Business Mathematics

Lecture 8

Mathematics of Finance

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Introduction to Lecture 8

This lecture 8 introduces you to mathematics of finance. The chapter will discuss compound and simple interest, annuities, and amortization.

Further Readings

The resources below are recommended for further reading to gain more insights on mathematics of finance (Jacques, 2006; Murray & Robert, 2009; Werner & Sotskov, 2006).

Intended Learning Outcomes

At the end of this lecture, you will be able to;

(i) Calculate simple and compound interest.
(ii) Define annuities and amortization.
(iii) Calculate annuities and amortization.

Compound and Simple Interest

Money deposited in a bank earns interest that may be simple or compounded. Interest depends on the principal amount deposited, the duration of the deposit, and the interest rate. The interest may be compounded annually, quarterly, monthly, daily etc.

Simple interest: This is interest that is calculated only on the amount borrowed or the principal given as

$$I = PRT$$

Where I is the simple interest, P is the principal, R is the annual interest rate mad T is the time in years.

Example 1: Onyango borrowed 3500 at 7% simple interest per annum to be paid in 10 months. How much interest did he pay? Determine the future value of the loan or the maturity value of the loan or total amount that he finally paid.

Solution: We know that $\mathrm{I = PRT} = 3500 \times 0.07 \times \frac{10}{12} \approx 204.17$ KES

The total amount of money he paid is given by $\mathrm{A = P + I = P + PRT = P(1 + RT)}$

$$\Rightarrow \mathrm{A} = 3500 + 204.17 = 3704.17 \text{ KES}$$

Example 2: An individual borrows 8800 KES at 7% per annum for 3 years. How much simple interest does he pay on the loan? Calculate the total amount paid.

Solution: Simple interest $I = PRT = 8800 \times 0.07 \times 3 = 1848$ KES

Total amount $\mathrm{A} = \text{Interest} + \text{Principal} = 1848 + 8800 = 10648$ KES

Example 3: Juma wants to borrow from Kama 9000 KES and he is ready to pay back 9390 KES in 3 months. Determine the simple interest rate that he will pay.

Solution: We apply the formula for calculating the maturity value A i.e. $\mathrm{A = P(1 + RT)}$

$\Rightarrow 9390 = 9000(1 + \frac{3}{12}\mathrm{r})$

$$9390 = 9000(1 + 0.25r) = 9000 + 2250r$$

$$9390 - 9000 = 2250r$$

$$\therefore 390 = 2250r \Rightarrow r = 0.1733$$

The simple interest rate is 17.33%

Compound Interest: Compound interest is charged on interest as well as the principal.

The future value A for compound interest after n interest period is given by

$$\mathrm{A = p(1 + r)^n}$$

Where p is the principal or present value and r is the interest rate per period in decimal point.

Example 1: Determine the future value A of principal value 10000 KES deposited for a period 2 years at 2% interest compounded (i) annually (ii) semiannually (iii) simple interest for the 2 at 2% p.a..

Solution

(i) Using the formula $\mathrm{A = p(1 + r)^n}$ we have;

$$\mathrm{A} = 10000(1 + 0.02)^2 = 10404 \text{ KES}$$

(ii) Using the formula $\mathrm{A = p(1 + r)^n}$ we have;

$$\mathrm{A} = 10000(1 + 0.02)^4 \approx 10824{,}32 \text{ KES}$$

(iii) We use the formula $\mathrm{I = PRT}$ where I is interest, p is the principal, R is the rate and T is the period. Hence we have,

$$I = 10000 \times 0.02 \times 2 = 400 \text{ KES}$$

Example 2: Find the amount of investment if 21000 KES is invested at rate of 3% per annum, compounded monthly for 4 years.

Solution: We use the formula $\mathrm{A = p(1 + r)^n} \Rightarrow A = 21000\left(1 + \frac{0.03}{12}\right)^{12\times4}$

$$= 21000(1.0025)^{48} \approx 23673.89 \text{ KES}$$

Effective Rate: Consider a 100 KES deposit at 5% compounded semiannually. Then the maturity value A is;

$$\mathrm{A} = 100\left(1 + \frac{0.05}{2}\right)^2 = 105.0625 \text{ KES}$$

Note that the actual increase is 5.0625% which is way above the 5%. We call the 5% the nominal or stated rate of interest while the 5.0625% is called the effective rate.

In general, the effective rate $\mathrm{r_E}$ corresponding to a stated rated of interest r compounded n times per year is given as;

$$\mathrm{r_E} = \left(1 + \frac{\mathrm{r}}{\mathrm{n}}\right)^{\mathrm{n}} - 1$$

Example 1: A woman intends to borrow money. Bank A charges 9% interest compounded semiannually. Bank B charges 8.9% interest compounded monthly. Which of the two banks charges less interest.

Solution: We can compare the effective rates

Bank A; $r_E = \left(1 + \frac{r}{n}\right)^n - 1 = \left(1 + \frac{0.09}{2}\right)^2 - 1 = 0.092025$

Bank B; $r_E = \left(1 + \frac{r}{n}\right)^n - 1 = \left(1 + \frac{0.089}{12}\right)^{12} - 1 \approx 0.092722$

Bank A has a lower effective rate even though it has a higher stated rate.

Example 2: Hanna must pay a lump sum of 10000 KES in 4 years. Determine the present amount or principal that she needs to deposit at 5.2 % compounded annually to achieve this lump sum.

Solution: We use the formula $\mathrm{A = P(1 + r)^n \Rightarrow P = \frac{A}{(1+r)^n}}$

$$P = \frac{10000}{(1 + 0.052)^4} \approx 8164.64 \text{ KES}$$

Example 3: Suppose the principal amount of 6540 KES is deposited at 4.25% compound quarterly until it reaches at least 15000 KES. How long will this take?

Solution: We use the formula $A = P(1 + r)^n$ our interest is to find n.

$$\Rightarrow 15000 = 6540\left(1 + \frac{0.0425}{4}\right)^n$$

$$\frac{15000}{6540} = (1 + 0.010625)^n$$

$$2.29358 = 1.010625^n$$

We introduce logarithms to get

$$\ln 2.29358 = n \ln 1.010625$$

$$\therefore n = \frac{\ln 2.29358}{\ln 1.010625} \approx 78.543 \text{ quarters}$$

This is approximately $\frac{78.543}{4} = 19.64$ years

Continuous Compounding

Consider the future value A of principal value 100 KES deposited for a period 2 years at 2% interest compounded (i) annually (ii) semiannually (iii) Quarterly (iv) monthly (v) daily (vi) hourly. The formula to use is $A = P(1 + r)^n$. We can display the above in a table as shown below;

n	Type of compounding	Calculation $A = P(1 + r)^{nt}$	Maturity value
1	Annually	$A = 100(1 + 0.02)^2$	104.04
2	Semiannually	$A = 100\left(1 + \frac{0.02}{2}\right)^4$	104.0604
4	Quarterly	$A = 100\left(1 + \frac{0.02}{4}\right)^8$	104.0707
12	Monthly	$A = 100\left(1 + \frac{0.02}{12}\right)^{24}$	104.0776
360	Daily	$A = 100\left(1 + \frac{0.02}{360}\right)^{720}$	104.08096
8640	Hourly	$A = 100\left(1 + \frac{0.02}{8640}\right)^{17280}$	104.08107

Note that as n increases the maturity value increases but by smaller margins. It can be shown that as n becomes infinitely large, $P\left(1+\frac{r}{n}\right)^{nt}$ approaches Pe^{rt} where constant $e \approx 2.718281828$

This type of compound interest where the number of times a year that the interest is compounded becomes infinite is known as continuous compounding.

In general, if a deposit of principal P is invested at a rate of interest r compounded continuously for t years, then the maturity value A is given by;

$$A = Pe^{rt}$$

Example 1: A principal of 3540 KES is deposited at the end of each year, at 2.25 % compounded continuously. Determine the compound amount and the interest earned after 7.5 years and the effective rate. How long will it take to mature to 10000 KES?

Solution: We know that $A = Pe^{rt}$ hence we have,

$$A = 3540 \times e^{0.0225\times 7.5} \approx 4192.31 \text{ KES}$$

Interest is $4192.31 - 3540 = 652.31$ KES

Note that the effective rate $r_E = e^r - 1 = e^{0.0225} - 1 \approx 0.02276\ (2.276\%)$

This is an increase 2.276 %.

Next we determine how long it will mature to 10000 KES. That is;

$$A = Pe^{rt}$$

$$10000 = 3540e^{0.0225t}$$

$$\frac{10000}{3540} = e^{0.0225t}$$

$$2.824859 = e^{0.0225t}$$

Introducing natural logarithms to get;

$$\ln 2.824859 = \ln e^{0.0225t}$$

$$\ln 2.824859 = 0.0225t \ln e$$

$$\frac{\ln 2.824859}{0.0225} = t \therefore t \approx 46.1537 \text{ years}$$

Annuities

Annuity is a sequence of equal payments made over a period. An ordinary annuity is one that is made such that the frequency of payments is like the frequency of compounding.

Example 1: Determine the maturity value of 12500 KES deposited at the end of each year for the next 10 years where it attracts 7% per annum compounded annually.

Solution: The payments and their interest can be viewed as geometric series

The first deposit of 12500 will earn interest for 6 years and will amount to $A = 12500(1.07)^6$

The second deposit of 12500 will earn interest for 5 years and will amount to $A = 12500(1.07)^5$

We can proceed like this until we get the last deposit of 12500 KES that will earn no interest since it will be deposited at the end of year 10.

Hence we have total among as a geometric series;

$$12500(1.07)^9 + 12500(1.07)^8 + \cdots + 12500$$

Using the formula

$$\mathrm{S_n} = \frac{\mathrm{a(r^n - 1)}}{\mathrm{r - 1}}$$

We have;

$$S_{10} = \frac{12500(1.07^{10} - 1)}{1.07 - 1} \approx 172705.599$$

In general, the maturity value of an annuity is given as;

$$\mathrm{S = t\left[\frac{(1 + r)^n - 1}{r}\right]}$$

Where S is the maturity value, R is the periodic payment, r is the interest rate per period, and n is the number of periods..

For instance, in the example above we have,

$$\mathrm{S = R\left[\frac{(1 + r)^n - 1}{r}\right]} = 12500\left[\frac{(1 + 0.07)^{10} - 1}{0.07}\right] \approx 172705.599 \text{ KES}$$

Example 2: Abdala deposits 36 000 KES at the end of each year for 9 years as a retirement saving scheme. The scheme is paying 5% compounded annually.

(i) How much will he have at the end of 9 years.

(ii) Suppose he deposits 3000 KES at the end of each month for the same period compounded annually. How much will he have at the end of the period.

Solution:

(i) Applying the formula

$$S = R\left[\frac{(1+r)^n - 1}{r}\right] = 36000\left[\frac{(1+0.05)^9 - 1}{0.05}\right] \approx 396{,}956.32 \text{ KES}$$

(ii) Applying the formula we have

$$S = R\left[\frac{(1+r)^n - 1}{r}\right] = 3000\left[\frac{\left(1+\frac{0.05}{12}\right)^{108} - 1}{\frac{0.05}{12}}\right] \approx 408{,}129.59 \text{ KES}$$

Note that in both cases Abdala invested $36000 \times 9 = 324000$ KES. However, the second option gives a better return.

Present value of an ordinary annuity

This is the amount that would have to be deposited in one lump sum today in order to produce exactly the same balance at the given period.

Note that if P is the principal or present value deposited today at compound interest r, then at the end of n periods, the future value will be

$$P(1+r)^n$$

This future value should be equal to the sum S given by;

$$S = R\left[\frac{(1+r)^n - 1}{r}\right]$$

$$\Rightarrow P(1+r)^n = R\left[\frac{(1+r)^n - 1}{r}\right]$$

Therefore

$$P = (1+r)^{-n}R\left[\frac{(1+r)^n - 1}{r}\right] = R\left[\frac{(1+r)^{-n}(1+r)^n - (1+r)^{-n}}{r}\right]$$

$$P = R\left[\frac{1-(1+r)^{-n}}{r}\right]$$

P is the present value of the annuity

Example 1: Kumar intends to contribute to an endowment fund. He prefers giving a lump sum today instead of giving 5000 KES at the end of each year for 10 years. If the endowment fund earns 6.5% compounded annually, determine the lump sum amount.

Solution: We need to determine the present value of the annuity i.e.

$$P = R\left[\frac{1-(1+r)^{-n}}{r}\right]$$

Where our R = 5000 KES, n = 10 years and r = 0.065, hence we have;

$$P = R\left[\frac{1-(1+r)^{-n}}{r}\right] = 5000\left[\frac{1-1.065^{-9}}{0.065}\right] \approx 33{,}280.52 \text{ KES}$$

Example 2: Anyango want to buy a car. The car costs 1,800,000 KES. She is expected to make a downpayment of 200,000 KES and the balance to paid off in 36 equal monthly payments with interest of 3% per annum on the unpaid balance. Determine the amount of each payment.

Solution: A single lump sum of 1,600,000 KES will pay off the car loan. Hence the present value of annuity of 36 monthly payment with interest of 3% or $\frac{0.03}{12} = 0.0025\ or\ 0.25\%$ per month.

$$\Rightarrow 1{,}600{,}000 = R\left[\frac{1-(1+r)^{-n}}{r}\right] = R\left[\frac{1-1.0025^{-36}}{0.0025}\right]$$

$$1{,}600{,}00 = 34.3865R \therefore R \approx 46{,}529.99 \text{ KES Monthly payment}$$

Amortization

Amortization refers to the process of spreading out the cost of a loan, mortgage, or otherwise over a specific period of time. It involves paying off debt through regular instalment payments, which typically include both principal and interest.

In the context of loans, such as mortgages or car loans, each instalment payment covers a portion of the principal amount borrowed as well as the interest accrued on the outstanding balance. Over time, as payments are made, the balance of the loan decreases until it is eventually paid off in full.

A loan of present value P at interest r per period can be amortized in n equal periodic payments of R amount made at the end of each period (say a month). Then

$$R = \frac{P}{\left[\frac{1-(1+r)^{-n}}{r}\right]} = \frac{Pr}{1-(1+r)^{-n}}$$

Example 1: Oti plans to buy a house for 13,750,000 KES that requires a down payment of 1,650,000 KES. They take a 15-year mortgage for 12,100,000 at annual interest rate of 3%. Find

(i) Determine the amount of the monthly payment needed to amortize this loan

(ii) Determine the total interest paid when the loan is amortized over 15 years.

(iii) Determine the part of the first payment that is interest and the part that is applied to reducing the debt (Exercise).

Solution:

(i) Determine the amount of the monthly payment needed to amortize this loan

$$R = \frac{Pr}{1-(1+r)^{-n}} = \frac{12{,}100{,}000 \times 0.0025}{1-(1+0.0025)^{-180}} \approx 83{,}560.38 \text{ KES}$$

(ii) Determine the total interest paid when the loan is amortized over 15 years.

Oti makes a monthly payment off 83,560.38 KES for 15 years i.e. a total of 15,040,868.4 KES. Thus the total interest is; $15{,}040{,}868.4 - 12{,}100{,}000 = 2{,}940{,}868.4 \text{ KES}$

Exercise

1) Atieno invest 112,300 KES in a bank, for 4 years. The bank offers a simple interest at rate of 7.6 %. Calculate the interest that the money earns after the end of the period.
2) Amina invests 1,712,360 KES in a bank, for 6 years. The bank offers a compound interest at rate of 5.3 %. She withdraws all her money at the end of the period. Find the amount she withdrew.
3) A young man decides to save 9,375 KES per month and deposits monthly into an account that pays interest at a rate of 3.5% p.a. Calculate the accumulated value after 7 years.
4) Determine the compound among and the interest earned after 7 years when 24500 KES is deposited at 3.25% compounded continuously. What is the effective rate?
5) Find the present value of 12 740 KES at 4.3% compounded annually for 5 years.
6) Describe eight areas where mathematics of finance discussed in this lecture is applied in the real world., besides loan repayments and savings.

References

Jacques, I. (2006). *Mathematics for economics and business* (5th ed.). Prentice Hall.

Murray, S., & Robert, M. (2009). *College Algebra*. McGraw-Hill.

Werner, F., & Sotskov, Y. N. (2006). *Mathematics of Economics and Business*. Routledge: Taylor & Francis Group.