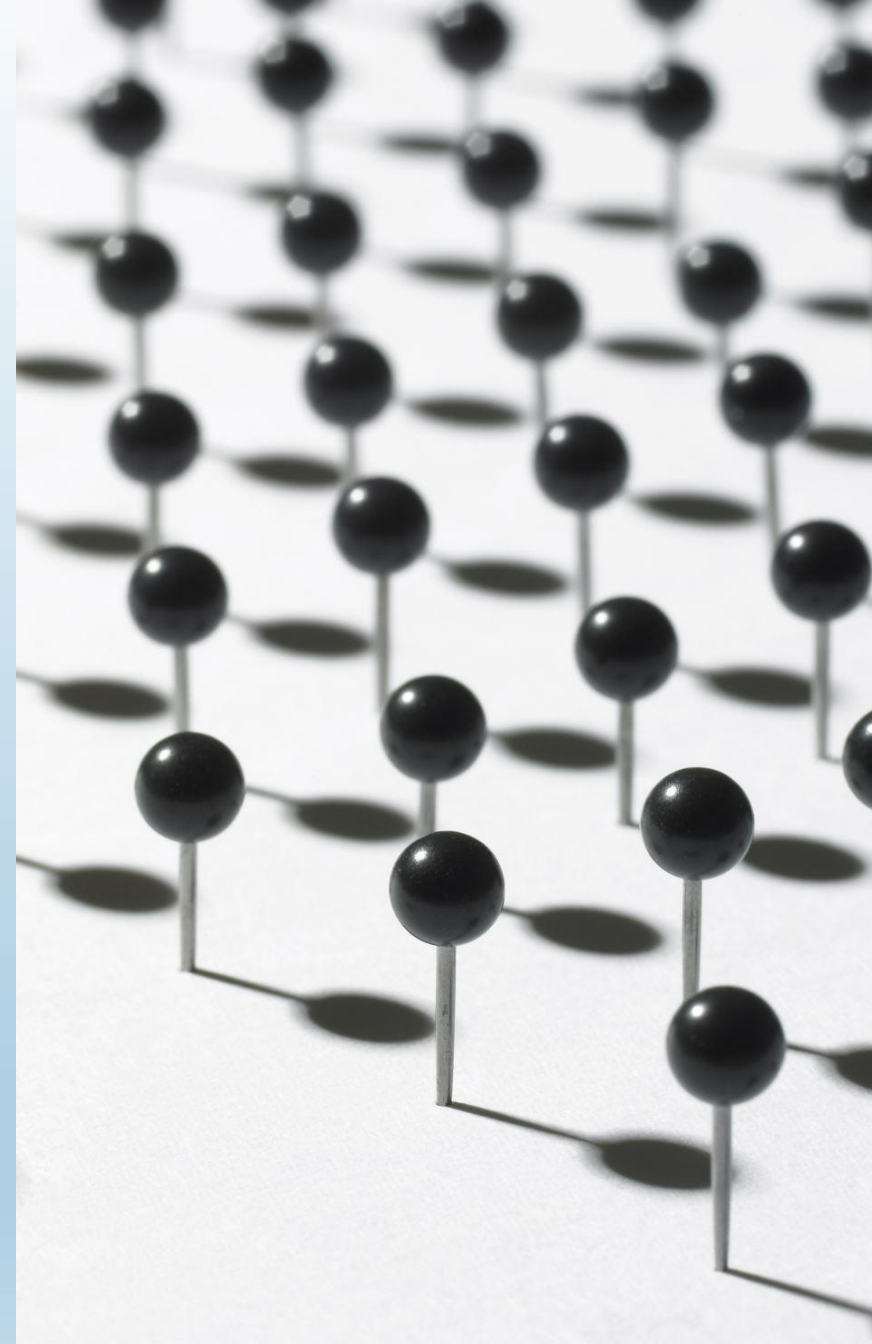


Probability Vector

A vector with non-negative entries that sum up to 1 is called a probability vector.

$$x = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$$

$$y = \begin{pmatrix} 0.9 \\ 0.1 \end{pmatrix}$$

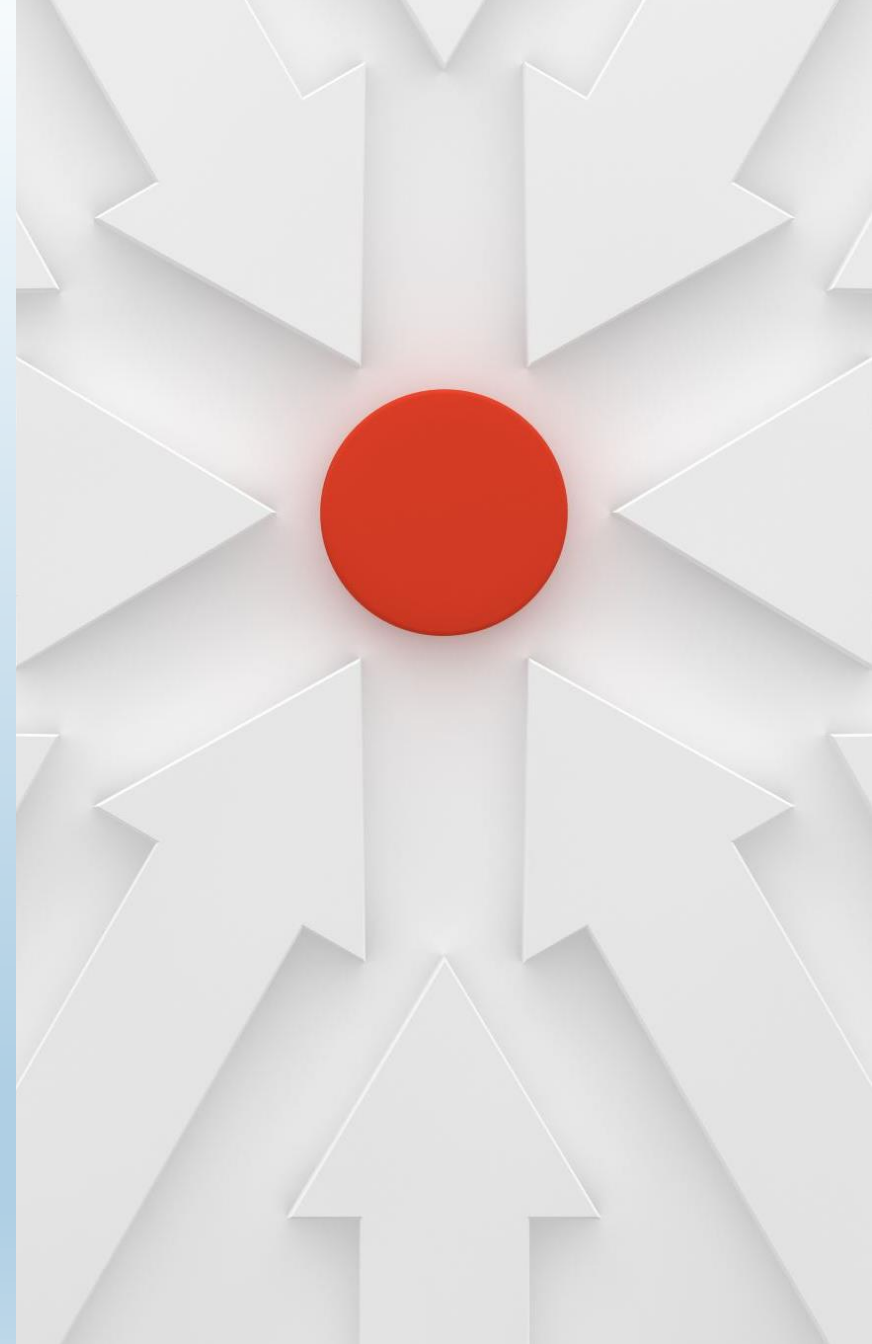


Stochastic Matrix

A stochastic matrix is a square matrix whose columns are probability vectors

$$X = \begin{pmatrix} 0.2 & 0.6 \\ 0.8 & 0.4 \end{pmatrix}$$

1.0 1.0



Markov Chain

A Markov chain is a sequence of probability vectors $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots$ together with a stochastic matrix M such that;

$$\mathbf{x}_1 = M\mathbf{x}_0, \mathbf{x}_2 = M\mathbf{x}_1, \mathbf{x}_3 = M\mathbf{x}_2, \dots$$

$$\Rightarrow \mathbf{x}_{i+1} = M\mathbf{x}_i \text{ with } i = 0, 1, 2, \dots$$

Where \mathbf{x}_i is called a state vector.

$$M = \begin{pmatrix} 0.1 & 0.3 \\ 0.9 & 0.7 \end{pmatrix}$$



Application

Markov chain can be used to model:

- Random processes such queueing systems.
- Economic phenomena such as interest rates, stock prices among others.
- Network protocols, machine learning algorithms, and natural language processing.
- Evolutionary process, populations dynamics, and genetic algorithms.



Example 1

Three TV stations A, B, and C commands 60%, 30%, and 10% of the market share in a given county respectively. It is noted that;

- TV station A retains 50% of its customer base but loses 10% to station B and 40% to station C.
 - TV station B retains 70% of its customer base but loses 5% to station A and 25% to station C
 - TV station C retains 80% of its customer base but loses 10% to station A and another 10% to station B
- (i) Determine the stochastic matrix or the transition probability matrix
- (ii) Determine each station share in the 1st and 2nd year.

Example 1... contd.

	A	B	C	
[0.5	0.05	0.1	A
[0.1	0.7	0.1	B
[0.4	0.25	0.8	C

$$\frac{0.1}{0.1} = 0.1$$

Note that from the stochastic matrix above,

- Station A retains 50% of the market while it gains 5% of B market base and 10% of C market base.
- Station B gained 10% of A market base, and 10% of C market base as it retains 70%.
- Station C retains 80% of the market while it gains 40% of A market base and gains 25% of B market base.

Example 1... contd.

First Year Share

$$\mathbf{x}_1 = \mathbf{M}\mathbf{x}_0 \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 0.5 & 0.05 & 0.1 \\ 0.1 & 0.7 & 0.1 \\ 0.4 & 0.25 & 0.8 \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.3 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 0.325 \\ 0.28 \\ 0.395 \end{bmatrix}$$

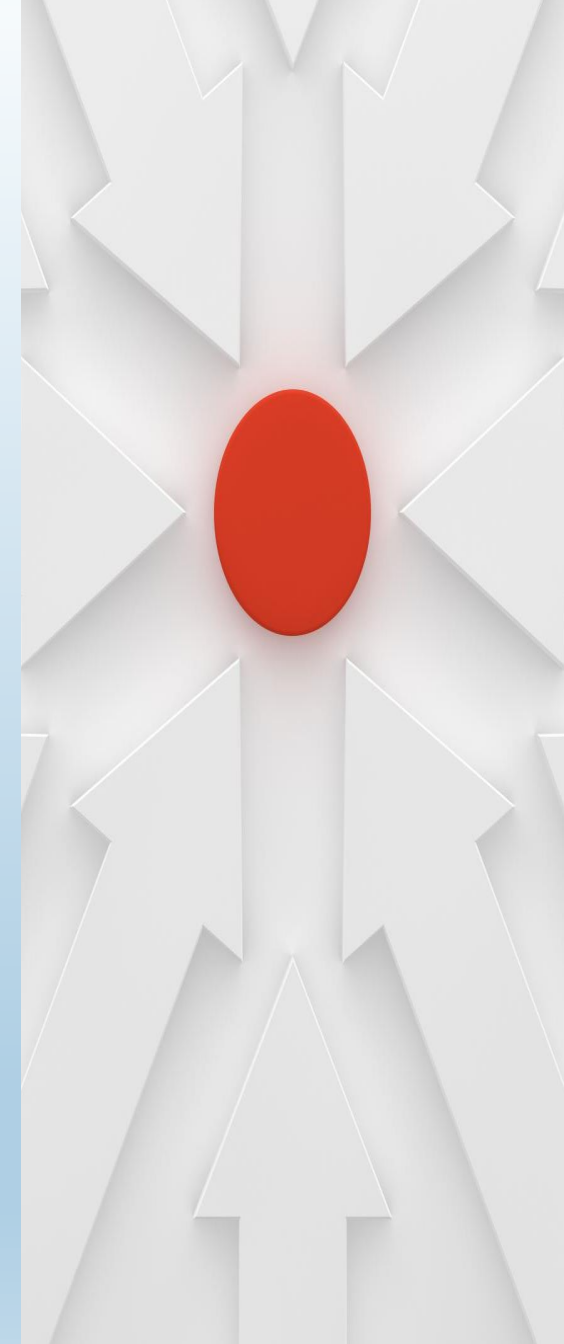
In the 1st year, TV station A will command 32.5%, TV station B will command 28% and TV station C will command 39.5%.

Example 1... contd.

Second Year Share

$$\mathbf{x}_2 = \mathbf{M}\mathbf{x}_1 \Rightarrow \mathbf{x}_2 = \begin{bmatrix} 0.5 & 0.05 & 0.1 \\ 0.1 & 0.7 & 0.1 \\ 0.4 & 0.25 & 0.8 \end{bmatrix} \begin{bmatrix} 0.325 \\ 0.28 \\ 0.395 \end{bmatrix} = \begin{bmatrix} 0.216 \\ 0.268 \\ 0.516 \end{bmatrix}$$

In the 2nd year, TV station A will command 21.6%, TV station B will command 26.8% and TV station C will command 51.6%.



Example 2

Consider the stochastic matrix $M = \begin{bmatrix} 0.6 & 0.1 & 0.4 \\ 0.4 & 0.5 & 0.1 \\ 0 & 0.4 & 0.1 \end{bmatrix}$ and vector $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Determine $\mathbf{x}_1, \dots, \mathbf{x}_{13}$

Solution:

$$\mathbf{x}_1 = M\mathbf{x}_0 \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 0.6 & 0.1 & 0.4 \\ 0.4 & 0.5 & 0.1 \\ 0 & 0.4 & 0.1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.5 \\ 0.4 \end{bmatrix}$$

$$\mathbf{x}_2 = M\mathbf{x}_1 \Rightarrow \mathbf{x}_2 = \begin{bmatrix} 0.6 & 0.1 & 0.4 \\ 0.4 & 0.5 & 0.1 \\ 0 & 0.4 & 0.1 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.5 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.27 \\ 0.33 \\ 0.40 \end{bmatrix}$$

Example 2... contd.

It can be shown that;

$x_1 \sim 4, x_2 \sim 5, \dots, x_{11}$

$$\mathbf{x}_{11} = M\mathbf{x}_{10} \Rightarrow \mathbf{x}_{11} = \begin{bmatrix} 0.6 & 0.1 & 0.4 \\ 0.4 & 0.5 & 0.1 \\ 0 & 0.4 & 0.5 \end{bmatrix} \begin{bmatrix} 0.3683 \\ 0.3509 \\ 0.2807 \end{bmatrix} = \begin{bmatrix} 0.3684 \\ 0.3509 \\ 0.2807 \end{bmatrix}$$

x_{12}

$$\mathbf{x}_{12} = M\mathbf{x}_{11} \Rightarrow \mathbf{x}_{12} = \begin{bmatrix} 0.6 & 0.1 & 0.4 \\ 0.4 & 0.5 & 0.1 \\ 0 & 0.4 & 0.5 \end{bmatrix} \begin{bmatrix} 0.3684 \\ 0.3509 \\ 0.2807 \end{bmatrix} = \begin{bmatrix} 0.3684 \\ 0.3509 \\ 0.2807 \end{bmatrix}$$

$$\mathbf{x}_{13} = M\mathbf{x}_{12} \Rightarrow \mathbf{x}_{13} = \begin{bmatrix} 0.6 & 0.1 & 0.4 \\ 0.4 & 0.5 & 0.1 \\ 0 & 0.4 & 0.5 \end{bmatrix} \begin{bmatrix} 0.3684 \\ 0.3509 \\ 0.2807 \end{bmatrix} = \begin{bmatrix} 0.3684 \\ 0.3509 \\ 0.2807 \end{bmatrix}$$

Example 2... contd.

$$\begin{aligned}x_3 &= M x_2 \\x_2 &= M x_1 \\x_1 &= M x_0\end{aligned}$$

Note that at some point the probabilities i.e.

$$\mathbf{x} = \begin{bmatrix} 0.3684 \\ 0.3509 \\ 0.2807 \end{bmatrix}$$

are hardly changing such that our system is $M\mathbf{x} = \mathbf{x}$

This state is referred to as steady state, long run or market equilibrium and vector \mathbf{x} is called a steady-state vector or equilibrium vector for the stochastic matrix M .



Example 3

Determine the steady-state vector given the matrix $\begin{bmatrix} 0.1 & 0.3 \\ 0.9 & 0.7 \end{bmatrix}$

Solution: Let the steady-state vector be $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then we have $M\mathbf{x} = \mathbf{x}$ then;

$$\begin{bmatrix} 0.1 & 0.3 \\ 0.9 & 0.7 \end{bmatrix} \mathbf{x} = \mathbf{x}$$

$$\begin{bmatrix} 0.1 & 0.3 \\ 0.9 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} 0.1x_1 + 0.3x_2 &= x_1 \\ 0.9x_1 + 0.7x_2 &= x_2 \end{aligned}$$

$$-0.9x_1 + 0.3x_2 = 0$$

$$0.9x_1 - 0.3x_2 = 0$$

From the above equations we have;

$$-0.9x_1 + 0.3x_2 = 0$$

$$0.9x_1 - 0.3x_2 = 0$$



Example 3... contd.

$$-0.9x_1 + 0.3x_2 = 0$$

$$0.9x_1 - 0.3x_2 = 0$$

Reducing the augmented matrix for the system into echelon form to get;

$$\begin{bmatrix} -0.9 & 0.3 & 0 \\ 0.9 & -0.3 & 0 \end{bmatrix} r_2 + r_1 \sim \begin{bmatrix} -0.9 & 0.3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(Handwritten notes: x_1 with an arrow pointing to the first column; -0.9 and 0.3 circled in the first row of the second matrix; -0.9 and 0.3 circled in the first row of the first matrix; 0.9 and -0.3 circled in the second row of the first matrix; a large circle around the second matrix; -0.9 and 0.3 circled in the first row of the augmented matrix; 0.9 and -0.3 circled in the second row of the augmented matrix.)

From the row echelon form x_2 is a free variable and hence we let it be α

From row 1 we have $-0.9x_1 = -0.3x_2 \therefore x_1 = \frac{1}{3}x_2 = \frac{1}{3}\alpha$

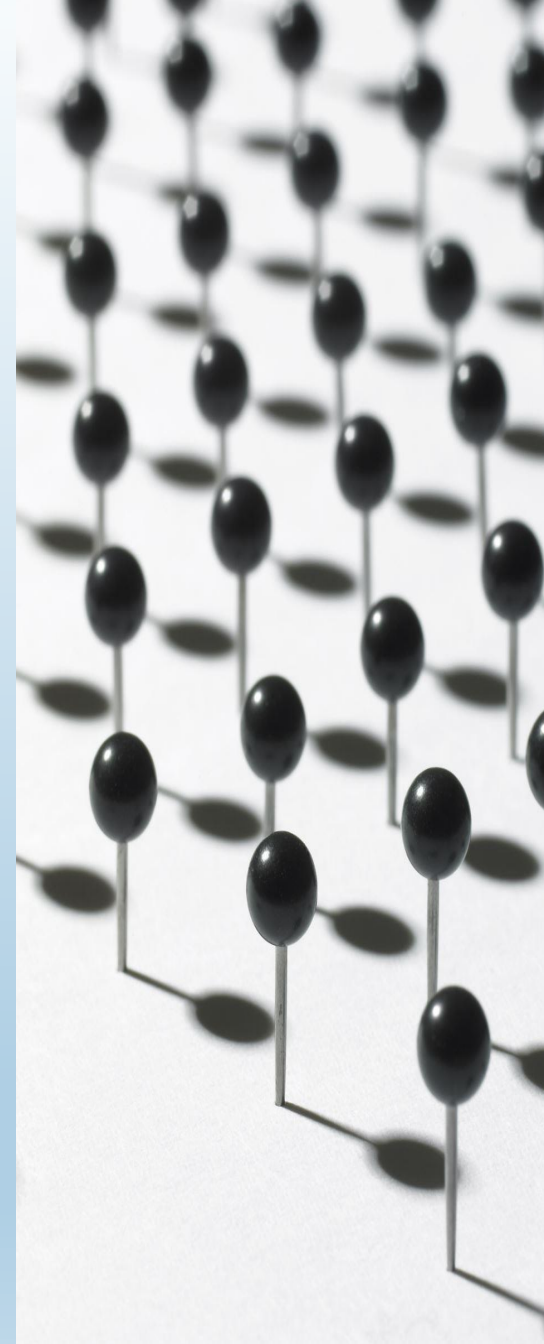
(Handwritten notes: -0.9 and -0.3 underlined; $\frac{1}{3}x_2$ circled.)

$$\Rightarrow \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

(Handwritten note: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ circled.)

We can say that the basis for the solution set is vector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

(Handwritten note: $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ circled.)



Example 3... contd.

$$\tilde{x} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

From this we can get the probability vector for the steady state i.e.

$$\text{vector } \mathbf{x} = \begin{bmatrix} 1 \\ 4 \\ 3 \\ 4 \end{bmatrix}$$

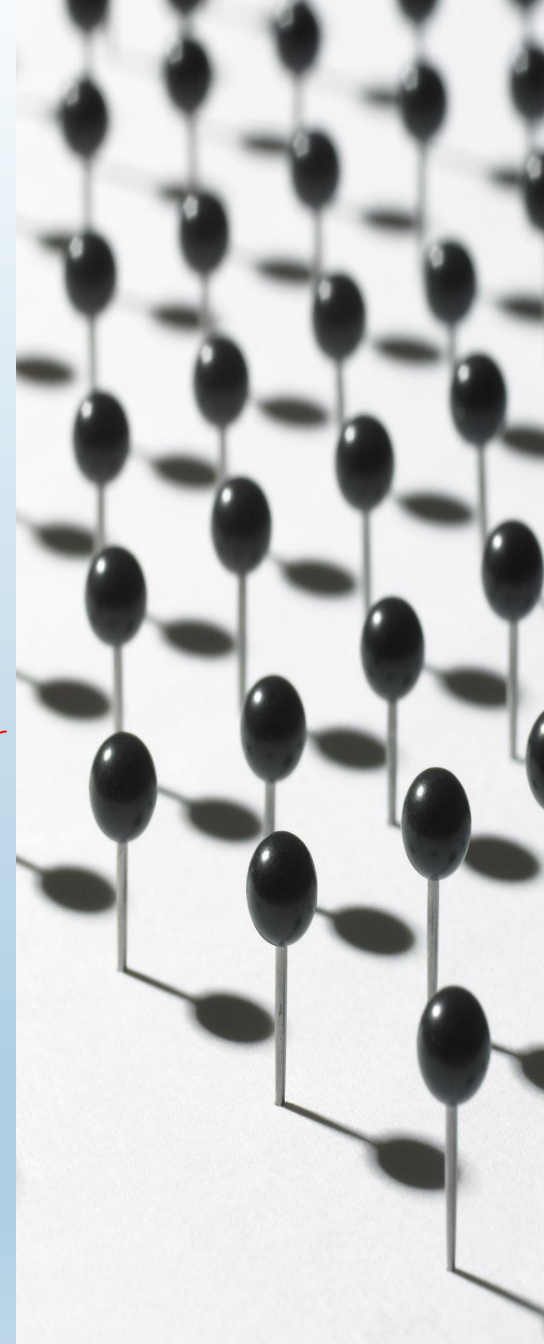
$$\tilde{x} = M \tilde{x} \quad \left(\begin{matrix} 1/4 & 3/4 \end{matrix} \right)$$

$$0.1 \times 0.25 + 0.3 \times 0.75 = 0.25$$

$$0.9 \times 0.25 + 0.7 \times 0.75 = 0.75$$

You can easily check that;

$$\begin{bmatrix} 0.1 & 0.3 \\ 0.9 & 0.7 \end{bmatrix} \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix}$$



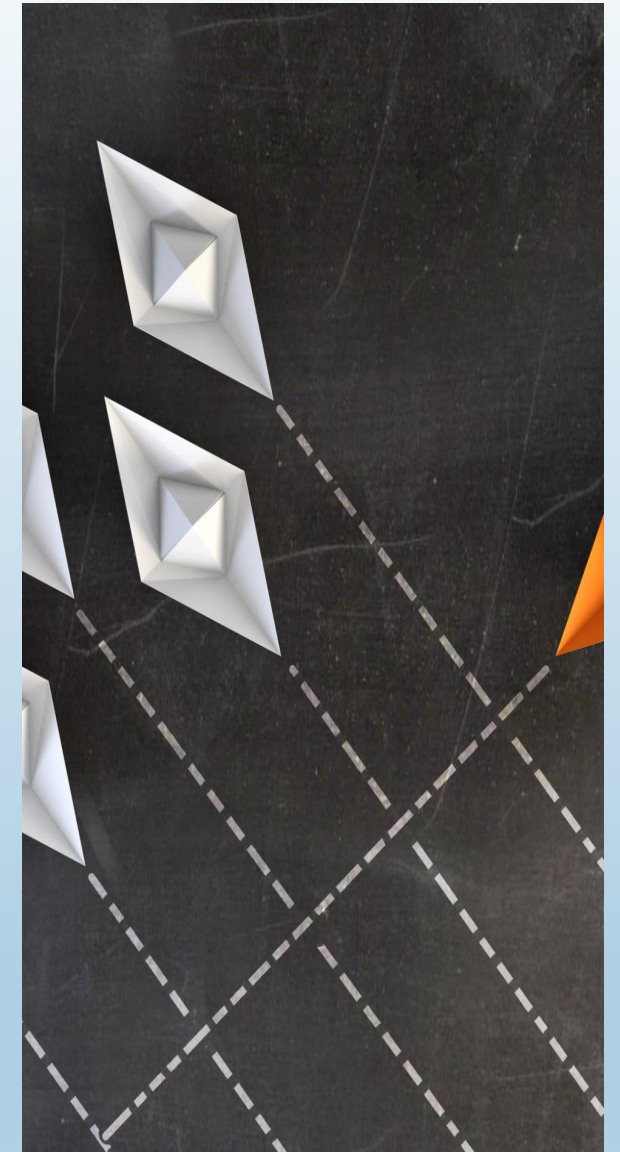
Decision Analysis

There exist different approaches in decision making.

In this lecture we shall look at some non-probability-based techniques such as the maximin minimax approach, and minimax regret approach.

Non-probability-based techniques

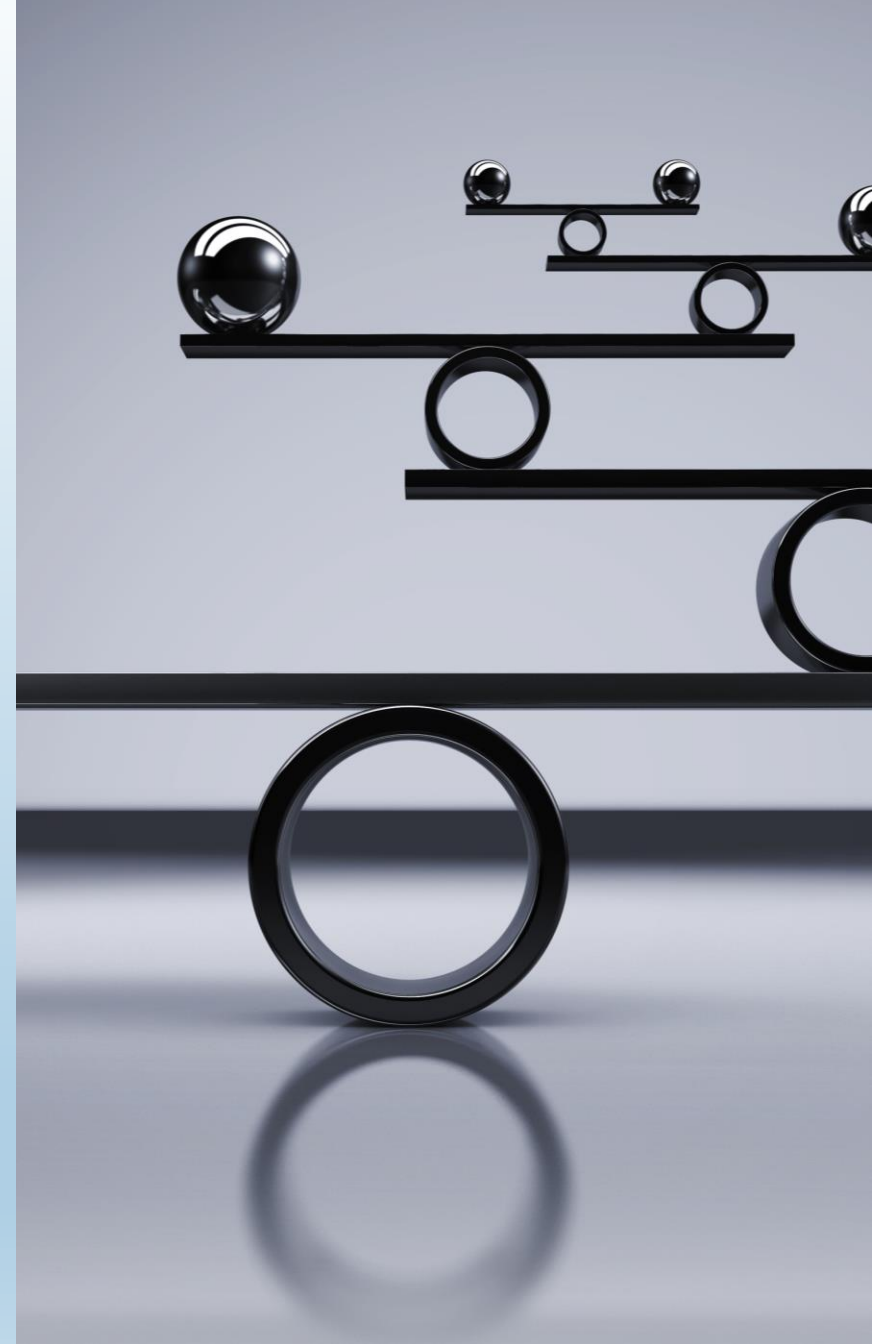
- Are easier to maintain
- Require less resources
- Are usually simple



Steps in Decision Analysis

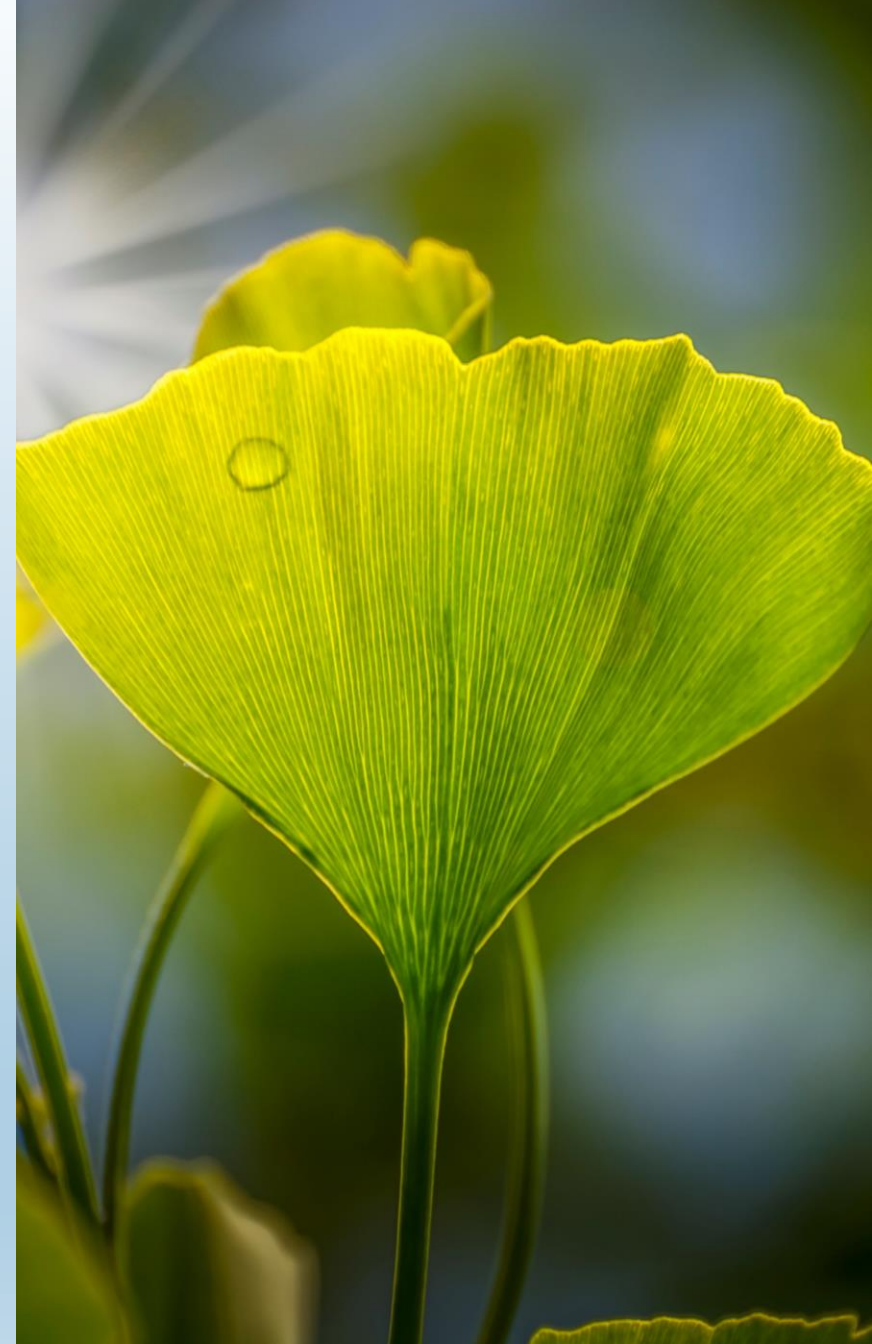
Steps in decision analysis in non-probability techniques

- Determining all the alternatives.
- Determining the possible scenarios for each alternative.
- Organize both the alternatives and scenarios in a meaningful way e.g. using payoff tables.



Payoff Tables

- Payoff tables help organize alternatives and scenario in a better way.
- One is best place to decide among alternatives, options, and actions. It easier to visualize the different possible scenarios.
- Scenarios can be categorized into three categories; Best possible, worst possible or most likely scenario.



PAYOFF TABLES

- In a payoff table, alternatives take the columns while the scenarios take the rows.
- The payoffs or outcomes or expected values or estimates are input into the table.
- The payoffs can be profits, returns, costs, revenues, or other values necessary for the decision-making process.



Payoff Tables

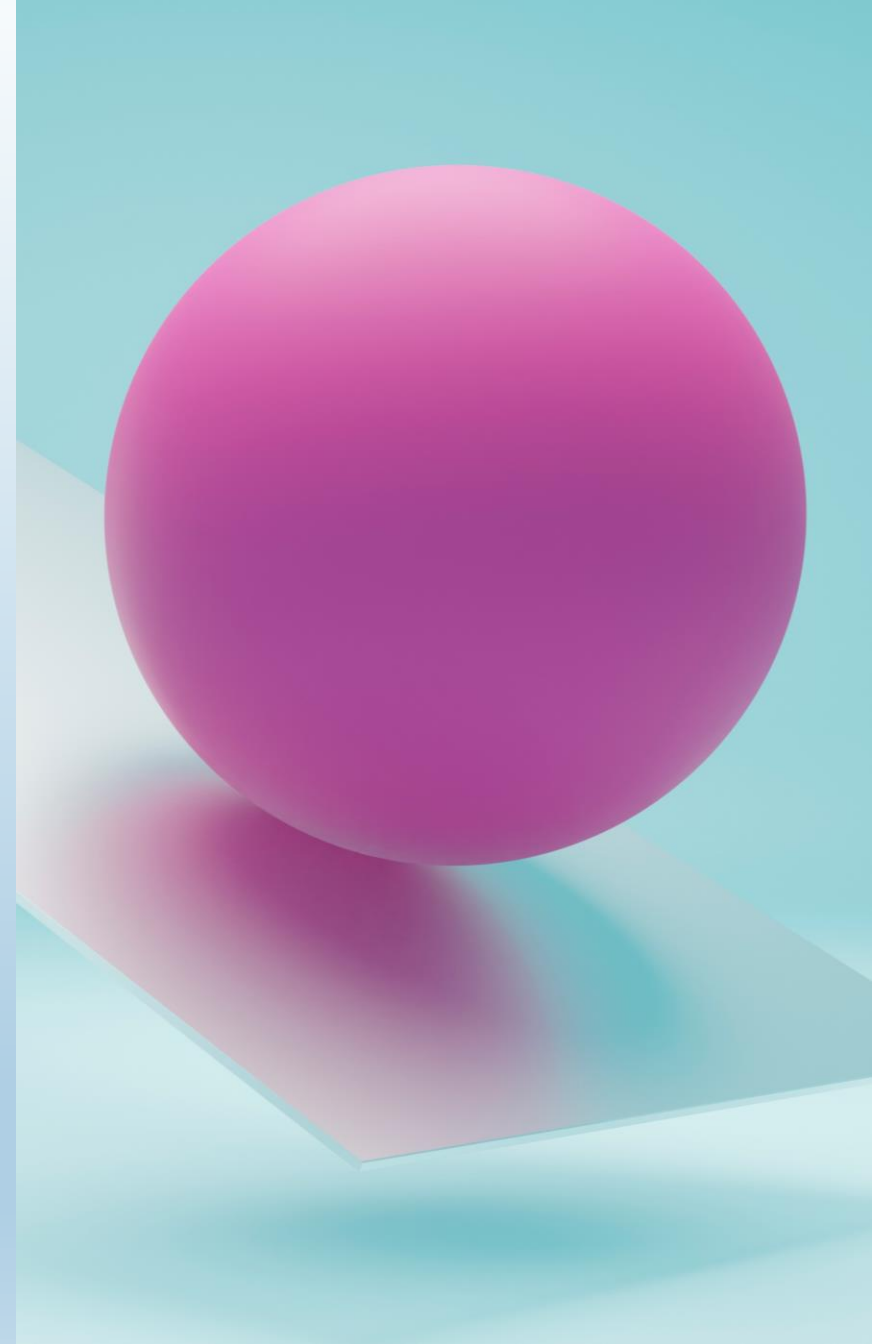
		Alternatives		
		A	B	C
Scenarios	1			
	2			
	3			

		Scenario		
		Best	Most likely	Worst
Alternatives	1			
	2			
	3			

Risk Taker

Analyzing different alternatives depends on the individual's attitude to risk or risk preferences.

- A risk taker is assumed to be an optimistic person and hence he will also think of the best can happen.
- A risk taker always considers the best outcomes.
- He will maximize the possible benefit by taking more risk.
- A risk taker has a decision rule of Maximax.



Risk Avoider

- A risk avoider is assumed to be pessimistic.
- He will think the worst can happen and hence is always considering the worst outcomes.
- He will aim to minimize the possible loss by taking less risk.
- A risk avoider has decision rule of Maximin.

RISK NEUTRAL

- A risk neutral person has a well-balanced approach and tends to remain neutral.
- This person will consider the best outcomes, the worst outcome, and the most likely outcomes.
- He will therefore tend to balance between benefit and loss with risk.
- The choice of a risk neutral person is minimax regret rule.



Example 1

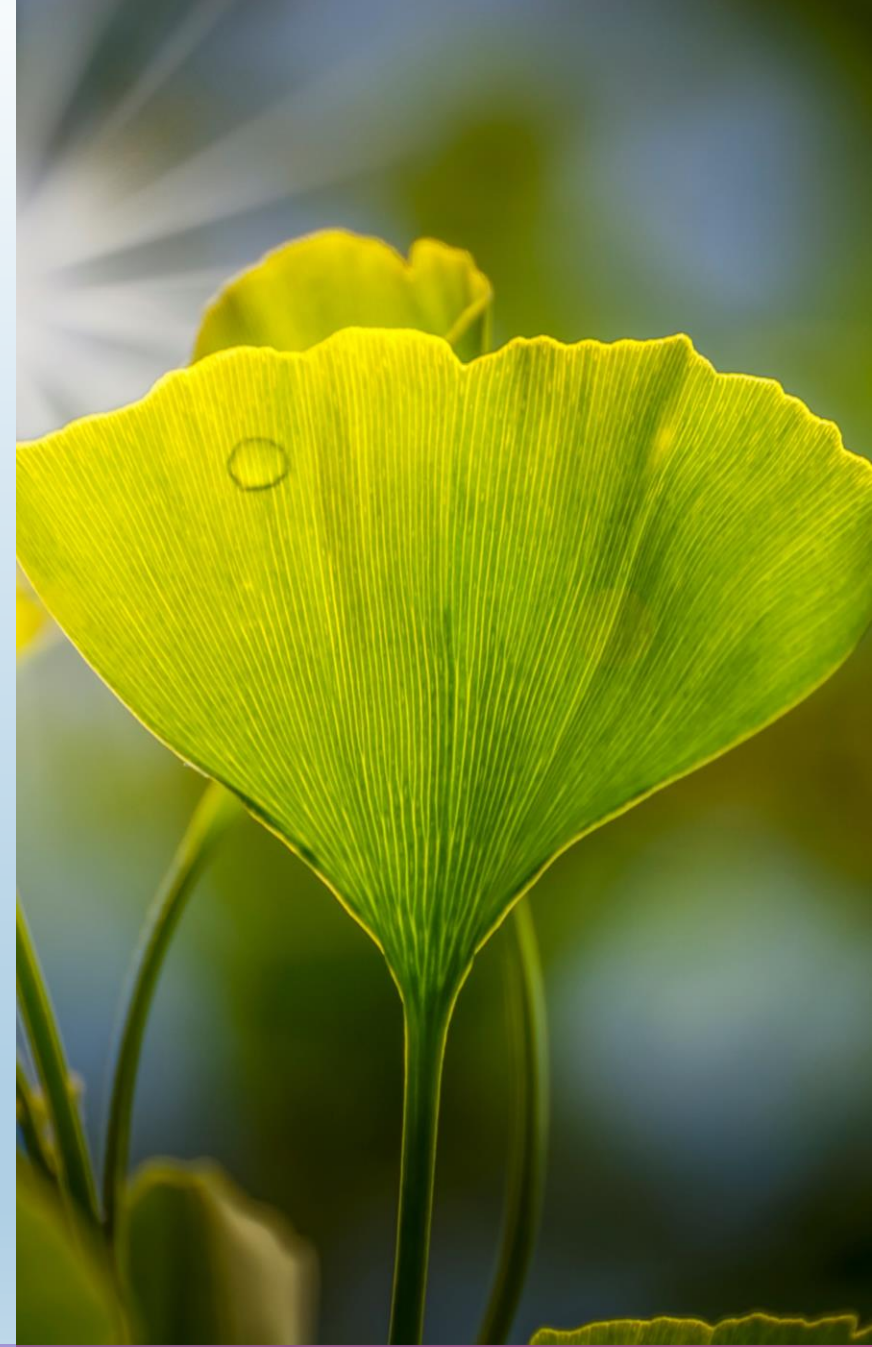
Risk takes

(Maximax – Maximising the Maximum Possible Outcome)

Consider the following payoff table.

		Alternatives		
		A	B	C
Scenarios	Best possible	120	130	98
	Most Possible	96	87	75
	Worst possible	20	74	53
	<u>Best</u>	120	130	98

Decision: Take up Alternative B



Example 1...contd

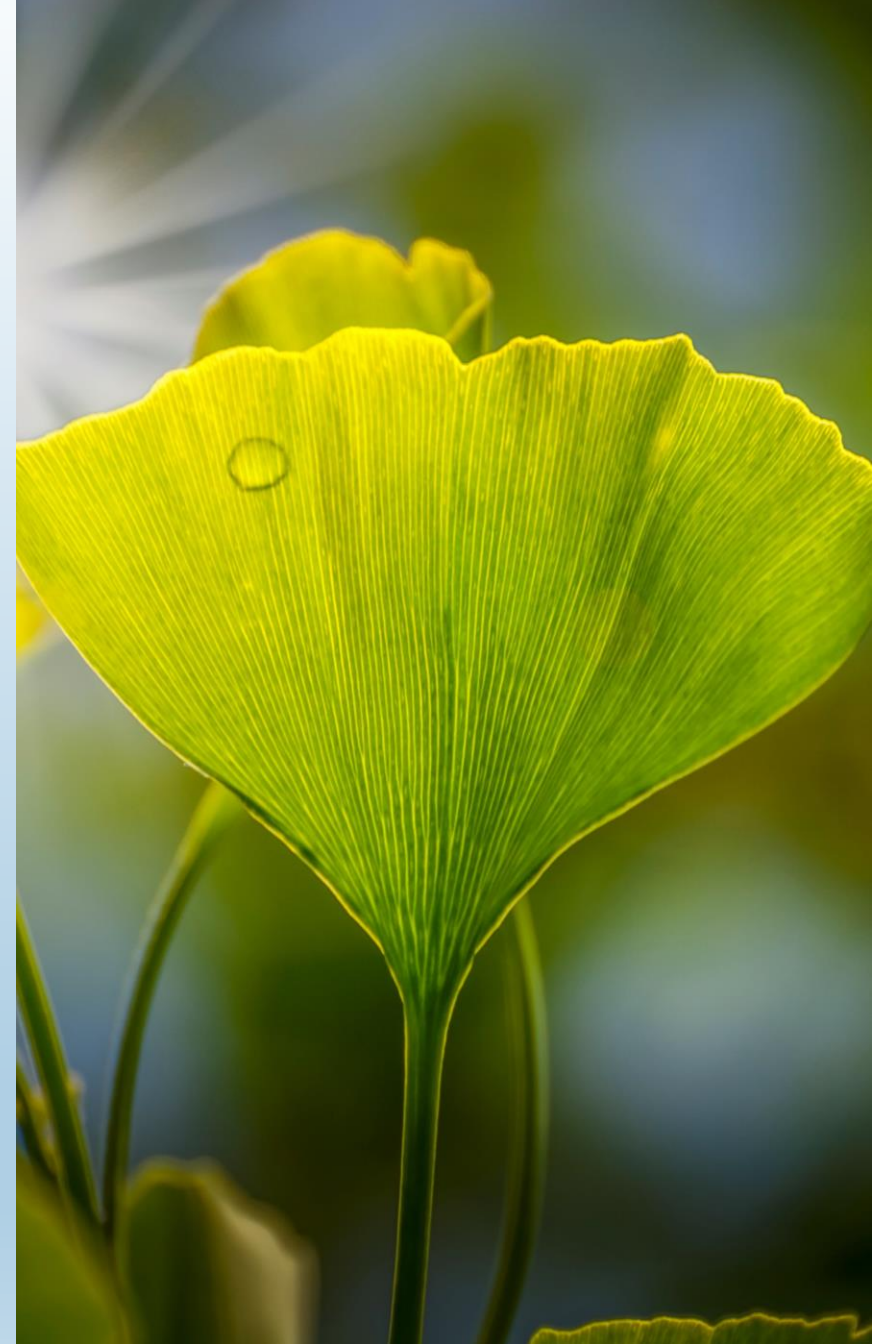
Maximin – Maximizing the minimum possible outcome

Maximin

		Alternatives		
		A	B	C
Scenarios	Best possible	120	130	98
	Most Possible	96	87	75
	Worst possible	20	74	53
	Worst	20	74	53

Maximum worst payoff is 74

Decision: Take up Alternative B



Minimax Regret

- Minimizing the maximum possible regret
- Regret is the opportunity of maximizing return lost.
- It is calculated for every option along the scenario.
- Regret for an option is the same as the difference between return from option opted and return from the best option i.e.

Regret = Best payoff – payoff received.

- Note then that the Regret for best option is zero

		Alternatives		
		A	B	C
Scenarios	Best possible	120 - 130 = -10	130 - 130 = 0	98 - 130 = -32
	Most Possible	96 - 96 = 0	87 - 96 = -9	75 - 96 = -21
	Worst possible	20 - 74 = -54	74 - 74 = 0	53 - 74 = -21

		Regret Table		
		A	B	C
Scenarios	Best possible	120 - 130 = -10 -10	130 - 130 = 0	98 - 130 = -32
	Most Possible	96 - 96 = 0	87 - 96 = -9	75 - 96 = -21
	Worst possible	20 - 74 = -54	74 - 74 = 0	53 - 74 = -21

Minimax Regret

	A	B	C
Maximum Regret	-54	-9	-21

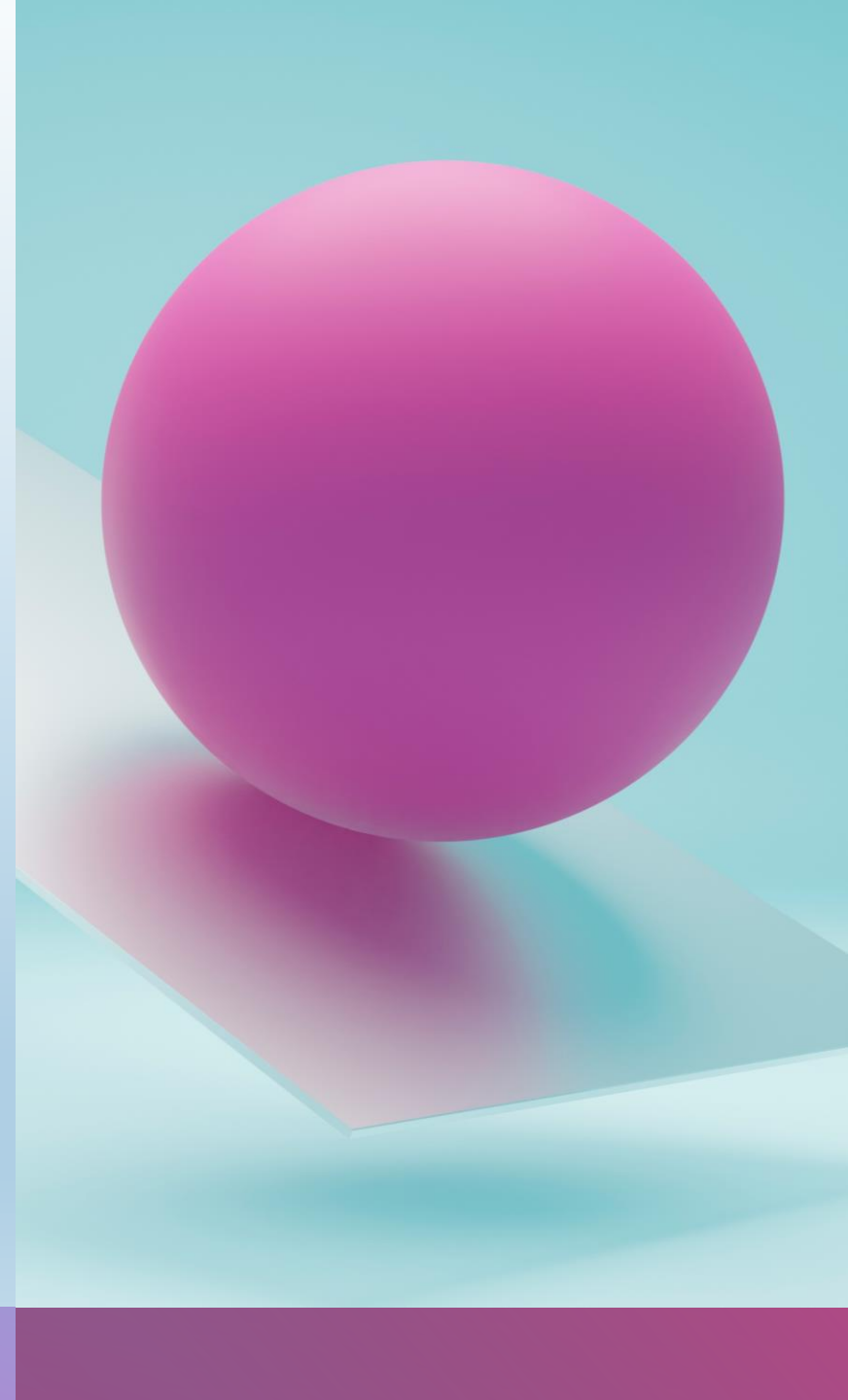
Then the minimum maximum regret is -9

Decision: Take up Alternative B

Example 2

Consider the following Payoff table where one wants to make a decision on which company to invest on. The payoffs are dividends.

		Economy Scenarios		
		Growing	Stable	Declining
Alternatives	SAFCOM	400	470	50
	KENPAN	700	350	-10
	KAS	530	470	-50

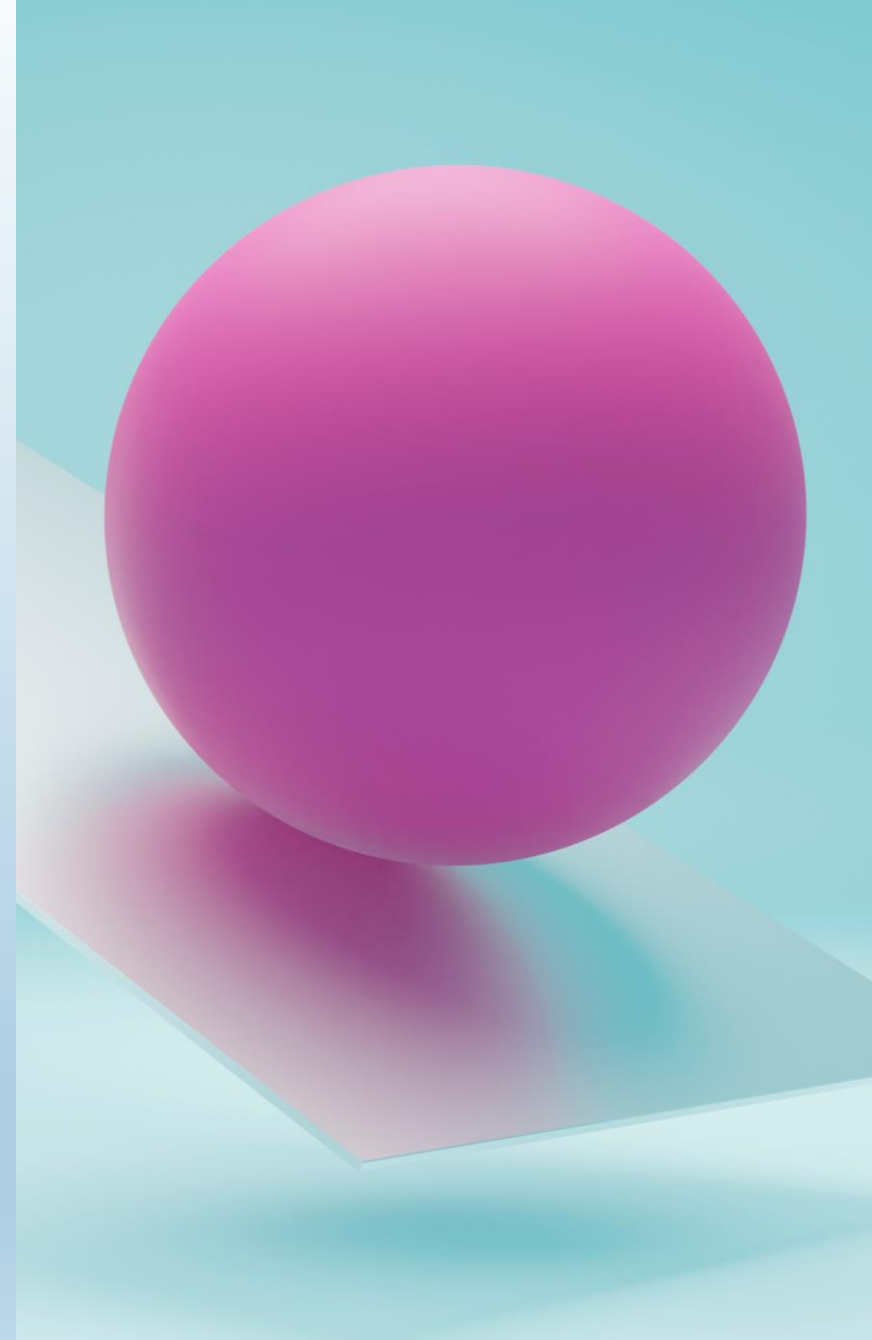


Example 2... contd.

- i) Consider the Maximax approach (Optimistic approach) – it maximizes the maximum payoff i.e. best of best

		Economy Scenarios			BEST
		Growing	Stable	Declining	
Alternatives	SAFCOM	400	470	50	470
	KENPAN	700	350	-10	700
	KAS	530	470	-50	530 530

Decision: Invest in KENPAN

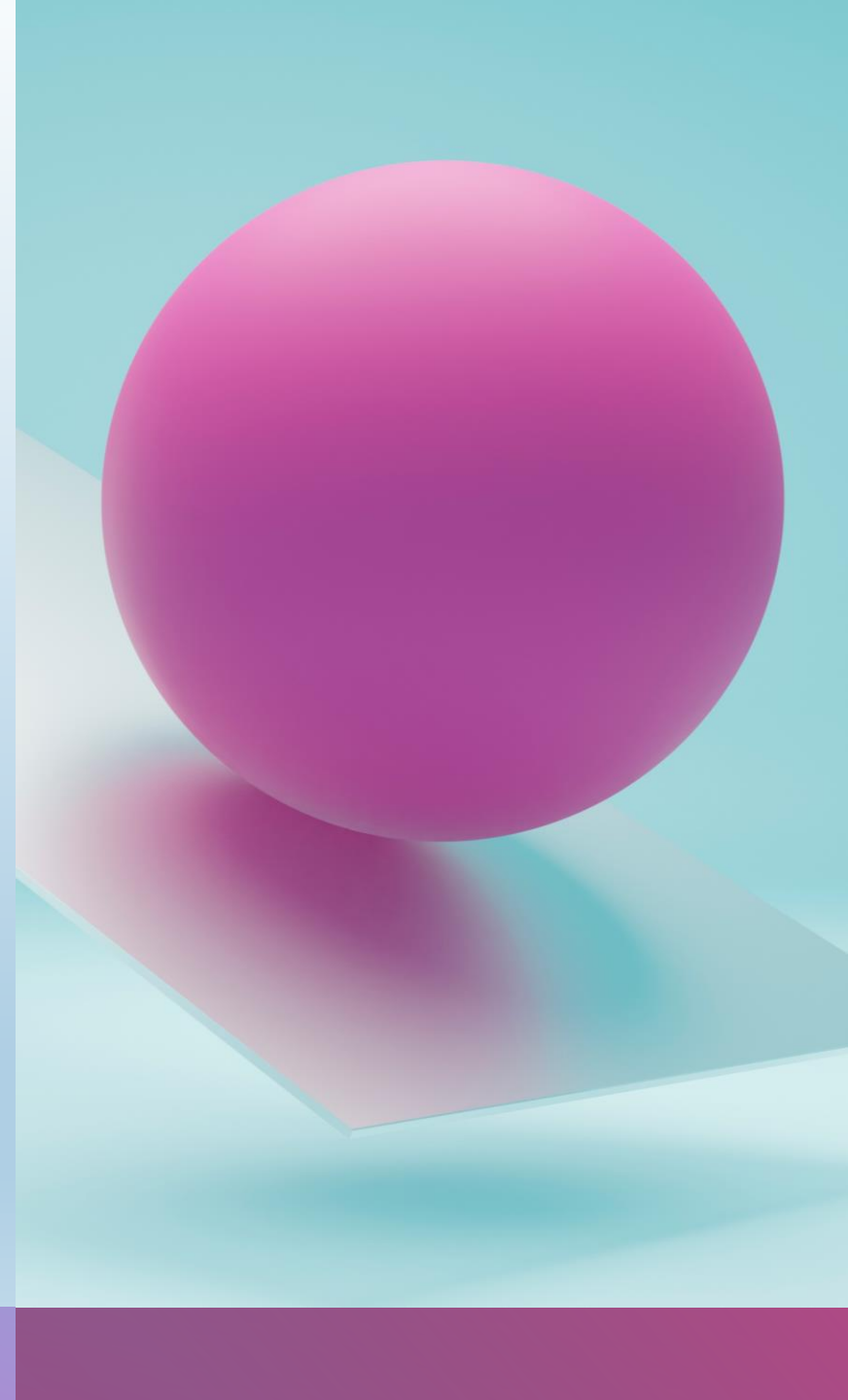


Example 2... contd.

- i) Maximin Approach (Pessimistic approach) – maximizes the minimum payoff i.e. best of the worst payoff

		Economy Scenarios			Worst
		Growing	Stable	Declining	
Alternatives	SAFCOM	400	470	50	50
	KENPAN	700	350	-10	-10
	KAS	530	470	-50	-50

Decision: Invest in SAFCOM



Example 2...contd.

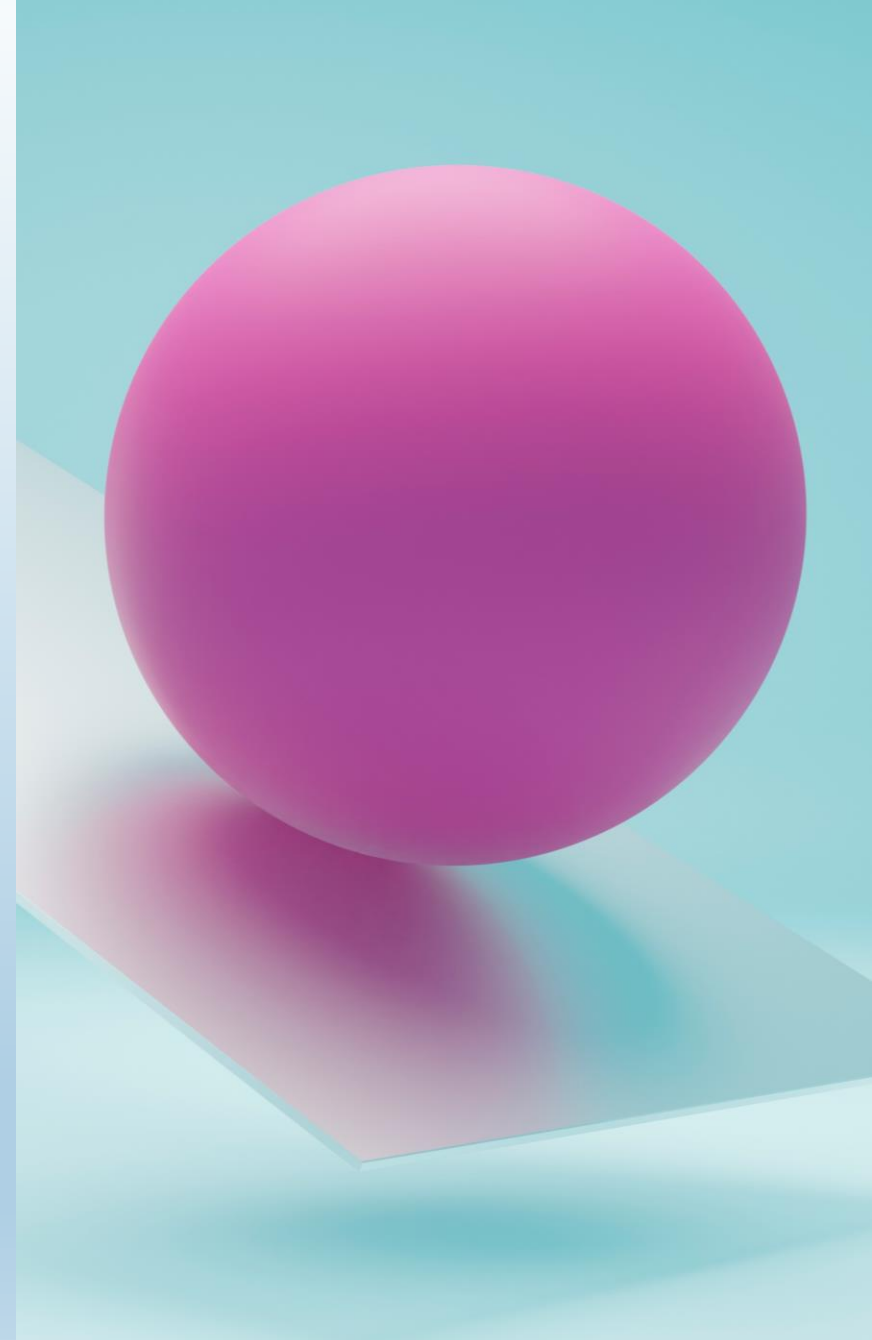
i) Minimax regret approach – Minimizes the maximum regret

		Economy Scenarios		
		Growing	Stable	Declining
Alternatives	SAFCOM	700-400=300	470-470=0	50-50=0
	KENPAN	700-700=0	470-350=120	50-(-10)=60
	KAS	700-530=170	470-470=0	50-(-50)=100

Regret table

		Economy Scenarios			Maximum Regret
		Growing	Stable	Declining	
Alternatives	SAFCOM	300	0	0	300
	KENPAN	0	120	60	120 (Minimum)
	KAS	170	0	100	170

Decision: Invest in KENPAN



References

Lay, D. C., Lay, S. R., &
McDonald, J. J. (2016). *Linear
Algebra and its Application* (5th ed.).
Pearson.

Decision Analysis 1.1 (Costs) -
Optimistic, Conservative,
Minimax Regret (youtube.com)



Thank You!

Intro & Outro music, 'Cross Your Mind' source: Content library ClipChamp ©.
Graphics, Source: Microsoft 365 Powerpoint