Business Mathematics

Lecture 10

Partial Differentiation

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Introduction to Lecture 10

This lecture introduces you to partial differentiation and its application to business mathematics. Partial differentiation is differentiation that involves more than one variable. It is a continuation of Lecture 9 on differentiation that involved only one variable.

Further Readings

The resources below are recommended for further reading to gain more insights on partial differentiation (Jacques, 2006; Sullivan & Miranda, 2019; Werner & Sotskov, 2006).

Intended Learning Outcomes

At the end of this lecture, you will be able to;

(i) Define partial differentiation.

(ii) Solve problems involving partial differentiation.

(iii) Apply partial differentiation to solve business related problems.

Introduction to Partial Derivatives

Given a function z which is a function of two variables x and y denoted $z = f(x, y)$ then one can find the derivative of f with respect to x as y is held constant referred to as the partial differentiation of f with respect to x denoted;

$$\frac{\partial z}{\partial x} \text{ or } \frac{\partial f}{\partial x} \text{ or } f_x$$

Again, one can find the derivative of f with respect to y as x is held constant referred to as the partial differentiation of f with respect to y denoted;

$$\frac{\partial z}{\partial y} \text{ or } \frac{\partial f}{\partial y} \text{ or } f_y$$

Functions of the form $z = f(x, y)$ represent surfaces e.g. The graph of $z = \frac{y^2}{(1+x^2)^3}$ is as shown below;

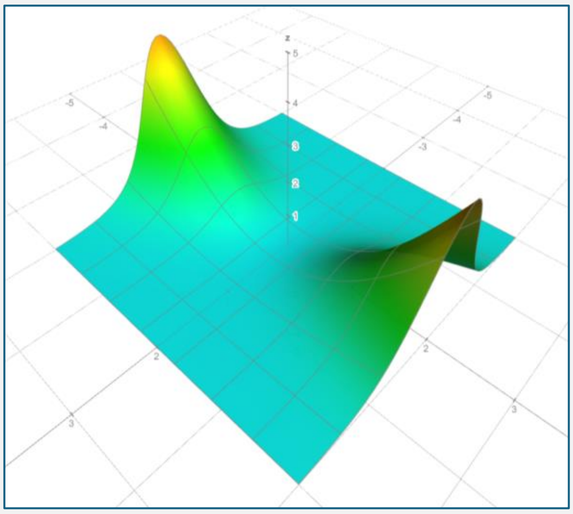

Figure 1: Draw using Math3d online graphing calculator at https://www.math3d.org/

Example 1: Determine the first-order partial derivatives of the function;

$$f(x, y) = x^3y^2 + 2x + 3y^2$$

Solution: We differentiate f with respect to x as we hold y constant, that is,

$$f_x = \frac{\partial f}{\partial x} = 3x^2y^2 + 2$$

Next we differentiate f with respect to y as we hold x constant, that is,

$$f_y = \frac{\partial f}{\partial y} = 2x^3y + 6y$$

Example 2: Determine $\frac{dy}{dx}$ given the implicit function $f(x, y) = x^3y^2 + 2x + 3y^2$ *(Implicit differentiation).*

Solution: Method 1: We let $f(x, y) = k$ where k is a constant say 0 i.e.

$$x^3y^2 + 2x + 3y^2 = 0$$

Then we differentiate each term separately with respect to x using the chain rule to get;

$$3x^2y^2 + 2x^3y\frac{dy}{dx} + 2 + 6y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2x^3y + 6y) = -3x^2y^2 - 2$$

$$\frac{dy}{dx} = -\frac{3x^2y^2 + 2}{2x^3y + 6y}$$

Method 2 (Using partial derivatives)

Given a function $z = f(x, y)$ then $\frac{dy}{dx} = -\frac{f_x}{f_y}$. From example 1 we have;

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{3x^2y^2 + 2}{2x^3y + 6y}$$

Remark 1: A function that can be easily expressed as $y = f(x)$ is said to be an explicit function. While a function of the form $f(x, y) = 0$ is an implicit function that cannot be expressed as $y = f(x)$.

For example, $y = 2x^2 + x + 1$, $y = 3x^4$, and $y = \ln x$ are explicit functions while $3xy + 2x^3 - y^2 = 0$, $4xy + 7y^2 = 9$, and $5x + zy - x^2z^2 = 9$ are implicit functions.

Example 3: Find $\frac{dy}{dx}$ given $y^3 + 5x^2y^2 - 7x = 9$

Solution: $\frac{\partial f}{\partial x} = 10xy^2 - 7$ and $\frac{\partial f}{\partial y} = 3y^2 + 10x^2y$. Hence;

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{10xy^2 - 7}{3y^2 + 10x^2y}$$

Definition (Small Increments): Consider the functions $z = f(x, y)$, suppose variable x changes by a small margin Δx while variable y is fixed, then z will change in a corresponding margin Δz given by;

$$\Delta z = \frac{\partial z}{\partial x}\Delta x$$

Again, if variable y changes by a small margin Δy while variable x is fixed, then z will change in a corresponding margin Δz given by;

$$\Delta z = \frac{\partial z}{\partial y}\Delta y$$

Hence the net change in z is given by;

$$\Delta z \approx \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$$

Example 1: It is given that $z = f(x, y) = xy^2 + x^2y$ is defined at point $A(1,2)$, then estimate the change in z when variable x increases from 2.0 to 2.01 and variable y increases from 3.0 to 3.1 simultaneously.

Solution: First we find the first order partial derivatives;

$$f_x = y^2 + 2xy; f_y = 2xy + x^2$$

Then at point A we $f_x = 4 + 4 = 8; f_y = 4 + 1 = 5$

The small changes in both variables x and y are 0.01 and 0.1 respectively and hence the small increment in z is given as

$$\Delta z \approx \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y = f_x \Delta x + f_y \Delta y = 8 \times 0.01 + 5 \times 0.1 = 0.58$$

Elasticity of Demand

In the real sense the demand Q for a good or service depends on many factors or conditions. A change in either of the factors or conditions will influence the demand for the commodity or service offered. Some of these conditions include the price P of the commodity, the price P_a of an alternative good or service that a customer may opt for, the disposable income of the consumer C, among others. That is demand Q is a function of all these variables;

$$Q = f(P, P_a, C)$$

Hence we shall have *price elasticity of demand*

$$E_p = -\frac{P}{Q} \times \frac{\partial Q}{\partial P}$$

Cross-price elasticity of demand will be

$$E_{p_a} = \frac{P_a}{Q} \times \frac{\partial Q}{\partial P_a}$$

Income elasticity of demand will be

$$E_C = \frac{C}{Q} \times \frac{\partial Q}{\partial C}$$

Note that E_c can be positive or negative. When the service or commodity is of good quality then demand for it will rise as income rises. This will mean that E_c is positive. Otherwise, if the quality is poor, the demand will fall as disposable income rises.

Example 1: Suppose the demand function $Q = 90 - 3P + 2P_a + 0.4C$ where $P = 5$ $P_a = 24$ $C = 2200$. Determine the price elasticity of demand, cross-price elasticity of demand, and disposable income elasticity of demand.

Solution:

Price elasticity of demand

Our value of $Q = 90 - 3 \times 5 + 2 \times 24 + 0.4 \times 2200 = 1003$

$$\frac{\partial Q}{\partial P} = -3$$

$$\Rightarrow E_p = -\frac{P}{Q} \times \frac{\partial Q}{\partial P} = -\frac{5}{1003} \times (-2) \approx 0.00997$$

Cross-price elasticity of demand

$$\frac{\partial Q}{\partial P_a} = 1$$

$$E_{p_a} = \frac{P_a}{Q} \times \frac{\partial Q}{\partial P_a} = \frac{24}{1003} \approx 0.0239 \text{ i.e. goods are substitutable}$$

Disposable income elasticity of demand

$$\frac{\partial Q}{\partial C} = 0.4$$

$$E_C = \frac{C}{Q} \times \frac{\partial Q}{\partial C} = \frac{2200}{1003} \times 0.4 \approx 0.877$$

Unconstrained Optimization

In linear programming we dealt with optimization within given constraints. Optimization of resources to maximize say profit or minimize cost subject to certain constraints such as budget or production capacity or available labor. In unconstrained optimization, there are no constraints on the given variables, and one only needs to determine the maximum or minimum value of a function within a given domain.

The optimization problem involves finding critical points i.e. points where the derivative is zero and proceeding by determining their nature. That is whether the critical points are points of local maxima, local minima, or points of inflection or saddle points.

Example 1: Given $f(x, y) = 5x^2y + y^3 - 2x^2y^3$ find $f_x, f_{xx}, f_y, f_{xy}, f_{yx}, f_{yy}$

Solution: We first find the first order partial derivatives are;

$$f_x = \frac{\partial f}{\partial x} = 10xy - 4xy^3; f_y = 5x^2 + 3y^2 - 6x^2y^2$$

The second-order partial derivatives we have;

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 10y - 4y^3$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = 10x - 12xy^2$$

$$f_{yx} = \frac{\partial^2 f}{\partial y\,\partial x} = 10x - 12xy^2$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = 6y - 12x^2y$$

Note that we can have a 2×2 matrix of the form below called the Hessian matrix

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

With $f_{xy} = f_{yx}$ and has determinant $f_{xx}f_{yy} - f_{xy}^2$. The surface will have a summit or a maximum stationary point if;

$$f_{xx} < 0, f_{yy} < 0 \text{ and } f_{xx}f_{yy} - f_{xy}^2 > 0$$

Example 2: Determine the nature of the stationary points given $f(x, y) = 3x^3 - 5x + 2xy^2$ *(Unconstrained optimization).*

Solution: The partial derivatives of the function are;

$$f_x = 9x^2 - 5 + 2y^2$$

$$f_y = 4xy$$

$$f_{xx} = 18x$$

$$f_{xy} = 4y$$

$$f_{yx} = 4y$$

$$f_{yy} = 4x$$

To determine the stationary points, we need to solve the simultaneous equations;

$$f_x = 0$$

$$f_y = 0$$

That is;

$$f_x = 0 = 9x^2 - 5 + 2y^2 \cdots \text{(i)}$$
$$f_y = 0 = 4xy \cdots \text{(ii)}$$

From equation (ii) we can let either $y = 0$ or $x = 0$

When $x = 0$ then from equation (i) we have $-5 + 2y^2 = 0 \therefore 2y^2 = 5 \Rightarrow y = \pm\sqrt{\frac{5}{2}}$

Hence points $\left(0, \sqrt{\frac{5}{2}}\right)$ and $\left(0, -\sqrt{\frac{5}{2}}\right)$ are stationary points

When y = 0 then from equation (i) we have $9x^2 - 5 = 0 \therefore 9x^2 = 5 \Rightarrow x = \pm\sqrt{\frac{5}{9}}$

Hence points $\left(\frac{\sqrt{5}}{3}, 0\right)$ and $\left(-\frac{\sqrt{5}}{3}, 0\right)$ are stationary points.

This surface has four stationary points. You can use a 3d calculator to view the stationary points.

To determine the nature of these points we evaluate the second-order derivatives at each point and note the signs.

$$f_{xx} = 18x;\ f_{xy} = 4y;\ f_{yy} = 4x$$

Point	$\mathbf{f_{xx} = 18x}$	$\mathbf{f_{yy} = 4x}$	$\mathbf{f_{xy} = 4y}$	$\mathbf{f_{xx}f_{yy} - f_{xy}^2}$	**Nature**
$\left(0, \sqrt{\frac{5}{2}}\right)$	0	0	$4\sqrt{\frac{5}{2}}$	-40<0	Saddle point
$\left(0, -\sqrt{\frac{5}{2}}\right)$	0	0	$-4\sqrt{\frac{5}{2}}$	-40<0	Saddle point
$\left(\frac{\sqrt{5}}{3}, 0\right)$	$6\sqrt{5}$	$\frac{4}{3}\sqrt{5}$	0	40>0	Local minimum
$\left(-\frac{\sqrt{5}}{3}, 0\right)$	$-6\sqrt{5}$	$-\frac{4}{3}\sqrt{5}$	0	40>0	Local maximum

Example 3: Given the total revenue function of a firm $TR = 3Q + S - Q^2 + SQ - S^2$ where Q represent the total output and S salary expenditure. Find the level of output for which TR will be maximized.

Solution: We first find $\frac{\partial}{\partial Q}(TR)$ and $\frac{\partial}{\partial S}(TR)$, that is,

$$\frac{\partial}{\partial Q}(TR) = 3 - 2Q + S \cdots \text{(i) and } \frac{\partial}{\partial S}(TR) = 1 + Q - 2S \cdots \text{(ii)}$$

Next we find the second-order partial derivatives;

$$\frac{\partial^2}{\partial Q^2}(TR) = -2; \frac{\partial^2}{\partial S^2}(TR) = -2; \frac{\partial}{\partial Q\, \partial S}(TR) = 1$$

From equation (i) and (ii) we have;

$$3 - 2Q + S = 0 \Rightarrow 2Q - S = 3 \cdots (*)$$
$$1 + Q - 2S = 0 \Rightarrow 2S - Q = 1 \cdots (**)$$

From equation (*) we have $S = 2Q - 3$ and then replace S in equation (**) with 2Q – 3 to get;

$$2(2Q - 3) - Q = 1$$

$$\Rightarrow 4Q - 6 - Q = 1 \therefore 3Q = 7 \Rightarrow Q = \frac{7}{3} \therefore S = 2\left(\frac{7}{3}\right) - 3 = \frac{5}{3}$$

Our turning point is $(Q, S) = \left(\frac{7}{3}, \frac{5}{3}\right)$

Our Hessian matrix H is given as;

$$H = \begin{bmatrix} \frac{\partial^2}{\partial Q^2}(TR) & \frac{\partial^2}{\partial Q\,\partial S}(TR) \\ \frac{\partial^2}{\partial S\,\partial Q}(TR) & \frac{\partial^2}{\partial S^2}(TR) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

$$\Rightarrow |H| = 4 - 1 = 3$$

This is a local maximum

Example 4: A competitive producer of two items A and B sells them each at 100 KES and 60 KES respectively. Suppose that the Total Cost function for producing these items is represented as below. Determine the maximum profit and the corresponding value of x and y, the respective output levels for items A and B;

$$TC = 3x^2 + 2xy + 2y^2$$

Definition: *In perfect competition, neither the buyers nor sellers in the market have the power to influence prices individually.*

Solution: The total revenue for the two items is; $TR_A = 100x$ and $TR_B = 60y$ respectively. Note that since the producer is operating in perfect competition then the price of each item is fixed by the market and hence does not depend on x and y.

Hence the total revenue for the two items is;

$$TR = TR_A + TR_B = 100x + 60y$$

Hence the total profit π is TR – TC i.e.

$$\pi = (100x + 60y) - (3x^2 + 2xy + 2y^2)$$

$$\pi = 100x + 60y - 3x^2 - 2xy - 2y^2$$

We need to determine the critical points of this function. Hence we have the following derivatives;

$$\frac{\partial \pi}{\partial x} = 100 - 6x - 2y$$

$$\frac{\partial \pi}{\partial y} = 60 - 2x - 4y$$

$$\frac{\partial^2 \pi}{\partial x^2} = -6$$

$$\frac{\partial^2 \pi}{\partial y^2} = -4$$

$$\frac{\partial^2 \pi}{\partial x \, \partial y} = -2$$

At the critical point the first-order derivatives are zero i.e.

$$\frac{\partial \pi}{\partial x} = 100 - 6x - 2y = 0$$

$$\frac{\partial \pi}{\partial y} = 60 - 2x - 4y$$

We solve the system of equations simultaneous to get;

$$\begin{matrix} 6x + 2y = 100 \\ 2x + 4y = 60 \end{matrix} \Rightarrow \begin{matrix} 3x + y = 50 \cdots (*) \\ x + 2y = 30 \cdots (**) \end{matrix}$$

From equation (**) we have $x = 30 - 2y$. Replacing x in equation (*) with $30 - 2y$ we have;

$$3(30 - 2y) + y = 50$$

$$90 - 6y + y = 50 \Rightarrow 40 = 5y \therefore y = 8$$

Hence $x = 30 - 2y = 30 - 16 = 14$

The turning point for the profit function is $(14,8)$

Note that

$$\frac{\partial^2 \pi}{\partial x^2} = -6 < 0; \; \frac{\partial^2 \pi}{\partial y^2} = -4 < 0; \text{ and}$$

$$\frac{\partial^2 \pi}{\partial x^2} \cdot \frac{\partial^2 \pi}{\partial y^2} - \left(\frac{\partial^2 \pi}{\partial x \, \partial y}\right)^2 = (-6)(-4) - (-2)^2 = 24 - 4 = 20 > 0$$

Implying that the point (14, 8) is a local maximum.

Therefore, the maximum profit is

$$\pi = 100x + 60y - 3x^2 - 2xy - 2y^2$$

$$= 100(14) + 60(8) - 3(14)^2 - 2(14)(8) - 2(8)^2 = 940$$

Exercise

1) It is given that $z = f(x, y) = 2x^3 + 3x^2 - 5y$ is defined at point $A(3,2)$, then estimate the change in z when variable x increases from 1.0 to 1.001 and variable y decreases from 3.0 to 2.99 simultaneously.

2) Evaluate $\frac{dy}{dx}$ given the equations below;

   a. $x^4 + 3xy^2 - y = 3x$
   b. $5xy^3 + 3x - 7y^2 = 9$
   c. $x^5y^3 - \frac{1}{y} + \frac{3}{x^3} = 0$

3) Given demand Q is a function of two variables, price P of the good, and price P_a of alternative good;

$$Q = f(P, P_a)$$

Then price elasticity of demand is given by

$$E_p = -\frac{P}{Q} \times \frac{\partial Q}{\partial P}$$

And c ross-price elasticity of demand will be

$$E_{p_a} = \frac{P_a}{Q} \times \frac{\partial Q}{\partial P_a}$$

Describe the conditions when $E_{p_a} < 0$ or $E_{p_a} > 0$

References

Jacques, I. (2006). *Mathematics for economics and business* (5th ed.). Prentice Hall.

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Werner, F., & Sotskov, Y. N. (2006). *Mathematics of Economics and Business*. Routledge: Taylor & Francis Group.