FINAL EXAM

ANSWER ALL THE QUESTIONS

QUESTION ONE: 25 Marks

Consider the general cubic polynomial

$$f(x) = \frac{1}{3}ax^3 + bx^2 + cx + d,$$

where a, b, c and d are real constants. If the stationary points of f are at x_1 and x_2, use MATLAB to show that

$$f(x_1) - f(x_2) = -\frac{a}{6}(x_1 - x_2)^3.$$

□

QUESTION TWO: 25 Marks

Show that the function $f = 1/r$, where $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$ and

a, b and c are constants, is a solution of Laplace's equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

QUESTION THREE: 25 Marks

The following two expressions are both approximations to π that were discov-

ered by the Indian mathematician Ramanujan (1887–1920):

$$\pi_1 = \frac{12}{\sqrt{190}} \ln((2\sqrt{2} + \sqrt{10})(3 + \sqrt{10}))$$

and

$$\pi_2 = \sqrt{\sqrt{9^2 + \frac{19^2}{22}}}.$$

Use VPA to find the absolute errors $|\pi_1 - \pi|$ and $|\pi_2 - \pi|$. Hence determine how good these approximations actually are. □

QUESTION FOUR: 25 Marks

Use `simplify` or `Simplify` to show that

$$\text{if} \quad y = \frac{1+\sin x}{1+\cos x} \qquad \text{then} \qquad \frac{dy}{dx} = \frac{\cos x + \sin x + 1}{1 + 2\cos x + \cos^2 x};$$

$$\text{if} \quad y = \ln\sqrt{\frac{1+x}{1-x}} \qquad \text{then} \qquad \frac{dy}{dx} = -\frac{1}{-1+x^2};$$

$$\text{if} \quad y = \ln\left(\frac{(1+x)^{1/2}}{(1-x)^{1/3}}\right) \qquad \text{then} \qquad \frac{dy}{dx} = \frac{-5+x}{6(-1+x^2)}.$$

QUESTION FIVE: 25 Marks

Use MATLAB to show that

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1),$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1),$$

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2.$$

QUESTION SIX: 25 Marks

The Golden Ratio is a number that occurs in nature and art. It is defined as

$$\varphi = \frac{1+\sqrt{5}}{2}. \tag{1}$$

It is related to the sequence of Fibonacci numbers F_n by taking the limit of ratios of successive Fibonacci numbers,

$$\varphi = \lim_{n\to\infty} \frac{F_n}{F_{n-1}}. \tag{2}$$

1. Write a function that generates the first n Fibonacci numbers.
2. Using your function for generating the Fibonacci numbers, write a function that outputs the n^{th} approximation to the Golden Ratio.
3. Use this function to compute an approximation to the Golden Ratio with accuracy 10^{-7}. Give your answer as a fraction.
4. Plot a graph of the error of your approximations against n. Your n values should range from $n = 1$ to the value corresponding to the error determined in Q3.