Applied Mechanics

Chapter 4

Center of Gravity, Centroid and Moment of inertia

Lecture 6 (week 6)

Second Moment of Area / Moment of inertia, Radius of Gyration and Parallel axis Theorem: Relevant Examples

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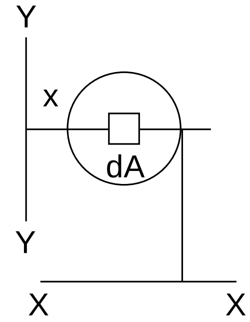
Learning Objectives:

- Understand second moment of area and radius of gyration in structural mechanics.
- Second moment of area for different shapes.
- Explore real-world applications of second moment of area in engineering design.
- Learn the Parallel Axis Theorem for calculating moment of inertia.
- Understand its application in finding moment of inertia for different shapes.
- Analyze practical examples demonstrating the use of the Parallel Axis Theorem.

4.2 Calculation of Second Moment of Area / Moment of inertia and Radius of Gyration: Relevant uses

4.2.1 Second Moment of Area / Moment of inertia

The cumulative product of area and square of its distance from an axis is called the moment of inertia of a section about that axis.



i.e. $I_X = \int y^2 dA$and

$I_Y = \int x^2 dA$

Or in other words

First moment of elemental area about y-y axis = X×dA.

(Moment of first moment) second moment of elemental area = (X ×dA) ×X = X^2dA

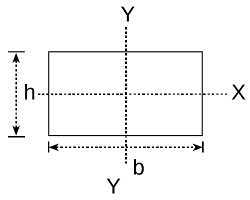
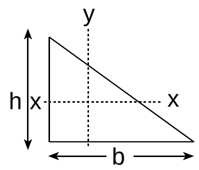
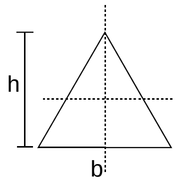
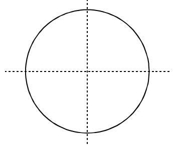
∴ 2^{nd} moment of entire area A w. r. to y-y axis is obtained by integrating above expression,

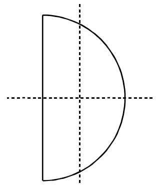
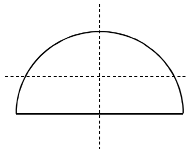
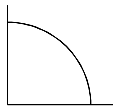
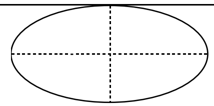
$I_Y = \int x^2 dA$ Similarly, $I_{xx} = \int y^2 dA$ ∴ And the second moment of an area is termed as M.O.I

4.2.1.1 Physical meaning of moment of Inertia

It gives the measure of resistance to bending of beams under the action of loads. The greater the second moment of area (M.O.I), the greater the resistance to bending So M.O.I of cross- section is essential for designing.

4.2.1.2 Moment of inertia of geometrical figure about centroidal axes

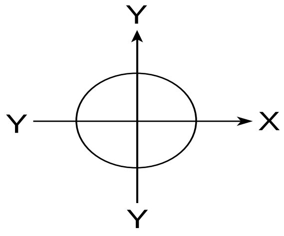
Geometrical Area (Name)	**Figure**	I_{X-X}	I_{Y-Y}
Rectangle		$\frac{bh^3}{12}$	$\frac{hb^3}{12}$
Triangle (Right angled)		$\frac{bh3}{36}$	$\frac{hb^3}{36}$
Isosceles triangle		$\frac{bh^3}{36}$	$\frac{hb^3}{48}$
Circle		$\frac{\pi r^4}{4}$	$\frac{\pi r^4}{4}$

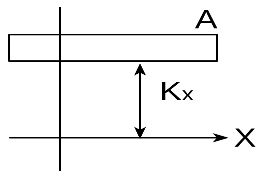
Semicircle		$\frac{\pi r^4}{8}$	$0.11r^4$
Semicircle		$0.11r^4$	$\frac{\pi r^4}{8}$
Quarter circle		$0.055r^4$	$0.055r^4$
Ellipse		$\frac{\pi ab^3}{4}$	$\frac{\pi ba^3}{4}$

4.2.2 Radius of Gyration

If we concentrate the entire area A of the lamina into thin strip (or into a point) such that there is no change in the moment of inertia about a given axis, then distance of that point from the given axis is called the radius of gyration.[1]

i.e $I_{XX} = Ak_x^2$ then the perpendicular distance K_x is termed as radius of gyration of the area w.r.t X – axis.

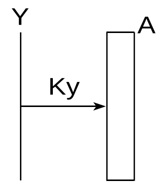




Similarly, $Iyy = A\,k_y^2$

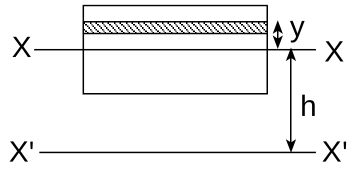
Ky = R.O. G w.r.t. Y – axis

Or In other words: R.O.G is the distance from the axis to a point where the entire area of the lamina could be concentrated into a thin strip and have the same M.O.I w.r.t to given axis.



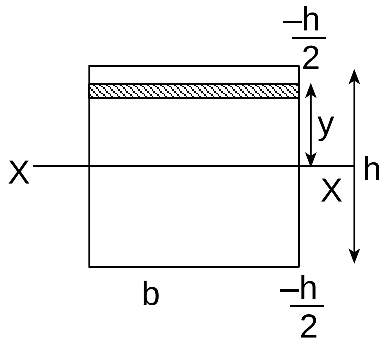
4.3 Statement and Provence of parallel axis theorem

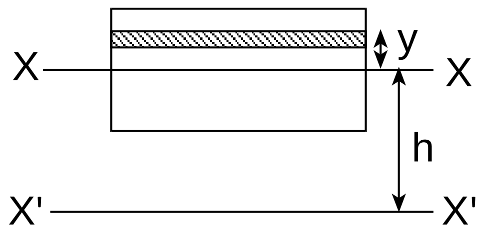
Parallel axis theorem states that the moment of inertia of a plane area about any axis parallel to the centroidal axis of that area is equal to the sum of moment of inertia about a parallel centroidal axis of that plane area and the product of the area and square of the distance between the two axes.[2]



Proof:

Consider a lamina having area 'A', X – X be the centroidal axis of this lamina and x'- x' be another axis which is parallel to the centroidal x – x axis and at a distance of h from centroidal axis. Consider an elemental area dA at a distance 'y' from the centroidal axis x – x.





Ixx = M.O.I of the lamina about x - x axis

we know, $I_{XX} = \Sigma y^2 dA$

Here,

$I_{X'X'}$ = moment of inertia of the lamina about x'-x' axis

so,

$I_{X'X'} = \Sigma(h+y)^2\, dA$ [(h+y) is the distance of elemental area from x' – x' axis.]

$=\Sigma(h^2+2hy+y^2)\, dA.$

$=\Sigma h^2 dA + 2h.\Sigma ydA + \Sigma y^2 dA$

$= Ah^2 + 2h\times 0 + I_{XX}$

$[I_{x'x'} = I_{xx} + Ah^2]$

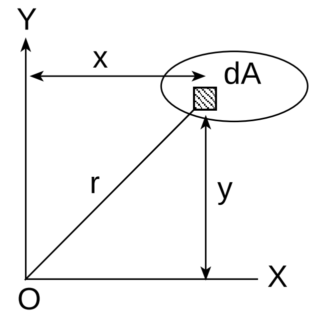
Hence proved the theorem

($\sum y.dA = 1^{st}$ moment of area = 0

i.e $\int ydA = \int_{-h/2}^{h/2} y.xdy = 0$)

4.4 Perpendicular axis theorem

It states that ' If I_{OX} and I_{OY} be the moment of inertia of a lamina about mutually perpendicular axes ox and OY in the plane of the lamina and I_{OZ} be the moment of inertia of the lamina about an axis (oz) normal to the lamina and passing through the point of intersection of the axes ox and OY, then $I_{OZ} = Iox + Ioy$



Let OX and OY be the two mutually perpendicular axes lying in the plane of the lamina. let oz be the axis normal to the plane of the lamina and passing through o.

Consider an elemental component of area dA of the lamina.

Let the distance of this elemental component from the axis oz, ie. from o be r.

M.o.I of the elemental component about oz, = $r^2 \times dA$ --------- (i)

if the coordinates of the elemental component be (x,y) referred to the axis ox and oy, we have.

$r^2 = x^2 + y^2$

hence equation (i) becomes

=M.O. I of elemental component about oz = $(x^2 + y^2)dA$

$= x^2 dA + y^2 dA$

$\therefore$ Total M.O.I of the lamina about the axis oz

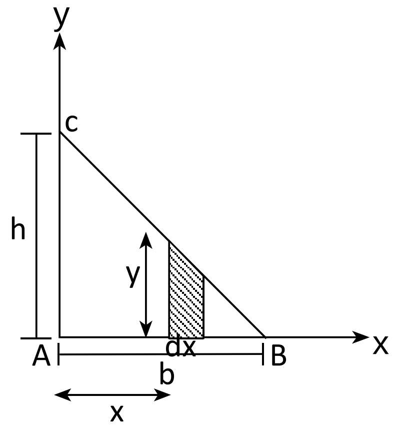
$I_{OZ} = \Sigma(x^2dA + y^2dA)$

$=\Sigma x^2dA + \Sigma y^2dA$

[$\therefore I_{OZ}=I_{OX} + I_{OY}$] Where $\Sigma x^2 dA$ = M.O.I of the lamina about ox axis and y^2dA =M.O.I of the lamina about oy axis]

Relevant Examples:

1. Derive an expression for M.O.I of triangular section having base 'b' and height 'h' about its centroidal axis, base axis and vertex.[2]



Solution:

1^{st} find the inclined line equation by similar triangle properties

Taking vertical strip

$$\frac{h}{b} = \frac{y}{(b-x)}$$

$$\left[\therefore y = h/b\ (b-x)\right]$$

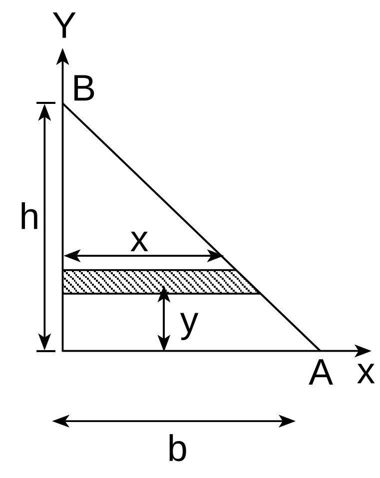
$$\therefore \left(\frac{by}{h} = b - x\right)$$

$$X = (b - b/hy)$$

M.O.I about any Axis y (I_Y) = $\int x_e^2 dA$

$$= \int_0^b x^2 y.dx = \int_0^b x^2.h/b\ (b-x)\ dx$$

$$= \frac{h}{b}\int_0^b (bx^2 - x^3)\, dx = \int_0^b \times \left[\frac{b^4}{3} - \frac{b^4}{4}\right]$$



$$= \frac{h}{b} \times \frac{b^4}{12} \quad (I_Y) = \frac{hb^3}{12}$$

And M.O.I about x

Taking strip parallel to x

$$I_x = \int y_e^2 dA$$

Ye = distance from x axis to the centroid of strip

$$I_x = \int_0^h y^2\, x\, dy$$

$$= \int_0^h y^2 \left(b - \frac{b}{h} y\right) dy = \int_0^h \left(by^2 - \frac{by^3}{h}\right) dy$$

$$= \left[\frac{bh^3}{3} - \frac{bh^4}{4h}\right]$$

$$I_x = \frac{bh^3}{12}$$

For M.O.I about centroidal axis (Ixx)

We know, (I any axis) = Ixx + A × h²

$$\frac{bh^3}{12} = I_{xx} + \frac{1}{2} \times b \times h \times \left(\frac{h}{3}\right)^2$$

$$\therefore \; \mathbf{Ixx} = \frac{\mathbf{bh^3}}{\mathbf{36}}$$

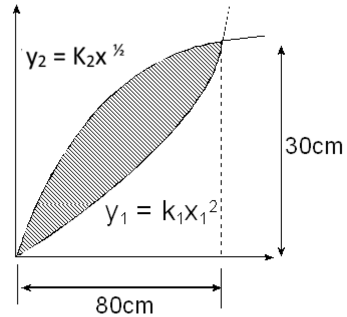
similarly

$Iy = I_{yy} + Ah^2$

$$\frac{hb^3}{12} = Iyy + \left(\frac{1}{2} \times b \times h\right) \times \left(\frac{b}{3}\right)^2$$

$$\therefore \; \mathbf{Iyy} = \frac{\mathbf{hb^3}}{\mathbf{36}}$$

2. **Determine the moment of inertia and radius of gyration of the common area as shown in figure below about 'x' and 'Y' axis.**



Solution: calculate the value of constant

(i.e value of constant by putting the limits value)

By putting the minimum and maximum value of x, y (intersecting point)

$y_1=K_1x^2$

$30 = K_1\times 80^2$ (i.e. x = 80, y= 30)

$K_1 = 30/80^2 = 0.00469$

And $y_2 = K_2x^{½}$ [x,y =(80,30)]

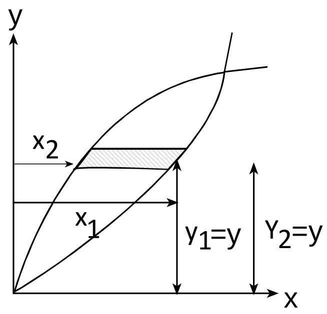
$K_2 = 3.354$

[Here, there is no values of constant by putting [x,y=(0,0)]

Hence the equation becomes

$[y_1 = 0.00469x_1^2]$ $[y_2 = 3.354\ x_2^{1/2}]$

For M.O.I, 1st of all we calculate the area between the curves taking either horizontal strip or vertical strip



In horizontal strips.

$A = \int dA = \int (x_1 - x_2) \times dy$

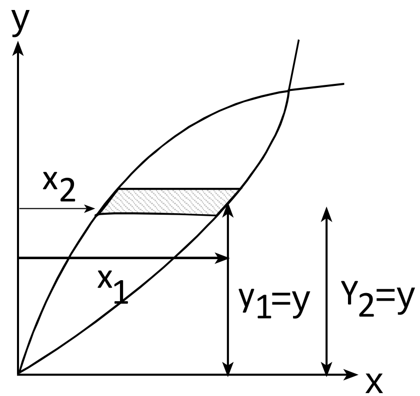
$\int$(length ×height)

$$= \int_0^{30} (x_1 - x_2) \times dy = \int_0^{30} \left[\frac{y^{1/2}}{0.00469^{1/2}} - \frac{y_2^2}{3.354^2} \right] \times dy \qquad [y_1 = y_2 = y]$$

$$= \left[\frac{y^{3/2}}{\frac{3}{2} \times 0.0685} - \frac{y^3}{3 \times 11.25} \right]_0^{30} = \left(\frac{30^{3/2}}{0.103} - \frac{30^3}{33.75} \right)$$

$A = 795.31 cm^2$

M.O.I. about x –x axis is calculated by using horizontal strip parallel to x – axis as shown in fig



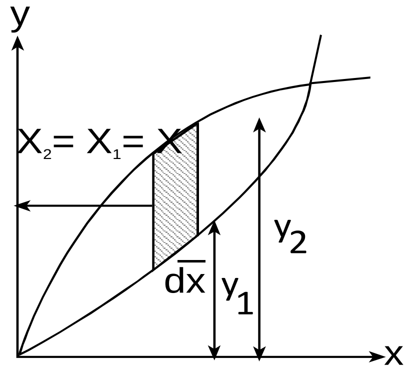
$$I_x = \int y_e^2 dA = \int_0^{30} y^2 \times (x_1 - x_2) \times dy$$

$$I_x = \int_0^{30} y^2 \times \left(\frac{y_1^{1/2}}{0.00469^{1/2}} - \frac{y_2^2}{3.354^2}\right) \times dy \quad I_x = \int_0^{30} \left(\frac{y^{\frac{5}{2}}}{0.0685} - \frac{Y^4}{11.25}\right) dy$$

$$I_x = \int_0^{30} \left[\frac{y^{7/2}}{\frac{7}{2} \times 0.0685} - \frac{y^5}{5 \times 11.25}\right]_0^{30} \quad I_x = \left[\frac{30^{7/2}}{7/2 \times 0.0685} - \frac{30^5}{5 \times 11.25}\right]$$

$I_x = 184830.41 cm^4$

Calculation of M.O.I about y axis by taking vertical strip parallel yo y axis



$Iy = \int x_e^2 dA$ (In horizontal strip $x_e = x_1 = x_2 = x$)

$$= \int_0^{80} x_e^2 \times (y_2 - y_1) dx$$

$$= \int_0^{80} x_e^2 \times \left(3.354 x_2^{1/2} - 0.00469 x_1^2\right) dx$$

$$= \int_{0}^{80} [3.354x^{5/2} - 0.00469x^4]dx$$

$$= \left[\frac{3.354 \times x^{\frac{7}{2}}}{\frac{7}{2}} - \frac{0.00469 \times x^5}{5}\right]_0^{80}$$

$$= \left[\frac{3.354 \times 80^{7/2}}{7/2} - \frac{0.00469 \times 80^5}{5}\right]$$

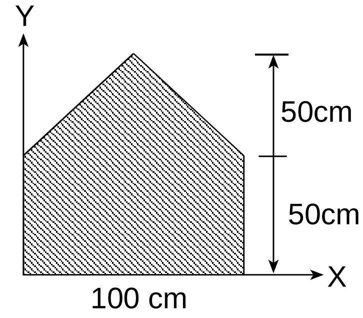
$I_y = 13148.00 cm^4$

Now, Radius of gyration can be calculated as,

$$K_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{184830.41}{795.31}} = 15.25cm$$

$$K_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{1314800}{795.31}} = 40.65cm$$

3. Determine the M.O.I. about the centroidal x and y axis of the shaded area shown in figures below.



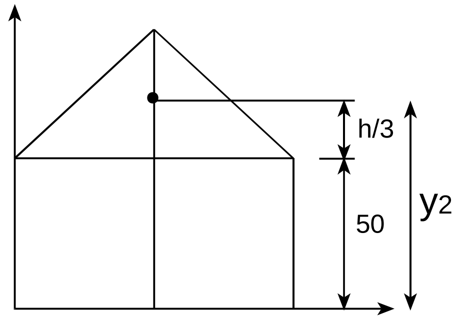
Solution:

whole figure divided into known geometrical figure

(1) Rectangle

(2) Triangle

Calculating the individual area and centroid of each figure



$A_1 = 100 \times 50 = 5000 cm^2$

$X_1 = 100/2 = 50$ cm

$Y_1 = 50/2 = 25$cm

$A_2 = ½ \times 100 \times 50 = 2500$ cm^2

$x_2 = 100/2 = 50$cm

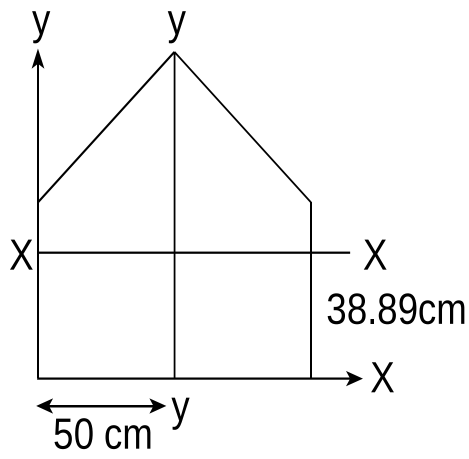
$Y_2 = 50 + \frac{50}{3} = 66.67$cm

Calculate the centroid of the whole given figure

$$\bar{x} = \frac{A_1X_1+A_2X_2}{A_1+A_2} = \frac{5000\times25+2500\times66.67}{(5000+2500)} = 50cm$$

$$\bar{y} = \frac{A_1Y_1 + A_2Y_2}{A_1+A_2} = \frac{5000\times25+2500\times66.67}{(500+2500)}$$

$= 38.89cm$



$(\bar{x}, \bar{y})$ =(50,38,.89)cm means

Now, According to question, we calculate the M .O. I about centroidal axis means about x – x and y- y axis.

*[**Notes** : for calculating M.O.I about centroidal axis ,using parallel axis theorem as]*

$$(I_{XX}) = \sum_{i=1}^{n}(IX)_i + A_i\,(Y_i - \bar{y})^2$$

(I_{XX}) = M. o. I. about centroidal axis of whole figure

(Ix_i) = M.o.I about own centroidal axis of Individual figure

Y_i = centroidal y distance from assume reference x axis of individual figure.

$\bar{y}$ = centroidal y distance from reference x axis of whole figure.

$$I_{xx} = [(I_x)_1 + A1(y_1 - \bar{y})^2] + [(I_x)_2 + A_2(y_2 - \bar{y})^2]$$

$$= \left[\frac{b_1 h_1^3}{12} + (b1 \times h1)(Y_1 - \bar{y})^2\right] + \left[\frac{b_2 h_2^3}{36} + (1/2 \times b_2 \times h_2) \times (Y_2 - \bar{y})^2\right]$$

$$I_{xx} = \left[\frac{100 \times 50^3}{12} + 500(25 - 38.89)^2\right] + \left[\frac{100 \times 50^3}{36} + 2500(66.67 - 38.89)^2\right]$$

$$= 2006327.167 + 2276543.22$$

$$= 4282870.38 cm^4$$

$$I_{yy} = [(I_{y1}) + A_1(x_1 - \bar{x})^2] + [(I_y)_2 + A_2 \times (x_2 - \bar{x})^2]$$

$$= \left[\frac{h_1 \times b_1^3}{12} + A_1(x_1 - \bar{x})^2\right] + \left[\frac{h_2 \times b_2^3}{48} + A_2(x_2 - \bar{x})^2\right]$$

$$= \left[\frac{50 \times 100^3}{12} + 5000(50-50)^2\right] + \left[\frac{50 \times 1003}{48} + 2500 \times (50-50)^2\right]$$

$$= 5555555.556 cm^4$$

($X_1 = x_2 = \bar{x}$) means the centroidal axis of whole figure lies in centroid of individual figure so distance between axis is 0 (i.e. h = 0)

References

[1] Kumar, D. (2019). *Engineering Mechanics.* New delhi: S.K Kataria and Sons.

[2] Neupane, P. a. (2024). *A Text book of Engineering Mechanics.* Bhotahity Kathmandu: Heritage Publisher and Distributors PVT .LTD.

Thank You!!!