Business Mathematics

Lecture 9

Differentiation

Lecturer: Kahenya, N.P

Introduction to Lecture 9

This lecture introduces you to differentiation and to its application in business mathematics.

Differentiation is a fundamental concept in calculus that deals with the study of how functions change in response to changes in their input variables. Differentiation involves finding the rate at which a quantity changes with respect to another quantity. It helps us understand how the output of a function changes as its input changes.

Differentiation is used in various fields, including physics, engineering, economics, and many others, to analyze and solve problems involving rates of change, optimization, and modeling of dynamic systems.

Differentiation is an important tool for understanding the behavior of functions and their relationships.

Further Readings

The resources below are recommended for further reading to gain more insights on differentiation (Jacques, 2006; Kahenya, 2022; Murray & Robert, 2009; Sullivan & Miranda, 2019).

Intended Learning Outcomes

At the end of this lecture, you will be able to;

(i) Describe the basic rules of differentiation.

(ii) Solve problems involving the basic rules of differentiation.

(iii) Apply differentiation to solve business related problems.

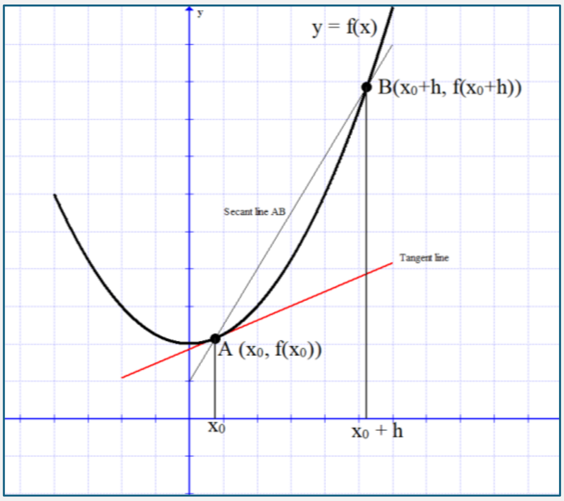
Basic rules of differentiation

Differentiation is the operation of finding the derivative function. The derivative function of $y = f(x)$ may be denoted as; $f^{/}(x)$ or $\frac{dy}{dx}$. The notation $\frac{dy}{dx}$ is called the Leibniz notation. This lecture will discuss the basic rules of differentiation and illustrate a few examples.

a) Differentiation by first principles

Consider the curve of the graph $y = f(x)$ below. The slope or gradient of the secant line AB is given as;

$$\frac{\text{change in y} - \text{coordinate}}{\text{change } in \text{ x} - \text{coordinate}} = \frac{\Delta y}{\Delta x} = \frac{f(x_0 + h) - f(x_0)}{(x_0 + h) - x_0} = \frac{f(x_0 + h) - f(x_0)}{h}$$



Note that as one reduces the value of h i.e. h approaches zero the secant line AB will tend to be the tangent line at point A. That is, the slope or gradient of the tangent line is the same as the gradient of the curve at that point A i.e.

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

If the limit exists we say that $f(x)$ is differentiable at point x_0. Using this notation to find the derivative of a function is called the method of first principles.

Example 1: Use the definition $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$ to find the derivative of $f(x) = x^2$ at point $x = 5$

Solution: Note out $f(x_0) = x^2$, $x_0 = 5$, then;

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} \frac{f(5 + h) - f(5)}{h} = \lim_{h \to 0} \frac{(5 + h)^2 - 25}{h}$$

$$= \lim_{h \to 0} \frac{25 + 10h + h^2 - 25}{h} = \lim_{h \to 0} \frac{10h + h^2}{h} = \lim_{h \to 0} h\left(\frac{10 + h}{h}\right)$$

$$= \lim_{h \to 0} (10 + h) = 10$$

Example 2: Use the definition $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$ to find the derivative of $f(x) = 3x^2 + 5$

Solution: Our $f(x_0) = 3x_0^2 + 5; f(x_0 + h) = 3(x_0 + h)^2 + 5$. Hence

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} \frac{(3(x_0 + h)^2 + 5) - (3x_0^2 + 5)}{h}$$

$$= \lim_{h \to 0} \frac{3x_0^2 + 6hx_0 + 3h^2 + 5 - 3x_0^2 - 5}{h}$$

$$= \lim_{h \to 0} \frac{6x_0h + 3h^2}{h} = \lim_{h \to 0} h\left(\frac{6x_0 + 3h}{h}\right)$$

$$= \lim_{h \to 0} (6x_0 + 3h) = 6x_0$$

$$\therefore \frac{dy}{dx} = 6x$$

b) Power Rule

Given the function $y = f(x) = x^n$ then $\frac{dy}{dx} = nx^{n-1}$ i.e. the derivative of y with respect to x.

For example, differentiate with respect to x the function $y = 7x^5$

$$\Rightarrow \frac{dy}{dx} = 7(5)x^{5-1} = 35x^4$$

c) Chain rule

Suppose y is a function of u i.e. $y = f(u)$ and that u is a function of x i.e. $u = f(x)$ then the derivative of y with respect to x is given by

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

For example, given the function $y = (3x^2 + 7x)^7$

Solution: One can expand the function and use the power rule. However, this is tedious and time-wasting. However, we can use the chain rule

Let $u = 3x^2 + 7x \Rightarrow \frac{du}{dx} = 6x + 7$

Again $y = u^7 \Rightarrow \frac{dy}{du} = 7u^6$

Therefore $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 7u^6 \times (6x + 7)$

$$\Rightarrow \frac{dy}{dx} = 7(6x + 7)(3x^2 + 7x)^6$$

d) Product rule

Suppose y is a product of u and v i.e. $y = f(uv)$ and u and v are functions of x i.e. $u = f(x)$and $v = f(x)$ then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

For example, find the derivative of $y = (5x^3 + 7x)(3x^{-5} - 9x^3 + 4)$

Solution: Let $u = 5x^3 + 7x \Rightarrow \frac{du}{dx} = 15x^2 + 7$.

Let $v = 3x^{-5} - 9x^3 + 4 \Rightarrow \frac{dv}{dx} = -15x^{-6} - 27x^2$

Therefore;

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = (5x^3 + 7x)(-15x^{-6} - 27x^2) + (3x^{-5} - 9x^3 + 4)(15x^2 + 7)$$

or

$$\frac{dy}{dx} = (3x^{-5} - 9x^3 + 4)(15x^2 + 7) - (5x^3 + 7x)(15x^{-6} + 27x^2)$$

e) Quotient Rule

Give the function $y = \frac{u}{v}$ where u and v are functions of x then

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

For example, differentiate $y = \frac{5x^2+2x}{x^3-8}$ with respect to x.

Solution: Let $u = 5x^2 + 2x \Rightarrow \frac{du}{dx} = 10x + 2$

Let $v = x^3 - 8 \Rightarrow \frac{dv}{dx} = 3x^2$. Therefore;

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(x^3 - 8)(10x + 2) - 3x^2(5x^2 + 2x)}{(x^3 - 8)^2}$$

$$\frac{dy}{dx} = \frac{-(5x^4 + 4x^3 + 80x + 16)}{(x^3 - 8)^2}$$

Optimization

Optimization is the process of determining the best solution to a given problem. This is either done by maximizing or minimizing some quantity. For example, a businessperson aims at minimizing the cost while maximizing profit.

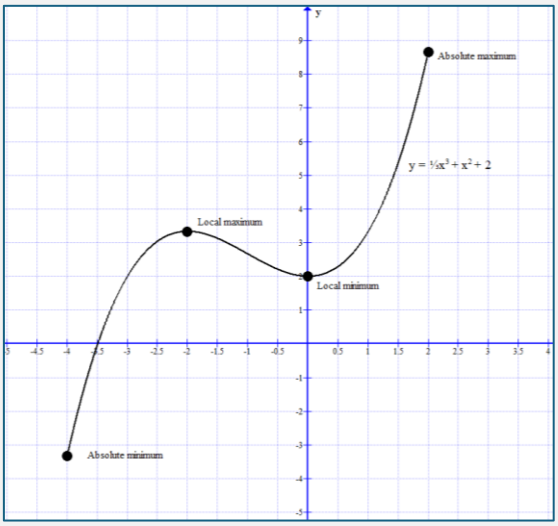
A function $f(x)$ may have a point where $\frac{d}{dx} f(x) = 0$, is such a point exists then the function is said to have a local maximum or a local minimum.

The local maximum (and minimum) are also referred to as critical points or stationary points or turning points.

The turning point may be a point of local maximum, local minimum, or inflexion/inflection.

Example 1: Given the function $y = \frac{1}{3}x^3 + x^2 + 2$ defined over the interval $-4 \leq x \leq 2$, determine the absolute maximum and minimum, local maximum and minimum.

Solution: We can plot the curve of the graph as shown below



The critical values occur when $\frac{dy}{dx} = 0 = x^2 + 2x \Rightarrow x(x + 2) = 0 \therefore x = 0 \text{ or } -2$

Hence $f(0) = 2$ and $f(-2) = \frac{10}{3}$

From the graph above, the local minimum is at point $(0,2)$while the local maximum is at point $\left(-2, \frac{10}{3}\right)$.

The values of the function at the endpoints are;

$$f(-4) = -\frac{10}{3}, f(2) = \frac{26}{3}$$

Hence the points $\left(-4, -\frac{10}{3}\right)$ and $\left(2, \frac{26}{3}\right)$ are the points of absolute minimum and absolute maximum respectively.

Definition: Marginal Revenue MR is the additional revenue generated from selling one more unit of a product or service. It is calculated as the change in Total Revenue TR resulting from selling one additional unit i.e. Change in TR with respect to Q

$$MR = \frac{d(TR)}{dQ}$$

Understanding the marginal revenue is a fundamental aspect for companies in making production and pricing decisions. It enables them to identify the optimal output level that maximizes their profits.

When the marginal revenue surpasses the marginal cost, it is usually advantageous for the company to increase its production. Conversely, when the marginal revenue is less than the marginal cost, the company should reduce its production to optimize profits.

Example 1: Suppose the demand function is given by $P = 140 - 5Q.$ Then find an expression for TR in terms of Q and MR when Q is 5. Determine the 1 unit increase approach.

Solution: From a previous lecture we saw that $TR = PQ$

$$\Rightarrow TR = (140 - 5Q)Q = 140Q - 5Q^2$$

However;

$$MR = \frac{d(TR)}{dQ} = 140 - 10Q \Rightarrow \text{at } Q = 5 \text{ then } MR = 140 - 50 = 90$$

Now when Q = 5 then $TR = 140Q - 5Q^2 = 140(5) - 5(5^2) = 575$

A change by 1 unit will give us Q = 6 then $TR = 140(6) - 5(36) = 660$

Hence the marginal revenue is $660 - 575 = 85$

Example 2: Consider the price of a good that increases from 20 KES to 24 KES

Solution: It is important to determine the effect on revenue with respect to the change in price. How demand of a good respond to price change is defined as price elasticity of demand E. It is given as;

$$E = -\frac{\text{Percentage change in demand}}{\text{Percentage change in price}}$$

When E is less than 1 we say that the demand is inelastic, when E is greater than 1 we say the demand is elastic, and it is unit elastic if E is equals to 1.

In our case the price change is $24 - 20 = 4\text{ KES} \Rightarrow \frac{4}{20} = \frac{1}{5} = 0.2$

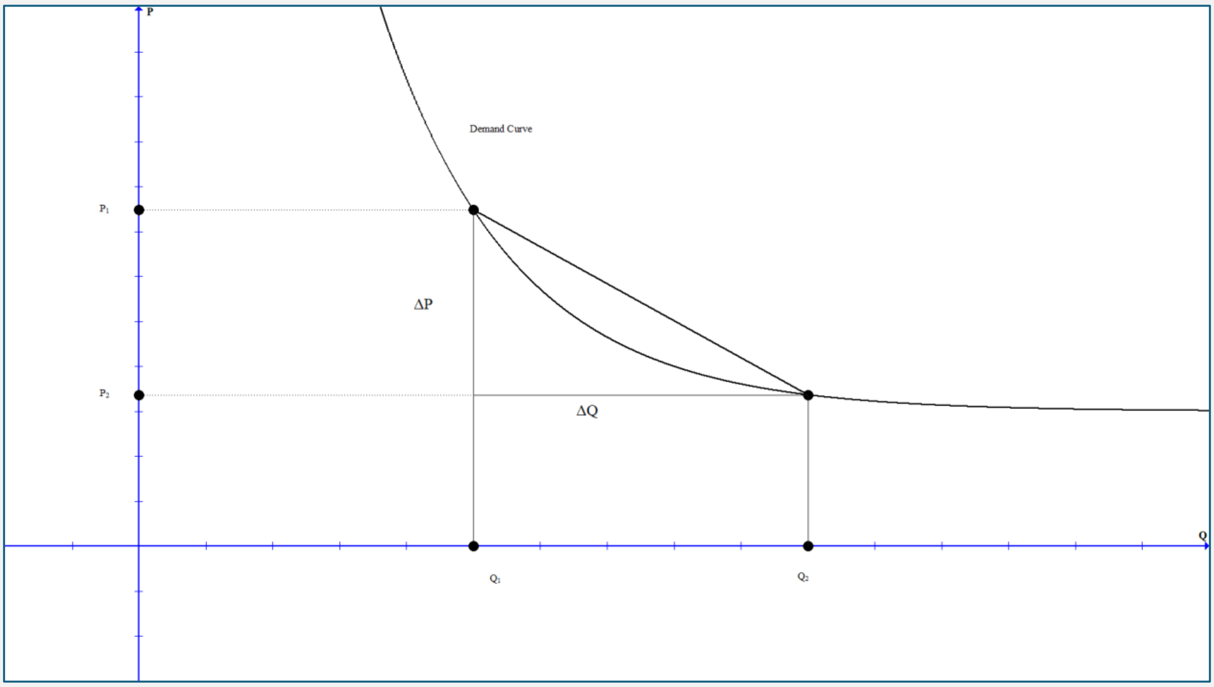
This 20% increase in price.

We can get this using the formula; $\frac{\Delta P}{p} \times 100$

We can equally get the percentage change in demand as $\frac{\Delta Q}{Q} \times 100$

Hence;

$$E = -\frac{\text{Percentage change in demand}}{\text{Percentage change in price}} = \frac{\frac{\Delta Q}{Q} \times 100}{\frac{\Delta P}{p} \times 100} = -\frac{P}{Q} \times \frac{\Delta Q}{\Delta P}$$



Example 3: Determine the elasticity of demand when the price P decreases from 120 KES to 99 KES, given the demand function $P = 220 - Q^2$

Solution: Using the diagram in the previous example we can assume that our $P_1 = 120$ and $P_2 = 99$. Hence we can find our Q_1 and Q_2. That is,

$$120 = 220 - Q^2 \Rightarrow Q^2 = 100 \therefore Q = \pm 10, \text{Hence } Q_1 = 10$$

Again, $99 = 220 - Q^2 \Rightarrow Q^2 = 121 \therefore Q = \pm 11$, Hence $Q_2 = 11$

We know that elasticity E is given by

$$E = -\frac{P}{Q} \times \frac{\Delta Q}{\Delta P}$$

Where $\Delta P = 99 - 120 = -21; \Delta Q = 11 - 10 = 1$

Now for what value of P and Q to take, it is advisable to use the average of the two, that is,

$$P = \frac{120 + 99}{2} = 109.5; Q = \frac{11 + 10}{2} = 10.5$$

Therefore;

$$E = -\frac{P}{Q} \times \frac{\Delta Q}{\Delta P} = -\frac{109.5}{10.5} \times \frac{1}{-21} = 0.497$$

Example 4: Suppose the supply curve is given by the function $P = 25 + \sqrt{Q}$, determine the price elasticity of supply averaged along the curve between $Q = 144$ and $Q = 152$ and at the point when Q = 144.

Solution: We first determine the corresponding prices P_1 and P_2, that is,

$$P_1 = 25 + \sqrt{144} = 25 + 12 = 37$$

$$P_2 = 25 + \sqrt{152} \approx 37.329$$

clearly $\Delta P = 37.329 - 37 = 0.329; \Delta Q = 152 - 144 = 8$

Our $P = \frac{1}{2}(37 + 37.329) = 37.1645$ and $Q = \frac{1}{2}(152 + 144) = 148$

Therefore, the elasticity is given by;

$$E = -\frac{P}{Q} \times \frac{\Delta Q}{\Delta P} = -\frac{37.1645}{148} \times \frac{8}{0.329} = 6.106$$

Next we find the elasticity when Q = 144. We need to find $\frac{dQ}{dP}$ i.e. first differentiate P with respect to Q.

$$P = 25 + \sqrt{Q} \Rightarrow \frac{dP}{dQ} = \frac{1}{2} Q^{-\frac{1}{2}}$$

Therefore

$$\frac{dQ}{dP} = 2\sqrt{Q}$$

When Q = 144 we get;

$$\frac{dQ}{dP} = 2\sqrt{144} = 24$$

The elasticity at this point is given by

$$E = \frac{P}{Q} \times \frac{dQ}{dP} = \frac{37}{144} \times 24 = 6.167$$

Note that this value is close to 6.106

Example 5: A small business has its production function as $Q = 9L^2 - 0.3L^3$ where w is the number of workers. Find;

(i) The number of workers needed to maximize output.

(ii) The number of workers needed to maximize the average product of labor.

(iii) The average product of labor AP and

(iv) The marginal product of labor MP.

Solution:

(i) To find the number of workers w that maximize output we need to find the stationary point of the function $Q = 9L^2 - 0.3L^3$ that is;

$$\frac{dQ}{dL} = 0$$

$$\Rightarrow \frac{dQ}{dL} = 18L - 0.9L^2 = 0$$

$$\therefore L(18 - 0.9L) = 0 \Rightarrow L = 0 \text{ or } 0.9L = 18 \therefore L = 20 \text{ workers}$$

The business entity must engage 20 workers to maximize productivity.

The corresponding output is;

$$Q = 9(20^2) - 0.3(20^3) = 1200$$

(ii) The *average product of labor* AP or *labor productivity* is the *Total output* Q divided by *labor* L i.e.

$$AP = \frac{Q}{L}$$

In our case

$$AP = \frac{Q}{L} = \frac{9L^2 - 0.3L^3}{L} = 9L - 0.3L^2$$

To maximize the *average product of labor or labor productivity* we differentiate AP with respect to L, that is,

$$\frac{d(AP)}{dL} = 9 - 0.6L = 0$$

$$\Rightarrow 0.6L = 9 \therefore L = 15 \text{ workers needed to maximize labor productivity.}$$

We can see that the corresponding productivity will be;

$$AP = 9L - 0.3L^2 = 9(15) - 0.3(15^2) = 67.5$$

(iii) Next we find the marginal product of labor MP, that is,

$$MP = \frac{dQ}{dL} = 18L - 0.9L^2$$

When L is 15, then $MP = 18(15) - 0.9(15^2) = 67.5$

Marginal product of labor MP is equal to the Average product of labor AP

Example 6: A firm that produces canned juice needs to produce cylindrical cans with a capacity of 300 cm^3. It is observed that the cost for production of top and bottom parts of the can is 0.02 KES per unit square cm, while the curved part will cost 0.012 KES per square cm. Determine the dimensions that will minimize the cost of producing the can.

Solution: The total surface area A of the can is given by;

$A = (\text{the circular top and bottom}) + (\text{the curved side })$

$A = 2\pi r^2 + 2\pi rh.$

Hence the cost C of producing the can in KES, is given by;

$$C = 2\pi r^2 + 2\pi rh = 0.04\pi r^2 + 0.024\pi rh \cdots \text{(i)}$$

Again $v = \pi r^2 h = 300 \Rightarrow h = \frac{300}{\pi r^2} \cdots \text{(ii)}$

Equation (i) becomes; $C = 0.04\pi r^2 + 0.024\pi r \left(\frac{300}{\pi r^2}\right) = 0.04\pi r^2 + \frac{7.2}{r}$

We need to minimize C. The firm must produce a can and therefore $r > 0$.

Hence;

$$\frac{dC}{dr} = 0.08\pi r - \frac{7.2}{r^2} = \frac{0.08\pi r^3 - 7.2}{r^2} = 0$$

$$\Rightarrow r^3 \approx 28.648\text{cm} \therefore r \approx 3.06 \text{ cm}$$

Our $h = \frac{300}{\pi r^2} = \frac{300}{\pi 3.06^2} \approx 10.2 \text{ cm}$

The minimum cost of the container is;

$$C = 2\pi r^2 + 2\pi rh = 0.04\pi r^2 + 0.024\pi rh = 0.04\pi 3.06^2 + 0.024\pi \times 3.06 \times 10.2 \approx 3.53 \text{ KES}$$

Example 7: A business production function is given by; $Q = L^2e^{-0.03L}$. Find the average product of labor AP. Determine the value of labor L that maximizes production.

Solution: In a previous example we had $AP = \frac{Q}{L}$. In this case we have,

$$AP = \frac{L^2e^{-0.03L}}{L} = Le^{-0.03L}$$

Next we determine the turning point of the curve, that is,

$$\frac{d}{dL}(AP) = 0$$

Note that AP is a product of two functions,

Let $u = L \Rightarrow \frac{du}{dL} = 1$ and let $v = e^{-0.03L} \Rightarrow \frac{dv}{dL} = -0.03e^{-0.03L}$

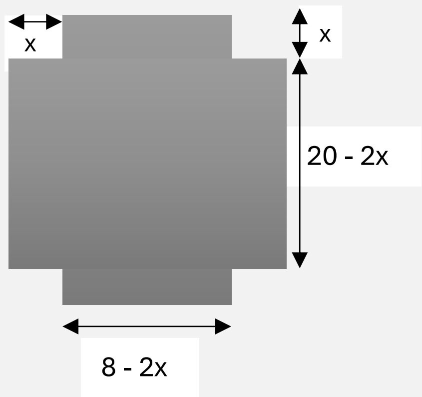
Then applying the product rule, we get;

$$\frac{d}{dL}(AP) = u\frac{dv}{dL} + v\frac{du}{dL} = -0.03Le^{-0.03L} + e^{-0.03} = e^{-0.03}(1 - 0.03L) = 0$$

$$\Rightarrow 1 - 0.03L = 0 \therefore L \approx 33.33$$

Example 8: A packaging company plans to create an open gift box from rectangular sheet of hard papers of sides 20 cm by 8 cm. Four equal square portions are cut off from the corners and the sides are turned up to form the open gift box. Determine the maximum volume of such a box.

Solution: Let assume that each square portion is of side x cm.



The expression for the volume V of the box is given by;

$$V = x(8 - 2x)(20 - 2x) = 4x^3 - 56x^2 + 160x$$

To maximize volume, we have

$$\frac{dV}{dx} = 0$$

$$\Rightarrow \frac{dV}{dx} = 12x^2 - 112x + 160 = 0$$

Using quadratic formula we have;

$$x = \frac{112 \pm \sqrt{(-112)^2 - 4(12)(160)}}{2(12)} = \frac{112 \pm \sqrt{4864}}{24} \approx 1.76 \text{ or } 7.57 \text{ cm}$$

We can use the second derivative to determine the nature of the turning points.

$$\frac{d^2V}{dx^2} = 24x - 112$$

When $x = 1.76, \frac{d^2V}{dx^2} = 24x - 112 = -69.76 < 0 -$ local maxima

When $x = 7.57, \frac{d^2V}{dx^2} = 24x - 112 = 69.68 > 0 -$ local minima

Hence to maximize volume we use $x = 1.76$

The maximum volume will be;

$$V = 4x^3 - 56x^2 + 160x = 4(1.76)^3 - 56(1.76)^2 + 160(1.76) \approx 129.94 \text{ cm}^3$$

Exercise

1. Differentiate the following functions with respect to x.
   a. $y = 3e^{5x}$
   b. $y = \ln x$
   c. $y = 3x^2 + 5x^{-3}$
   d. $y = \frac{5x^2-2}{3x^3+x}$
   e. $y = (5x^4 + 2x^3 + 2)^7$
2. A business entity notes that the total revenue for selling Q goods is given by $TR = \ln(3 + 100Q^2)$. Find marginal revenue when it sells 27 items.
3. The production function is given as Q= $50Le^{-0.1L}$. At what value of L is the production maximized.

References

Jacques, I. (2006). *Mathematics for economics and business* (5th ed.). Prentice Hall.

Kahenya, P. N. (2022). *Calculus 1*. https://www.hufocw.org/Course/971

Murray, S., & Robert, M. (2009). *College Algebra*. McGraw-Hill.

Sullivan, M., & Miranda, K. (2019). *Calculus: Early Transcendentals* (second). W.H. Freeman and Company.