Business Mathematics

Lecture 11

Integration

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Introduction to Lecture 11

This lecture introduces you to Integration and its application to business mathematics. Integration is the reverse of differentiation and therefore this lecture is a continuation to lecture 9 on Differentiation.

Further Readings

The resources below are recommended for further reading to gain more insights on partial differentiation (Jacques, 2006; Menz & Mulberry, 2018; Sullivan & Miranda, 2019; Werner & Sotskov, 2006).

Intended Learning Outcomes

At the end of this lecture, you will be able to;

(i) Describe the basic integration techniques.

(ii) Solve problems involving integration.

(iii) Apply Integration to solve business related problems.

Introduction to Integration

Suppose you are given the function $f(x) = x^4$ then the derivative of f with respect to x is given as;

$$f^{/}(x) = 4x^3$$

Now suppose you are required to find a function $F(x)$ which differentiate to $f(x) = 4x^3$. You can see that $F(x) = x^4$. We can write this as;

$$F^{/}(x) = 4x^3 = f(x)$$

Then $F(x)$ is called the integral or anti-derivative of $f(x)$ denoted;

$$F(x) = \int f(x)dx$$

That is the integral of $f(x)$ with respect to x.

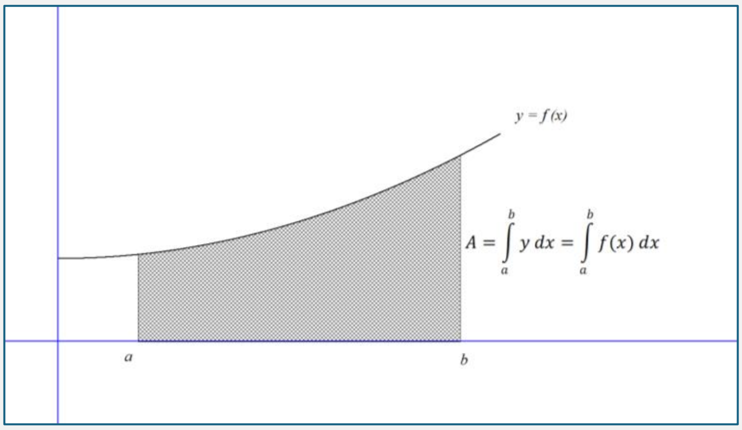
In general, if $F(x)$ is any function that differentiates to $f(x)$ then $F(x) + c$ where c is a constant will also differentiate to $f(x)$. Therefore we have the indefinite integral;

$$\int f(x)\,dx = F(x) + c$$

Where c is referred to as the constant of integration.

Geometrically, the integral of a function $y = f(x)$ over the given interval $a \le x \le b$ is the area below the curve. That is,

$$A = \int_a^b f(x)dx$$



Given the function $y = kx^n$ where $k, n \in \mathbb{R}$, are known real numbers with $n \neq -1$ then;

$$\int y\,dx = \int kx^n dx = \frac{kx^{n+1}}{n+1} + c - \text{Power rule}$$

Example 1: Integrate $y = 7x^4$ with respect to x.

Solution:

$$\int ydx = \int 7x^4 dx = \frac{7x^5}{5} + c$$

Given the function $y = x^n$ where $n \in \mathbb{R}$, is a known real number with $n = -1$ then;

$$\int ydx = \int kx^{-1}dx$$

$$= \int \frac{k}{x}dx = \ln|x| + c \text{ i.e. Natural logarithm of the absolute value of x plus c}$$

Example 2: Evaluate $\int \frac{3}{x}\,dx = 3\int \frac{1}{x}\,dx \Rightarrow 3\ln|x| + c$

Remark 1: Recall that given $y = e^x$ then $\frac{dy}{dx} = e^x$ therefore,

$$\int e^x dx = e^x + c$$

Given $y = e^{bx}$ where $b \in \mathbb{R}$ then $\int y\,dx = \int e^{bx}dx = \frac{1}{b}e^{bx} + c$

For instance, evaluate $\int e^{5x}dx$

Solution: $\int e^{5x}dx = \frac{1}{5}e^{5x} + c$

Again evaluate,

$$\int e^{7x+3}dx$$

Solution: Let $u = 7x + 3 \Rightarrow \frac{du}{dx} = 7$ (and $dx = \frac{1}{7}du$)

Also;

$$\int e^{7x+3}dx = \int \frac{e^u}{7}du = \frac{1}{7}\int e^u du = \frac{1}{7}e^u + c$$

But $u = 7x + 3$ hence we have

$$\int e^{7x+3}dx = \frac{1}{7}e^{7x+3} + c$$

Example 3: Given the Marginal Cost function $MC = 2Q^2 + Q + 3$. Determine the Total Cost TC function if the fixed costs are 13.

Solution: Recall from the lecture on differentiation that $MC = \frac{d}{dQ}(TC)$

$$\Rightarrow MCdQ = d(TC)$$

Integrating both sides to get;

$$\int d(TC) = \int MC\,dQ + c$$

$$TC = \int (2Q^2 + Q + 3)dQ = \frac{2Q^3}{3} + \frac{Q^2}{2} + 3Q + c$$

But our c is the fixed costs hence; $TC = \frac{2Q^3}{3} + \frac{Q^2}{2} + 3Q + 13$

Example 4: Determine the Total Revenue TR and the demand function given the Margin Revenue function $MR = 25 - 3Q$.

Solution: Recall that marginal revenue $MR = \frac{d(TR)}{dQ}$

$$\Rightarrow MR\, dQ = d(TR)$$

Integrating both sides to get;

$$\int d(TR) = \int MRdQ$$

$$TR = \int MR\, dQ + c$$

$$= \int (25 - 3Q)dQ = 25Q - \frac{3Q^2}{2} + c \cdots (*)$$

Note that if there is no production i.e. Q = 0 then the total revenue TR = 0. Hence equation (*) becomes $TR = 25(0) - \frac{3}{2}(0) + c = 0 \therefore c = 0$. Hence $TR = 25Q - 1.5Q^2$.

Recall in previous lecture total revenue is a product of price and quantity sold i.e. $TR = PQ$.

Hence $P = \frac{TR}{Q}$

$$\Rightarrow P = \frac{25Q - 1.5Q^2}{Q} = \frac{Q(25 - 1.5Q)}{Q} \therefore P = 25 - \frac{3}{2}Q$$

Area under the curve

Consider the area below the curve of the graph $y = f(x)$ over the domain $a \leq x \leq b$ as shown below. Then the area below the graph is given by;

$$A = \int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

This is the definite integral where a and b are the limits of integration.

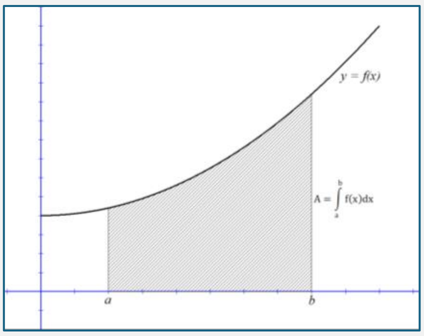


Figure 1

Example1 : Evaluate;

$$\int_1^3 6x^4 dx$$

Solution:

$$\int_1^3 6x^4 dx = \left[\frac{6x^5}{5}\right]_1^3 = \frac{6}{5}(3)^4 - \frac{6}{5}(1)^4 = 97.2 - \frac{6}{5} = 96$$

Example 2: Suppose the revenue function is given as $R(t) = 0.2t^3 - 2t^2 + 3t + 12$ where t is period in months. Evaluate the revenue during the 3rd month.

Solution:

$$\int_2^3 (0.2t^3 - 2t^2 + 3t + 12)dt = \left[\frac{1}{20}t^4 - \frac{2}{3}t^3 + \frac{3}{2}t^2 + 12t\right]_2^3$$

$$= \left(\frac{1}{20} \cdot 3^4 - \frac{2}{3} \cdot 3^3 + \frac{3}{2} \cdot 3^2 + 12 \cdot 3\right) - \left(\frac{1}{20} \cdot 2^4 - \frac{2}{3} \cdot 2^3 + \frac{3}{2} \cdot 2^2 + 12 \cdot 2\right)$$

$$= 35.55 - 25\frac{7}{15} \approx 10.083$$

Surplus in Consumption and Production

Consumer surplus measures the difference between what consumer is willing to pay for a good or service and what they actually pay. Consumer plus represents the benefit consumers receive from purchasing a product at a price lower than the maximum they are willing to pay.

For example, if a consumer is willing to pay 50 KES for an item but buys them for 20 KES, then the consumer surplus is 30 KES.

Willingness to pay varies among individuals based on say their desires or preferences or income. While actual payment is the market price that consumers actually pay for the good. Actual payment is determined by the equilibrium price where supply meets demand in a competitive market.

Consumer surplus is;

- used to measure consumer welfare and the benefits they receive from market transactions. If it is higher it means that consumers are benefiting more.
- Used to help in assessing market efficiency. If the market is functioning efficiently then consumer surplus is maximised.

- Used by policy analysis to evaluate the impact of regulations, taxes, and subsidies. For instance a tax imposed on a good may reduce consumer surplus as it will raise the price that consumers have to pay.

The consumer surplus is the area between the demand curve and the market price level, up to the quantity bought.

The demand curve represents consumers' willingness to pay, while the horizontal line at the market process shows the actual payment.

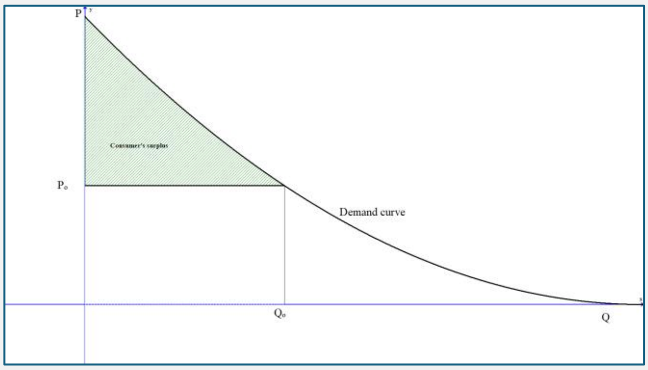


Figure 2

The consumer's surplus area is equal to the demand curve over the range $0 \leq Q \leq Q_0$ less the area represented by the rectangle with vertices $(0,0), (Q_o, 0), (Q_0, P_0)$, and $(0, P_0)$. That is;

$$\text{Consumer's surplus} = \int_0^{Q_0} P(Q)dQ - Q_0P_0$$

Example 1: Determine the consumer's plus at $Q = 20$ for the demand function $P = 45 - 2Q$

Solution: Our $Q_0 = 20, \Rightarrow \ P_0 = 45 - 2(20) = 5$

Hence the consumer's surplus is;

$$\int_0^{Q_0} P(Q)dQ - Q_0P_0 = \int_0^{20} (45 - 2Q)dQ - 20 \times 5$$

$$= [45Q - Q^2]_0^{20} - 100$$

$$= (45 \times 20 - 20^2) - 100 = 400$$

Producer's Surplus

It is the difference between what producers are willing to accept for a good or service and what they actually receive.

It is the additional benefit producers gain by selling at market price higher than their minimum acceptable price.

Market price is the price at which goods or service are actually sold in the market.

For instance;

Consider a simplified market for a product where the supply curve shows that producers are willing to sell the product at a minimum price of 100 KES.

If the market price is 150 KES, the producer surplus for each unit sold is 50 KES

i.e. 150 market price - 100 minimum acceptable price.

In the diagram below that represents the supply curve, the quantity Q_0 of goods sold price P_0 gives the total revenue received as P_0Q_0. This is represented by the area of the rectangle with vertices $(0,0), (Q_0, 0), (Q_0, P_o), (0, P_0)$.

Note that P_0 is the *price that the producer is ready to supply the last item i.e. the Q_0 good.* The producer is comfortable to receive the lower price given in the supply curve. Hence the shaded region is the benefit to the producer selling at a fixed price P_0. It is called the *producer's surplus.*

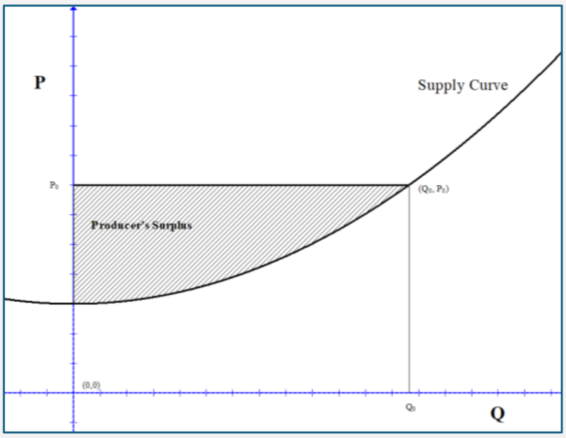


Figure 3

Therefore the *producer's surplus* is difference between the area of the rectangle with vertices $(0,0), (Q_0, 0), (Q_0, P_o), (0, P_0)$ and the area below the supply curve over the range $0 \leq Q \leq Q_0$. That is,

$$Q_0P_0 - \int_0^{Q_0} P(Q)dQ$$

Example 1: Given the supply curve $P = 10 + Q^2$ determine the producer's surplus at $Q = 12$

Solution: Our $Q_0 = 12 \Rightarrow P_0 = 10 + 12^2 = 154$. Hence the producer's surplus at $Q_0 = 12$ is given as

$$Q_0P_0 - \int_0^{Q_0} P(Q)dQ = 12 \times 154 - \int_0^{12} (10 + Q^2)dQ$$

$$= 1848 - \left[10Q + \frac{1}{3}Q^3\right]_0^{12} = 1848 - 696 = 1152$$

Example 2: Suppose the demand and supply functions are $P = 35 - 2Q_d$ and $P = 5 + 3Q_s$ respectively. Determine the consumer's and producer's surplus giving that the two are operating in pure competition where the price is determined by the market.

Solution: First determine the market equilibrium price and quantity. At equilibrium point both quantities are the same i.e. $Q_d = Q_s = Q; 35 - 2Q = 5 + 3Q \Rightarrow 5Q = 30 \therefore Q = 6$

$$\Rightarrow P = 35 - 2 \times 6 = 23$$

The equilibrium price P and quantity Q is 23 and 6 respectively.

Therefore the consumer's surplus is

$$\int_0^{Q_0} P(Q)dQ - Q_0P_0 = \int_0^{6} (35 - 2Q)dQ - 6 \times 23 = [35Q - Q^2]_0^6 - 138$$

$$(35 \times 6 - 36) - 138 = 36$$

The producer's surplus is;

$$Q_0P_0 - \int_0^{Q_0} P(Q)dQ = 6 \times 23 - \int_0^{6} (5 + 3Q)dQ = 138 - \left[5Q + \frac{3}{2}Q^2\right]_0^6 =$$

$$= 138 - \left[5 \times 6 + \frac{3}{2} \times 36\right] = 54$$

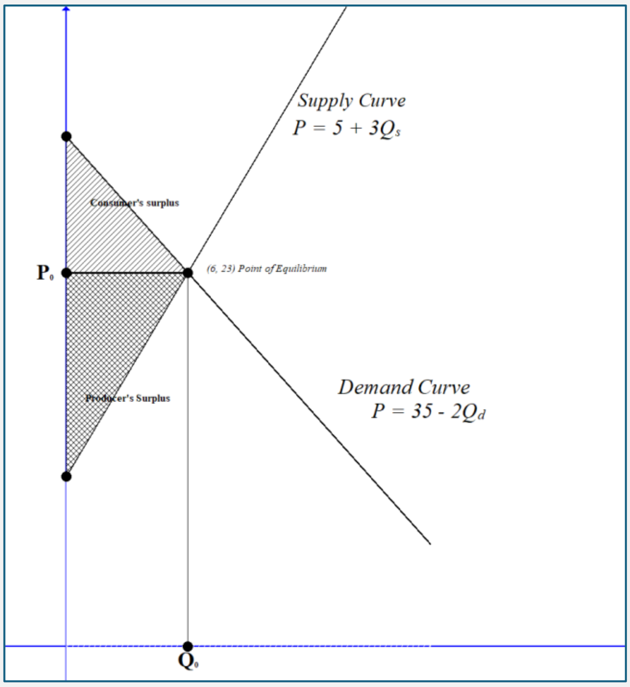


Figure 4

Exercise

1) Given the marginal revenue and marginal cost function as $MR = 12 - \frac{1}{4}Q^2$ and $MC = 16 + \frac{2}{5}Q^2$. Determine the values of Q that leads to a profit and determine the maximum profit.
2) Evaluate the following indefinite integrals
   a. $\int 7x^3 dx$
   b. $\int \frac{5}{x} dx$
   c. $\int (5x^8 + 9x + \ln x) dx$
   d. $\int e^{4x} dx$
3) Evaluate the following definite integrals
   a. $\int_0^3 \frac{7}{x} dx$
   b. $\int_1^2 (5x^2 - 7x + 4) dx$
   c. $\int_{-1}^5 (3x^5 - 5x + 2) dx$
   d. $\int_1^4 e^{3x+2} dx$
4) Determine the total revenue and the demand functions given the marginal revenue $MR = 130 - 15Q$
5) Determine the consumer's and producer's surplus given the demand and supply functions $P = 24 - 3Q_d$ and $P = 8 + 2Q_s$ respectively.
6) Determine the capital stock from end of year one to end of year 2, given the net investment $I(t) = 300\sqrt{t}$ where t is time.

References

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