Business Mathematics

Lecture 12

Dynamics

Lecturer: Kahenya, N.P

Introduction to Lecture 12

This lecture introduces you to the concept of dynamics. That is, how economic variables change over time and factors that influence these changes. Dynamics involves the analysis of time-dependent processes and the evolution of economic systems. The lecture will introduce dynamic models such as differential and difference equations.

Further Readings

The resources below are recommended for further reading to gain more insights on partial differentiation (Jacques, 2006; Menz & Mulberry, 2018; Munem & Foulis, 1988; Sullivan & Miranda, 2019; Werner & Sotskov, 2006).

Intended Learning Outcomes

At the end of this lecture, you will be able to;

(i) Describe dynamics in economics.
(ii) Describe difference and differential equations.
(iii) Apply difference and differential equations to solve economic problems.

Importance of Dynamics in Economics

In economics we have such dynamics as dynamics in pricing. That is how prices adjust over time in response to supply and demand changes. We also have labor market dynamics i.e. employment , unemployment, and wage adjustments over economic cycles. Capital accumulation is another economic dynamic that involves how investment in capital goods leads to economic growth over time.

Dynamics models in economics help in;

- Understanding fluctuations i.e. analyzing why economies experience periods of boom and bust and how to mitigate such cycles.
- Formulating policies that can help stabilize the economy or promote sustainable growth.

- Predicting future economic conditions.
- Evaluating long term effects of economic policies and structural changes.

Definition 1: Differential and difference equations are mathematical formulations that are applied to model how economic variables change over continuous time and discrete time.

Definition 2: Discrete time is one that progresses in distinct steps or intervals. For example, occasions, events, or actions that are taken at specific points in time. It can be daily, monthly, or annually. Here time advances in fixed increments e.g. month 1, month 2. For instance, the discrete-time version of compound interest.

It can be used in economic data reporting e.g. monthly unemployment rates; financial analysis such as daily stock prices.

Discrete time models are often represented using difference equations or recurrence relations e.g. $Y_i = \alpha Y_{i-1} + \beta$

Definition 3: Continuous time refers to a model where time progresses in a seamless, unbroken pattern. Time flow is considered a continuous variable.

In continuous time scenarios, differential equations are normally used to model phenomena. In economics it can be used to model continuous compounding and growth models.

A differential equation is of the form $\frac{dy}{dt} = y$ where $y = f(t)$.

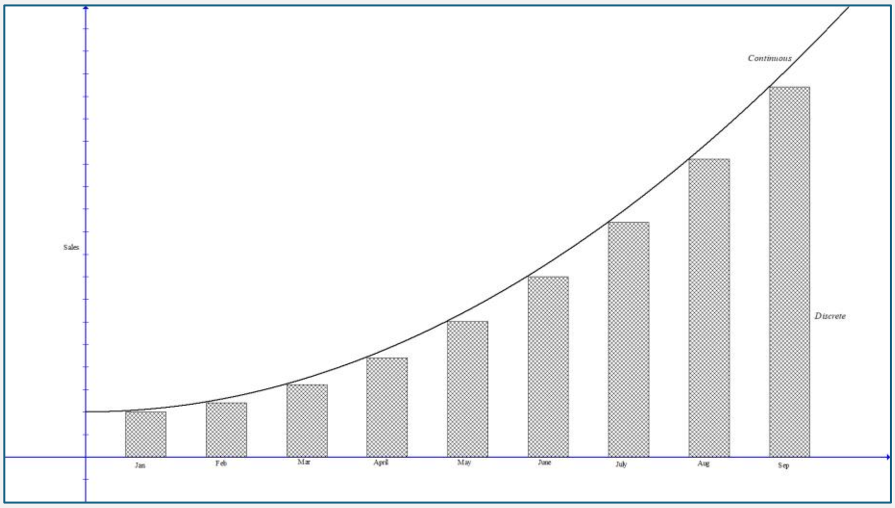

Figure 1

Definition 4: First-Order Difference Equation

A difference equation of first order that is linear and homogeneous is of the form

$$Y_i = \alpha Y_{i-1}, \qquad Y_0 = k - \text{initial condition, with } i = 1,2,\cdots$$

Example 1: Let $Y_i = \alpha Y_{i-1}$, $Y_0 = 4$ then

$$Y_1 = \alpha Y_0 = 4\alpha$$

$$Y_2 = \alpha Y_1 = \alpha(4\alpha) = 4\alpha^2$$

$$Y_3 = \alpha Y_2 = \alpha(4\alpha^2) = 4\alpha^3$$

It can be shown that the solution $Y_i = 4\alpha^i$

Remark: Given the general equation $Y_i = \alpha Y_{i-1}$, $Y_0 = k$ then the solution of the general equation is $Y_i = k\alpha^i$. It is also referred to as the complementary function.

Example 2: Let $Y_i = \alpha Y_{i-1} + \beta$ where $\alpha, \beta \in \mathbb{R}$. Then

$Y_1 = \alpha Y_0 + \beta$

$Y_2 = \alpha Y_1 + \beta = \alpha(\alpha Y_0 + \beta) + \beta = \alpha^2 Y_0 + \alpha\beta + \beta = \alpha^2 Y_0 + \beta(1 + \alpha)$

$Y_3 = \alpha Y_2 + \beta = \alpha\left(\alpha^2 Y_0 + \beta(1 + \alpha)\right) + \beta = \alpha^3 Y_0 + \alpha\beta(1 + \alpha) + \beta$

$$= \alpha^3 Y_0 + \beta(1 + \alpha + \alpha^2)$$

It can be shown that;

$$Y_4 = \alpha^4 Y_0 + \beta(1 + \alpha + \alpha^2 + \alpha^3)$$

Note that;

$$(1 + \alpha + \alpha^2 + \alpha^3) = \sum_{k=0}^{i-1} \alpha^k$$

In general, we have;

$$Y_i = \alpha^i Y_0 + \beta \sum_{k=0}^{i-1} \alpha^k$$

Now consider the series $1 + \alpha + \alpha^2 + \alpha^3 + \cdots$ that keeps growing as i increases.

If $\alpha = 1$ then the series $1 + \alpha + \alpha^2 + \alpha^3$ becomes 1+1+1+1. Hence in general

$$Y_i = Y_0 + \beta i$$

If $\alpha < 1$ then the series $1 + \alpha + \alpha^2 + \alpha^3 + \cdots$ is a geometric series with common ratio αless than 1 i.e. $S_n = \frac{a(1-r^n)}{1-r}$

For our case we have $a = 1, r = \alpha, n = i$

$$S_i = \frac{1 - \alpha^i}{1 - \alpha}$$

Therefore, we have : $Y_i = \alpha^i Y_0 + \beta\left(\frac{1-\alpha^i}{1-\alpha}\right)$ as the general solution for the first-order linear difference equation $Y_i = \alpha Y_{i-1} + \beta$.

Remark: Given the equation $Y_i = \alpha Y_{i-1} + \beta$ with the initial condition Y_0 then the general solution is the sum of complementary function and particular solution i.e.

$$Y_i = Y_0\alpha^i + \beta\left(\frac{1 - \alpha^i}{1 - \alpha}\right)$$

Where $Y_0\alpha^i$ is the complementary function and $\beta\left(\frac{1-\alpha^i}{1-\alpha}\right)$ is the particular solution.

Example 3: Compute the next three terms of the sequence; $x_n = x_{n-1} + 3$, with $x_0 = 7$

Solution: $x_1 = x_0 + 3 = 7 + 3 = 10, x_2 = x_1 + 3 = 10 + 3 = 13, x_3 = x_2 + 3 = 16$

Example 4: Solve the difference equation below given the initial condition, $x_n = 2x_{n-1}, x_0 = 3$ and hence determine x_7.

Solution:

$x_1 = 2x_0 = 6,$

$x_2 = 2x_1 = 2 \cdot 2x_0 = 2^2 \cdot x_0 = 12,$

$x_3 = 2x_2 = 2 \cdot 2 \cdot 2x_0 = 2^3 \cdot x_0 = 24, \cdots$

$\Rightarrow x_i = 3 \cdot 2^i$ is the general solution to the difference equation.

We can determine say $x_7 = 3 \cdot 2^7 = 384$

Example 5: Solve the difference equation $x_i = 3x_{i-1} + 8$ with the initial condition $x_0 = 3$

Solution: Our equation $x_i = 3x_{i-1} + 8$ is of the form $X_i = \alpha X_{i-1} + \beta$.

The complementary function is the general solution of $x_i = 3x_{i-1}$, which is $k(3^i)$.

On the other hand, the particular solution is solution to $x_i = 3x_{i-1} + 8$

We can let $x_i = T$ where T is a sequence that when replaced in $x_i - 3x_{i-1}$ gives the constant 8.

That is;

$$T - 3T = 8 \Rightarrow -2T = 8 \therefore T = -4$$

Therefore, the general solution is

$$x_i = k(3^i) - 4$$

The specific solution that satisfies the initial condition $x_0 = 3$ is

$$x_0 = k(3^0) - 4 = 3 \Rightarrow k = 7$$

$$\therefore\ x_i = 7(3^i) - 4$$

Example 6: Consider a two-sector economy with the income function $Y_i = C_i + I_i$ where C is Consumption function, and I is the investment function. Suppose $C_i = 20 + 0.75Y_{i-1}$ and $I_i = 40$. Determine Y_i when $Y_0 = 340$.

Solution: $Y_i = C_i + I_i = 20 + 0.75Y_{i-1} + 40 = 60 + 0.75Y_{i-1}$

The complementary function is $k(0.75^i)$

For particular solution we let $Y_i = T$ such that

$$T - 0.75T = 60 \Rightarrow T = 240$$

Hence the general solution for $Y_i = 60 + 0.75Y_{i-1}$ is

$$Y_i = k(0.75^i) + 240$$

But the initial condition is $Y_0 = 340$ which gives us

$$Y_0 = k(0.75^0) + 240 = 340$$

$$k + 240 = 340 \therefore k = 100$$

Our solution is

$$Y_i = 100(0.75^i) + 240$$

Note that as i increases 0.75^i tends to zero and hence Y_i approaches 240 meaning that the system is stable.

Definition 5 (Solution to Differential Equations): The general solution to the first-order ordinary differential equation; $\frac{dy}{dx} = ky$ is $y = Ke^{tx}$ where K is a constant.

Example 1: Given $\frac{dy}{dx} = t(x)$ where y and t are functions, we can solve by integrating both sides to get;

$$\int \frac{dy}{dx}\,dx = \int t(x)dx$$

$$y(x) = \int t(x)dx + c$$

the function $y(x)$ is the general solution with c as an arbitrary constant.

Example 2: Given $\frac{dy}{dx} = \alpha y + \beta, \alpha, \beta \in \mathbb{R}$ then the solution of such differential equations involves the exponential function e^{bx}, for some b. Since given $y = e^x$ then $\frac{dy}{dx} = e^x$ i.e. $\frac{dy}{dx} = y$

Example 3:

Given $y = 5e^x$ then $\frac{dy}{dx} = 5e^x$

Given $y = 3e^{7x}$ then $\frac{dy}{dx} = 21\,e^{7x}$

Example 4: Determine the solution to the differential equation $\frac{dy}{dx} = 5y$ that satisfies the initial condition $y(0) = 2$

Solution: Any function of the form $y(x)$ is a solution. Indeed, any function of the form $y = Ke^{5x}$ will satisfies our ODE since

$$\frac{dy}{dx} = 5(K\,e^{5x}) = 5y$$

To determine K we have $y(0) = 2$

$$\Rightarrow y(x) = Ke^{5x}$$

$$\therefore y(0) = Ke^0 = K$$

That is, $K = 2$. Hence our solution is $y(x) = 2e^{5x}$

Remark 1: Note that given the differential equation $\frac{dy}{dx} = \alpha y$ then the complementary function or solution is $Ae^{\alpha x}$.

Example 5: Solve the ODE $\frac{dy}{dx} = 3y + 13$ which satisfies the initial condition $y(0) = 5$.

Solution: The complementary function for $\frac{dy}{dx} = 3y$ is Ae^{3x}

Next we need to determine the particular solution for $\frac{dy}{dx} = 3y + 13$. Let consider the function $y(x) = T$ that will replace $\frac{dy}{dx} - 3y$ to get;

$$T = 13 \Rightarrow \frac{dy}{dx} = 0$$

Therefore;

$$0 = 3T + 13 \therefore T = -\frac{13}{3}$$

Hence our particular solution will be

$$y(x) = Ae^{3x} - \frac{13}{3}$$

Then when $y(0) = 5$ then $y(x) = Ae^{3x} - \frac{13}{3}$ becomes; $y(0) = Ae^{0} - \frac{13}{3} = 5$

$$\therefore A = 5 + \frac{13}{3} = \frac{28}{3}$$

The solution is: $y(t) = \frac{28}{3}e^{3x} - \frac{13}{3}$

Example 6 (Difference equation): Suppose in a market in equilibrium, it is given that $Q_i = 6P_{i-1} - 15$ and $q_i = -5P_i + 25$ as supply and demand functions respectively. Determine P_i when $P_0 = 9$.

Solution:

In an equilibrium $Q_i = q_i \Rightarrow 6P_{i-1} - 15 = -5P_i + 25$

$$-5P_i = 6P_{i-1} - 40$$

$$P_i = -1.2P_{i-1} + 8 \cdots (*)$$

Hence the complementary function is $K(-1.2)^i$

For particular solution we let $P_i = T$ then (*) becomes

$$T = -1.2T + 8 \Rightarrow 2.2P = 8 \therefore T = \frac{40}{11}$$

The general solution will then be;

$$P_i = K(-1.2)^i + \frac{40}{11}$$

However, our initial condition is $P_0 = 9$

$$9 = K(-1.2)^0 + \frac{40}{11} \Rightarrow K = \frac{59}{11} \therefore P_i = \frac{59}{11}(-1.2)^i + \frac{40}{11}$$

Example 7: Find the general solution of the differential equation below and hence determine the particular solution satisfying the condition $x = 1$ when $y = 2$

$$\frac{dy}{dx} = 3x^2y^2$$

Solution: We can rewrite the differential equation, by putting like terms on one side, as;

$$\frac{dy}{y^2} = 3x^2dx$$

Then integrate both sides to get;

$$\int \frac{dy}{y^2} = \int 3x^2dx$$

$$\frac{y^{-1}}{-1} + C_0 = x^3 + C_1$$

$$\frac{1}{y} + C_0 = x^3 + C_1$$

$$\frac{1}{y} - x^3 = C_1 - C_0 = C_2 \therefore y = \frac{1}{x^3} + C_2 - \text{General Solution}$$

When $x = 1, y = 2$ we have $2 = 1 + C_2 \Rightarrow C_2 = 1$

$$\therefore y = \frac{1}{x^3} + 1$$

Exercise

1) Compute the next 5 terms of the sequences below;
   a. $y_i = y_{i-1} + 7, y_0 = -1$
   b. $y_i = 2y_{i-1} - 21, y_0 = 3$
   c. $y_i = \frac{1}{4}y_{n-1} + 2, y_0 = 14$
2) Sove the following difference equations, given the initial condition and hence use your solution to find y_{11}.
   a. $y_i = \frac{1}{4}y_{i-1}\ , y_0 = 21$
   b. $y_i = 2.4y_{(i-1)}\ , y_0 = 7$
   c. $y_i = \frac{1}{3}y_{i-1}, y_0 = 13$
3) Sove the following difference equations, given the initial condition
   a. $y_i = \frac{1}{3}y_{i-1} + 2, y_0 = 10$

b. $y_i = \frac{1}{2}y_{i-1} + 7, y_0 = 3$

c. $y_{i+1} = 5y_i + 2, y_0 = 6$

4) Consider a two-sector economy with the income function $Y_i = C_i + I_i$ where C is Consumption function, and I is the investment function. Suppose $C_i = 100 + 0.5Y_{i-1}$ and $I_i = 20$. Determine Y_i when $Y_0 = 500$.

5) Determine the solution of the following differential equations which satisfies the given initial condition.

a. $\frac{dy}{dx} = 7y, y(0) = 1$

b. $\frac{dy}{dx} = -9y, y(0) = 3$

c. $\frac{dy}{dx} = 3x, y(0) = 5$

6) Determine $Y(t)$ when the initial condition $Y(0) = 3000$ for a 2-sector economy is modeled by $\frac{dy}{dt} = \frac{1}{4}(2C + I - Y)$, $C = 0.75Y + 120$, and $I = 450$

References

Jacques, I. (2006). *Mathematics for economics and business* (5th ed.). Prentice Hall.

Menz, P., & Mulberry, N. (2018). *Calculus Early Transcendentals: Integral & Multivariable Calculus for Social Sciences*. https://www.sfu.ca/math-coursenotes/Math 158 Course Notes/sec_applications-to-business.html

Munem, M. ., & Foulis, D. J. (1988). *Calculus with Analytic eometry*. Worth Publishers, Inc.

Sullivan, M., & Miranda, K. (2019). *Calculus: Early Transcendentals* (second). W.H. Freeman and Company.

Werner, F., & Sotskov, Y. N. (2006). *Mathematics of Economics and Business*. Routledge: Taylor & Francis Group.