Business Mathematics

Lecture 7

Decision Analysis

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Introduction to Lecture 7

This lecture is continuation of lecture 6 on Probability and Decision-making. It introduces the Markov chain and Decision Analysis.

Further Readings

You can get more insight into Markov chain and decision analysis in the following resources; (Lay et al., 2016).

Intended Learning Outcomes

At the end of this lecture, you will be able to;

(i) Explain Markov chain.
(ii) Describe basic decision analysis.
(iii) Determine the maximin minimax and minimax regret approach

Introduction

Decision analysis is a systematic, quantitative, and visual approach that is used to evaluate complex decisions. This involves breaking down a decision problem into its components, such as alternatives, uncertainties, and objectives, and then applying mathematical and analytical techniques to identify the best course of action. It helps determine optimal strategies when one is faced with several options or alternatives and an uncertain risk-filled pattern of future happenings.

Components of decision analysis

Identifying the decision: One should identify the problem or opportunity and determine the objectives to be achieved, then clearly define the decision to be made.

Identifying the alternatives: Decision analysis involves considering different courses of action or alternatives that could be taken to address the decision problem. What other options exists?

Identifying uncertainties: One should identify the uncertainties that could affect the outcome of each alternative. What are the conditions of uncertainties under which the decisions are being made?

Defining consequences: One should define the consequences for each alternative opted and its outcome.

Each combination of course of action and events or outcome is normally associated with a *payoff*. It is ideally the net benefit in decision making that arises from a combination of decision alternatives and events. We can also refer them as conditional economic consequences or conditional profit values.

Quantitative analysis: Decision analysis requires the use of mathematical and statistical techniques to model the relationships between alternatives, uncertainties, and outcomes. Probability theory and decisions trees are some of the techniques that one can use to model for decision-making.

Visual representation: Visual tools are used to represent the decision problem and its components in a clear and intuitive manner.

Markov Chain

Definition 1: A Markov chain is a mathematical model used to describe a sequence of events where the probability of transitioning from one state to another depends only on the current state and not on the sequence of events that preceded it.

Markov process model is a stochastic model that is time based and is used to make prediction of various states in the next future given their current state.

Definition 2: A vector with non-negative entries that sum up to 1 is called a probability vector.

Definition 3: A stochastic matrix is a square matrix whose columns are probability vectors.

Definition 4: A Markov chain is a sequence or probability vectors $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ together with a stochastic matrix M such that;

$$\mathbf{x_1} = \mathrm{M}\mathbf{x}_0, \mathbf{x}_2 = \mathrm{M}\mathbf{x}_1, \mathbf{x}_3 = \mathrm{M}\mathbf{x}_2, \cdots$$

$$\Rightarrow \mathbf{x_{i+1}} = \mathrm{M}\mathbf{x_i} \text{ with i} = 0,1,2,\cdots$$

Where $\mathbf{x_i}$ is called a state vector.

Components of a Markov Chain

States: A state represent a situation or condition that the system can be in.

Transition: Transition probabilities characterize the likelihood of moving from one state to another in a single step.

Markov property: This is probability of transitioning to a future state that depends entirely on the current state and not on the past history of the system.

Transition matrix: Transition probability is organised into a square matrix called the transition matrix or stochastic matrix. Where each entry represents the probability of transitioning from one state to another.

Chains: This is the sequence of transitions of events.

Application

Markov chain can be used to model:

- Random processes such queueing systems.
- Economic phenomena such as interest rates, stock prices among others.
- Network protocols, machine learning algorithms, and natural language processing.
- Evolutionary process, populations dynamics, and genetic algorithms.

Example 1: Three TV stations A, B, and C commands 60%, 30%, and 10% of the market share in a given county respectively. It is noted that;

- TV station A retains 50% of its customer base but loses 10% to station B and 40% to station C.
- TV station B retains 70% of its customer base but loses 5% to station A and 25% to station C
- TV station C retains 80% of its customer base but loses 10% to station A and another 10% to station B

(i) Determine the stochastic matrix or the transition probability matrix

(ii) Determine each station share in the 1st and 2nd year.

Solution:

(i)
$$\begin{matrix} & A & B & C & \\ & \left[\begin{matrix} 0.5 \\ 0.1 \\ 0.4 \end{matrix}\right. & \begin{matrix} 0.05 \\ 0.7 \\ 0.25 \end{matrix} & \left.\begin{matrix} 0.1 \\ 0.1 \\ 0.8 \end{matrix}\right] & \begin{matrix} A \\ B \\ C \end{matrix} \end{matrix}$$

Note that from the stochastic matrix above,

- Station A retains 50% of the market while it gains 5% of B market base and 10% of C market base.
- Station B gained 10% of A market base, and 10% of C market base as it retains 70%.
- Station C retains 80% of the market while it gains 40% of A market base and gains 25% of B market base.

(ii) $$\mathbf{x_1} = \mathrm{M}\mathbf{x_0} \Rightarrow \mathbf{x_1} = \begin{bmatrix} 0.5 & 0.05 & 0.1 \\ 0.1 & 0.7 & 0.1 \\ 0.4 & 0.25 & 0.8 \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.3 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 0.325 \\ 0.28 \\ 0.395 \end{bmatrix}$$

In the 1st year, TV station A will command 32.5%, TV station B will command 28% and TV station C will command 39.5%.

(iii) $$\mathbf{x_2} = \mathrm{M}\mathbf{x_1} \Rightarrow \mathbf{x_2} = \begin{bmatrix} 0.5 & 0.05 & 0.1 \\ 0.1 & 0.7 & 0.1 \\ 0.4 & 0.25 & 0.8 \end{bmatrix} \begin{bmatrix} 0.325 \\ 0.28 \\ 0.395 \end{bmatrix} = \begin{bmatrix} 0.216 \\ 0.268 \\ 0.516 \end{bmatrix}$$

In the 2nd year, TV station A will command 21.6%, TV station B will command 26.8% and TV station C will command 51.6%.

Example 2: Consider the stochastic matrix $\mathrm{M} = \begin{bmatrix} 0.6 & 0.1 & 0.4 \\ 0.4 & 0.5 & 0.1 \\ 0 & 0.4 & 0.1 \end{bmatrix}$ and vector $\boldsymbol{x}_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Determine $\mathbf{x}_1, \cdots, \mathbf{x}_{10}$

Solution:

$$\mathbf{x_1} = \mathrm{M}\mathbf{x_0} \Rightarrow \mathbf{x_1} = \begin{bmatrix} 0.6 & 0.1 & 0.4 \\ 0.4 & 0.5 & 0.1 \\ 0 & 0.4 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.5 \\ 0.4 \end{bmatrix}$$

$$\mathbf{x_2} = \mathrm{M}\mathbf{x_1} \Rightarrow \mathbf{x_2} = \begin{bmatrix} 0.6 & 0.1 & 0.4 \\ 0.4 & 0.5 & 0.1 \\ 0 & 0.4 & 0.5 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.5 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.27 \\ 0.33 \\ 0.40 \end{bmatrix}$$

$$\mathbf{x_3} = \mathrm{M}\mathbf{x_2} \Rightarrow \mathbf{x_3} = \begin{bmatrix} 0.6 & 0.1 & 0.4 \\ 0.4 & 0.5 & 0.1 \\ 0 & 0.4 & 0.5 \end{bmatrix} \begin{bmatrix} 0.27 \\ 0.33 \\ 0.40 \end{bmatrix} = \begin{bmatrix} 0.355 \\ 0.313 \\ 0.332 \end{bmatrix}$$

$$\mathbf{x_4} = \mathrm{M}\mathbf{x_3} \Rightarrow \mathbf{x_4} = \begin{bmatrix} 0.6 & 0.1 & 0.4 \\ 0.4 & 0.5 & 0.1 \\ 0 & 0.4 & 0.5 \end{bmatrix} \begin{bmatrix} 0.355 \\ 0.313 \\ 0.332 \end{bmatrix} = \begin{bmatrix} 0.3771 \\ 0.3317 \\ 0.2912 \end{bmatrix}$$

It can be shown that

$$\mathbf{x_{10}} = \mathrm{M}\mathbf{x_9} \Rightarrow \mathbf{x_{10}} = \begin{bmatrix} 0.6 & 0.1 & 0.4 \\ 0.4 & 0.5 & 0.1 \\ 0 & 0.4 & 0.5 \end{bmatrix} \begin{bmatrix} 0.3682 \\ 0.3510 \\ 0.2807 \end{bmatrix} = \begin{bmatrix} 0.3683 \\ 0.3509 \\ 0.2807 \end{bmatrix}$$

$$\mathbf{x_{11}} = \mathrm{M}\mathbf{x_{10}} \Rightarrow \mathbf{x_{11}} = \begin{bmatrix} 0.6 & 0.1 & 0.4 \\ 0.4 & 0.5 & 0.1 \\ 0 & 0.4 & 0.5 \end{bmatrix} \begin{bmatrix} 0.3683 \\ 0.3509 \\ 0.2807 \end{bmatrix} = \begin{bmatrix} 0.3684 \\ 0.3509 \\ 0.2807 \end{bmatrix}$$

$$\mathbf{x_{12}} = \mathrm{M}\mathbf{x_{11}} \Rightarrow \mathbf{x_{12}} = \begin{bmatrix} 0.6 & 0.1 & 0.4 \\ 0.4 & 0.5 & 0.1 \\ 0 & 0.4 & 0.5 \end{bmatrix} \begin{bmatrix} 0.3684 \\ 0.3509 \\ 0.2807 \end{bmatrix} = \begin{bmatrix} 0.3684 \\ 0.3509 \\ 0.2807 \end{bmatrix}$$

$$\mathbf{x_{13}} = \mathrm{M}\mathbf{x_{12}} \Rightarrow \mathbf{x_{13}} = \begin{bmatrix} 0.6 & 0.1 & 0.4 \\ 0.4 & 0.5 & 0.1 \\ 0 & 0.4 & 0.5 \end{bmatrix} \begin{bmatrix} 0.3684 \\ 0.3509 \\ 0.2807 \end{bmatrix} = \begin{bmatrix} 0.3684 \\ 0.3509 \\ 0.2807 \end{bmatrix}$$

Note that at some point the probabilities i.e. $\mathbf{x} = \begin{bmatrix} 0.3684 \\ 0.3509 \\ 0.2807 \end{bmatrix}$ are hardly changing such that our system is $\mathrm{M}\mathbf{x} = \mathbf{x}$

This state is referred to as steady state, long run or market equilibrium and vector $\mathbf{x}$ is called a steady-state vector or equilibrium vector for the stochastic matrix M.

Example 3: Determine the steady-state vector given the matrix $\begin{bmatrix} 0.1 & 0.3 \\ 0.9 & 0.7 \end{bmatrix}$

Solution: Let the steady-state vector be $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then we have $\mathrm{M}\mathbf{x} = \mathbf{x}$ then;

$$\begin{bmatrix} 0.1 & 0.3 \\ 0.9 & 0.7 \end{bmatrix} \mathbf{x} = \mathbf{x}$$

$$\begin{bmatrix} 0.1 & 0.3 \\ 0.9 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{aligned} 0.1x_1 + 0.3x_2 &= x_1 \\ 0.9x_1 + 0.7x_2 &= x_2 \end{aligned}$$

From the above equations we have;

$$\begin{aligned} -0.9x_1 + 0.3x_2 &= 0 \\ 0.9x_1 - 0.3x_2 &= 0 \end{aligned}$$

Reducing the augmented matrix for the system into echelon form to get;

$$\begin{bmatrix} -0.9 & 0.3 & 0 \\ 0.9 & -0.3 & 0 \end{bmatrix} r_2 + r_1 \sim \begin{bmatrix} -0.9 & 0.3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From the row echelon form $x_2$is a free variable and hence we let it be α

From row 1 we have $-0.9x_1 = -0.3x_2 \therefore x_1 = \frac{1}{3}x_2 = \frac{1}{3}\alpha$

$$\Rightarrow \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

We can say that the basis for the solution set is vector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

From this we can get the probability vector for the steady state i.e. vector $\mathbf{x} = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix}$.

You can easily check that;

$$\begin{bmatrix} 0.1 & 0.3 \\ 0.9 & 0.7 \end{bmatrix} \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix}$$

Decision Analysis

There exist different approaches in decision making. In this lecture we shall look at some non-probability-based techniques such as the maximin minimax approach, and minimax regret approach.

Non-probability-based techniques

- Are easier to maintain
- Require less resources
- Are usually simple

Steps in decision analysis in non-probability techniques

- Determining all the alternatives.
- Determining the possible scenarios for each alternative.
- Organize both the alternatives and scenarios in a meaningful way e.g. using payoff tables

Payoff tables help organize alternatives and scenario in a better way. One is best place to decide among alternatives, options, and actions. It easier to visualize the different possible scenarios. Scenarios can be categorized into three categories; Best possible, worst possible or most likely scenario.

In a payoff table, alternatives take the columns while the scenarios take the rows. The payoffs or outcomes or expected values or estimates are input into the table. The payoffs can be profits, returns, costs, revenues, or other values necessary for the decision-making process.

		Alternatives		
		A	B	C
Scenarios	1			
	2			
	3			

Note that scenarios and alternatives can swap places.

		Scenario		
		Best	Most likely	Worst
Alternatives	1			
	2			
	3			

Analysing different alternatives depends on the individual's attitude to risk or risk preferences. For example;

- A risk taker is assumed to be an optimistic person and hence he will also think of the best can happen. A risk taker always considers the best outcomes. He will maximize the possible benefit by taking more risk. A risk taker has a decision rule of Maximax.

- A risk avoider is assumed to be pessimistic. He will think the worst can happen and hence is always considering the worst outcomes. He will aim to minimize the possible loss by taking less risk. A risk avoider has decision rule- Maximin.

- A risk neutral person has a well-balanced approach and tends to remain neutral. This person will consider the best outcomes, the worst outcome, and the most likely outcomes. He will therefore tend to balance between benefit and loss with risk. The choice of a risk neutral person is minimax regret rule.

Example 1: (Maximax – Maximising the maximum possible outcome)

Consider the following payoff table.

		Alternatives		
		A	B	C
Scenarios	Best possible	120	130	98
	Most Possible	96	87	75
	Worst possible	20	74	53

Best	120	130	98

130 is the best among the maximum

Decision: Take up Alternative B

Maximin – Maximizing the minimum possible outcome

		Alternatives		
		A	B	C
Scenarios	Best possible	120	130	98
	Most Possible	96	87	75
	Worst possible	20	74	53

Worst	20	74	53

Maximum worst payoff is 74

Decision: Take up Alternative B

Minimax Regret – Minimizing the maximum possible regret

Regret is the opportunity of maximizing return lost. It is calculated for every option along the scenario. Regret for an option is the same as the difference between return from option opted and return from the best option i.e. Regret = Best payoff – payoff received.

Note then that the Regret for best option is zero.

		Alternatives		
		A	B	C
Scenarios	Best possible	120	130	98
	Most Possible	96	87	75
	Worst possible	20	74	53

We create a Regret table

		Regret Table		
		A	B	C
Scenarios	Best possible	120-130=-10 **-10**	130-130=0	98-130=-32
	Most Possible	96 -96=0	87-96=-9	75-96=-21
	Worst possible	20-74=-54	74-74=0	53-74=-21

	A	B	C
Maximum Regret	-54	-9	-21

Then the minimum maximum regret is -9

Decision: Take up Alternative B

Example 2: Consider the following Payoff table where one wants to make a decision on which company to invest on. The payoffs are dividends.

		Economy Scenarios		
		Growing	Stable	Declining
Alternatives	SAFCOM	400	470	50
	KENPAN	700	350	-10
	KAS	530	470	-50

i) Consider the Maximax approach (Optimistic approach) – it maximizes the maximum payoff i.e. best of best

		Economy Scenarios			
		Growing	Stable	Declining	BEST
Alternatives	SAFCOM	400	470	50	470
	KENPAN	700	350	-10	**700**
	KAS	530	470	-50	530

Decision: Invest in KENPAN

ii) Maximin Approach (Pessimistic approach) – maximizes the minimum payoff i.e. best of the worst payoff

		Economy Scenarios			
		Growing	Stable	Declining	Worst
Alternatives	SAFCOM	400	470	50	**50**
	KENPAN	700	350	-10	-10
	KAS	530	470	-50	-50

Decision: Invest in SAFCOM

iii) Minimax regret approach – Minimizes the maximum regret

		Economy Scenarios		
		Growing	Stable	Declining
Alternatives	SAFCOM	700-400=300	470 - 470=0	50-50=0
	KENPAN	700-700=0	470-350=120	50-(-10)=60
	KAS	700-530=170	470-470=0	50-(-50)=100

Regret Table

		Economy Scenarios			
		Growing	Stable	Declining	Maximum Regret
Alternatives	SAFCOM	300	0	0	300
	KENPAN	0	120	60	**120 (Minimum)**
	KAS	170	0	100	170

Decision: Invest in KENPAN

Exercise

1) Research on decision trees and create a decision tree for example 1 for the alternatives SAFCOM, KENPAN, and KAS.
2) Research on decision making with probabilities and then recommend the best decision alternative using the optimistic, pessimistic, bayes, and minimax regret approach for the following case given the payoff table below.

Action	Scenario 1	Scenario 2	Scenario 3
Alternative 1	10	15	−8
Alternative 2	20	5	18
Alternative 3	60	0	52
Probability	0.2	0.5	0.3

3) Draw a decision tree for the case where one is faced with three decision alternatives and two scenarios or states of nature as shown in the profit payoff table below.

■	Scenario 1	Scenario 2
Alternative 1	260	300
Alternative 2	450	125
Alternative 3	280	340

4) Suppose one is faced with 4 decision alternatives A_i and 4 states of nature S_i as shown in the table below. Suppose one is not aware of the probabilities of the 4 states, what is the recommended decision using the approaches; Laplace, optimistic, conservative, and minimax regret.

■	S_1	S_2	S_3	S_4
A_1	18	7	12	9
A_2	13	10	6	20
A_3	8	14	14	6
A_4	6	12	19	17

References

Lay, D. C., Lay, S. R., & McDonald, J. J. (2016). *Linear Algebra and its Application* (5th ed.). Pearson.

Decision Analysis 1.1 (Costs) - Optimistic, Conservative, Minimax Regret (youtube.com)