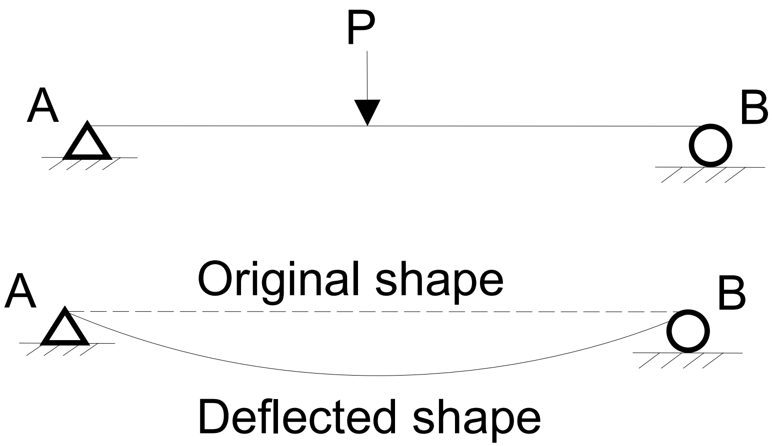
Theory of Structures - I

Chapter 2. Analysis by Strain Energy Method

Lecturer: Dr. Sanjeema Bajracharya

Contents

2.1 Strain energy and complementary strain energy

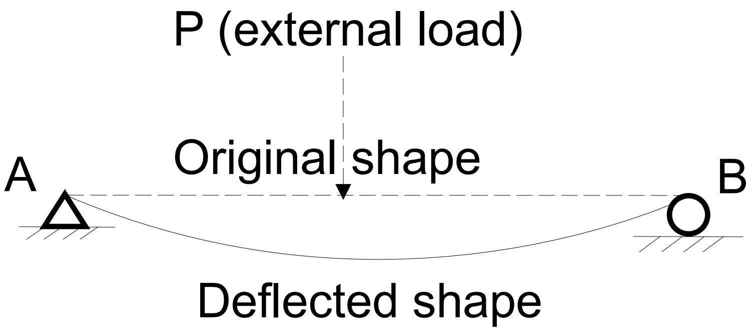


When external load acts on a structure, the structure undergoes deformation. Hence, ***external work*** is done.

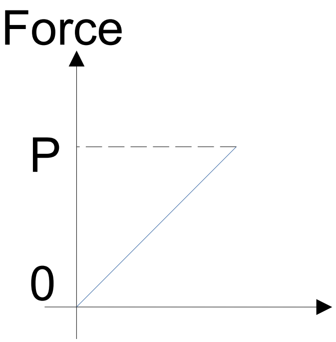
External Work Done (EWD) = External load *deformation

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

2.1 Strain energy and complementary strain energy



To resist these external forces, internal forces develop in the structure gradually from zero to their final value.

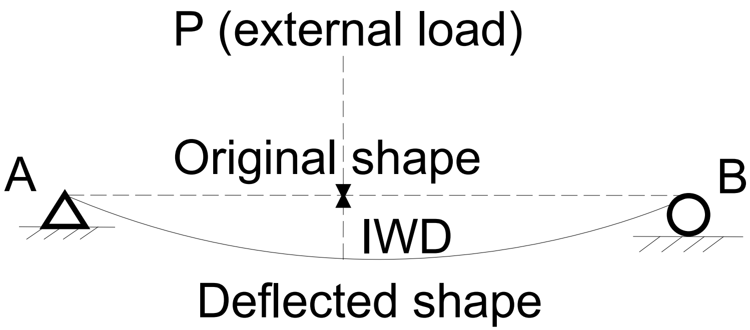


Hence, Internal Work is done.

This Internal Work Done (IWD) is stored as *energy* in the structure.

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

2.1 Strain energy and complementary strain energy



This internal work done helps the structure to spring back to its original shape and size, when these external loads are removed.

Definition: The internal work done, which is stored as energy, due to the straining of the material, is called **strain energy**.

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

2.1 Strain energy and complementary strain energy

From law of conservation of energy, work done by the external forces must equal the strain energy stored, i.e.

External Work Done (EWD) = Internal Work Done (IWD)
= Strain Energy

This concept of energy balance is utilized in the structural analysis to develop a number of methods to find the deflection of structures.

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

2.1 Strain energy and complementary strain energy

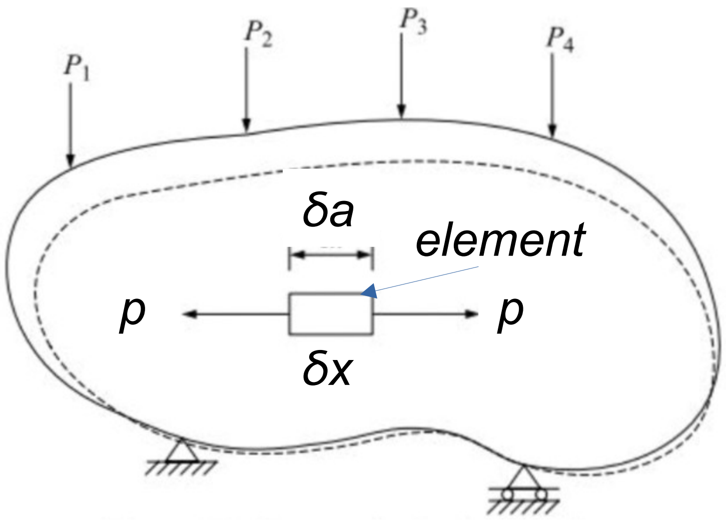


Fig. A general structural system

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

Let us consider a general structural system provided with supports and a system of external forces P_i acting on it.

Let us take a small element with cross-sectional area *δa* and length *δx*.

2.1 Strain energy and complementary strain energy

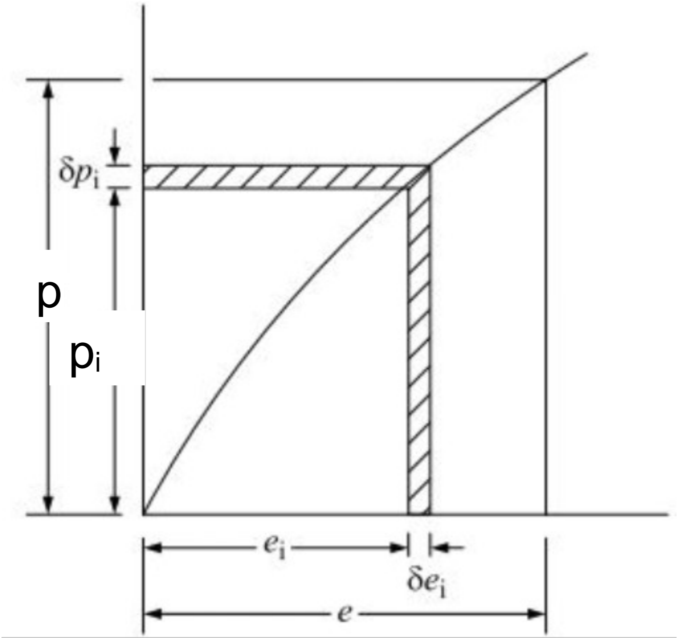


Fig. Stress-strain relation

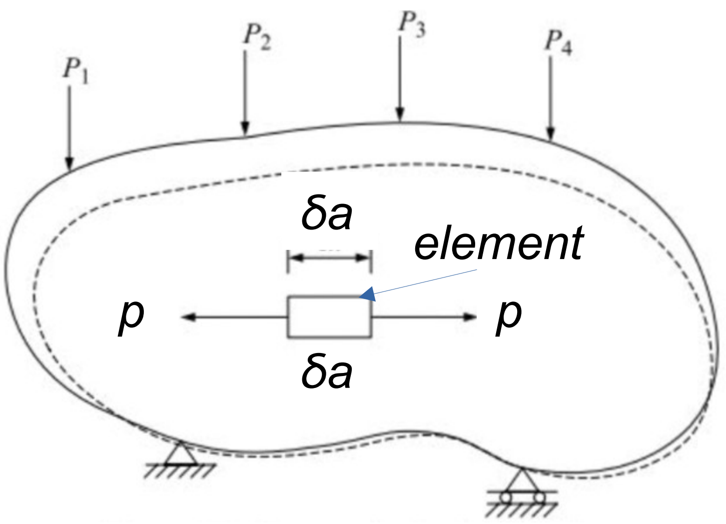


Fig 1. A general structural system

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

Let the stress in the element gradually increases from zero to its final value p as strain increases from zero to its final value e.

2.1 Strain energy and complementary strain energy

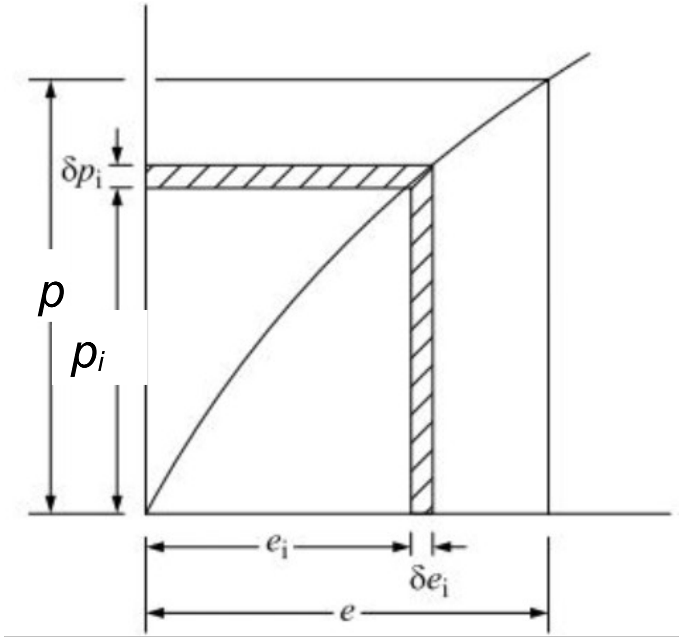


Fig. Stress-strain relation

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

Let the stress at any instant be p_i. When the strain δe_i takes place,

Work done on the element
= Force * Displacement
= $(p_i * \delta a) * (\delta e_i * \delta x)$

[Force = Stress * Area;
Displacement = Strain * Original length]

Here, $\delta a * \delta x = \delta v$,
Where,
v = the volume of the element

2.1 Strain energy and complementary strain energy

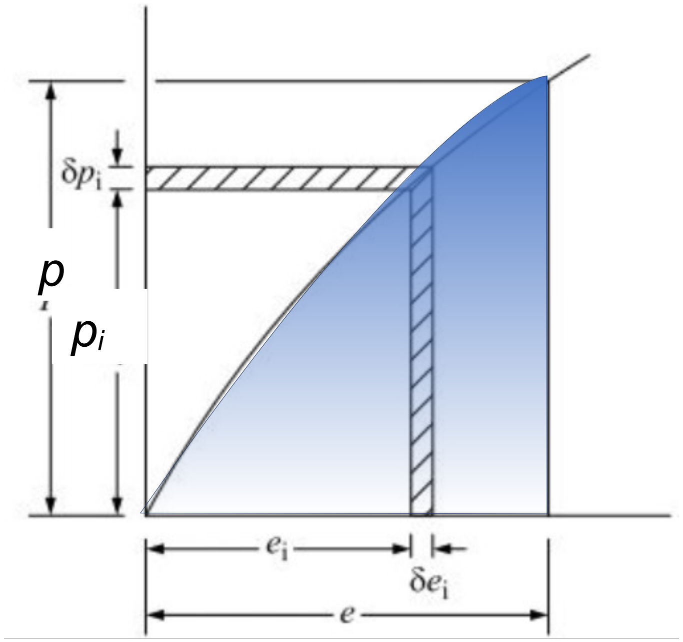


Fig. Stress-strain relation

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

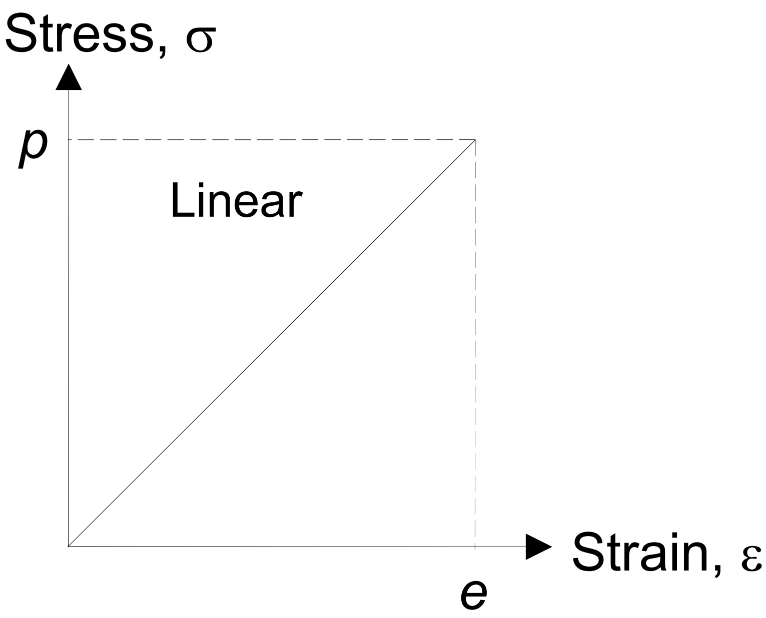
Work done on the element
$= (p_i * \delta a) * (\delta e_i * \delta x)$
$= p_i * \delta e_i * \delta v$

Strain Energy stored in the element = **Internal Work Done** =

$$\int_0^e p_i * \delta e_i * \delta v$$

= Area under the stress-strain curve * *dv*

2.1 Strain energy and complementary strain energy



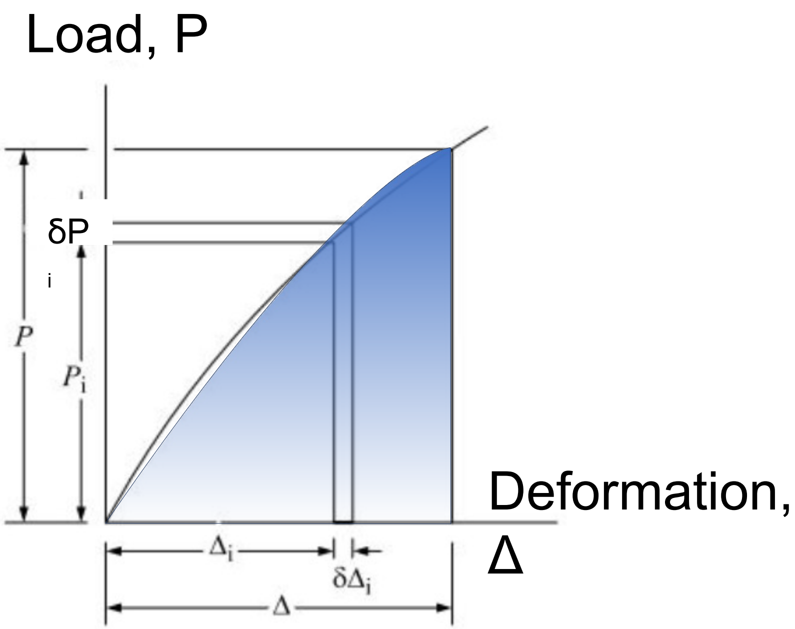
If the stress-strain curve is linear, then the **Strain Energy** stored in the element is:

$$U = \int \frac{1}{2} pedv$$

$$U = \int \frac{1}{2} stress * strain * dv$$

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

2.1 Strain energy and complementary strain energy



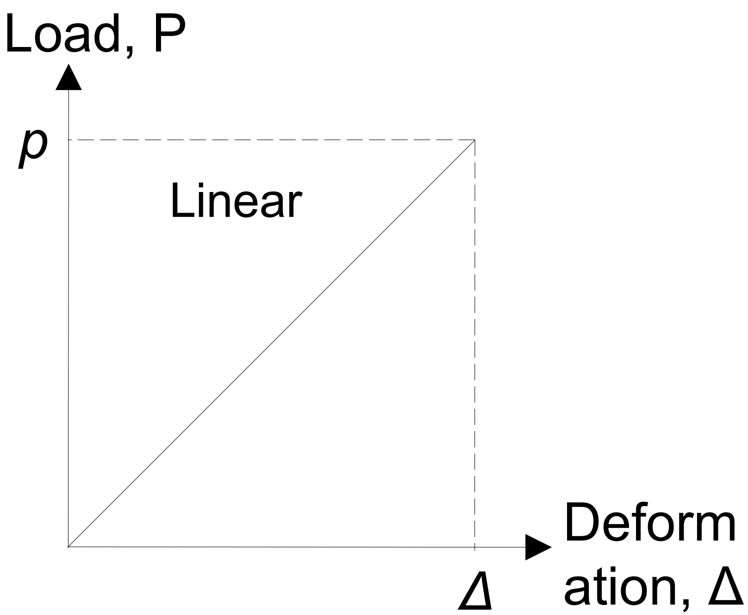
Let the deformation due to load P_i acting at any instant be $\delta\Delta_i$. Then,

Work done by external load (on the whole structure) = $\int P_i \delta\Delta_i$

= Area under the load- deformation curve

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

2.1 Strain energy and complementary strain energy



Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

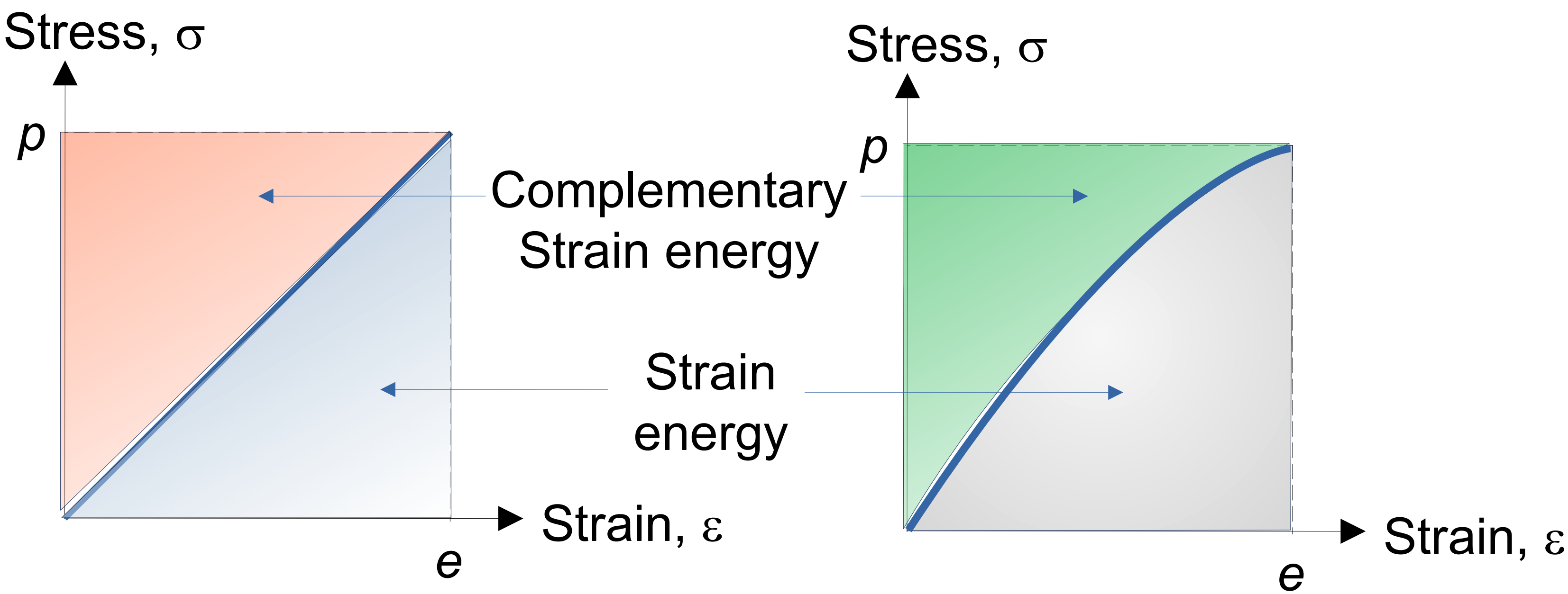
For linear elastic problems,

External Work Done = ½ *P*Δ

If there are 'n' no. of loads, total work done by external loads is the summation of work done by each load, i.e.

Total External Work Done by 'n' no. of loads = Σ½ *P*Δ

2.1 Strain energy and complementary strain energy



Linear Elastic material

Non-Linear Elastic material

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

2.1 Strain energy and complementary strain energy

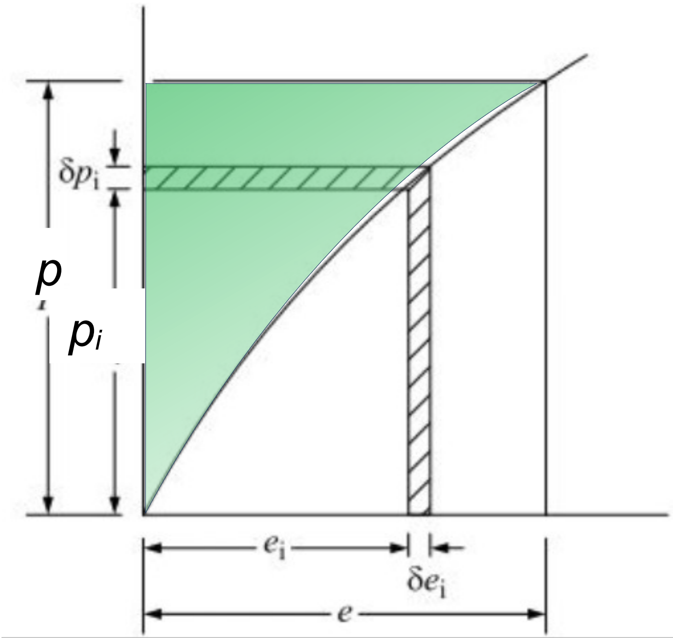


Fig. Stress-strain relation

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

Complementary strain energy at any instant during deformation of the element

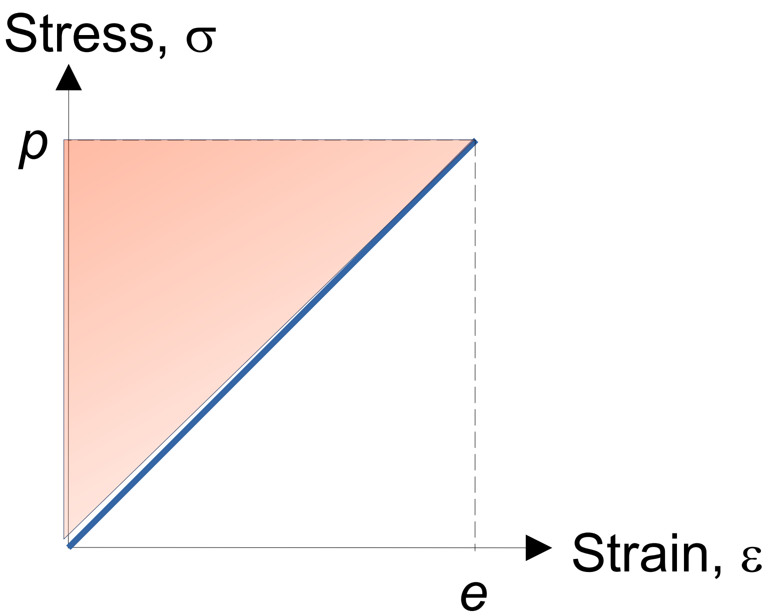
$= e_i * \delta p_i * \delta v$

Complementary strain energy of the element when final deformation takes place =

$$\int_0^p e_i * \delta p_i * \delta v$$

= Area above the stress-strain curve * *dv*

2.1 Strain energy and complementary strain energy



For linear-elastic problems,

Complementary strain energy of the element when final deformation takes place
= ½ * p*e*dv

Complementary strain energy of the entire structure,

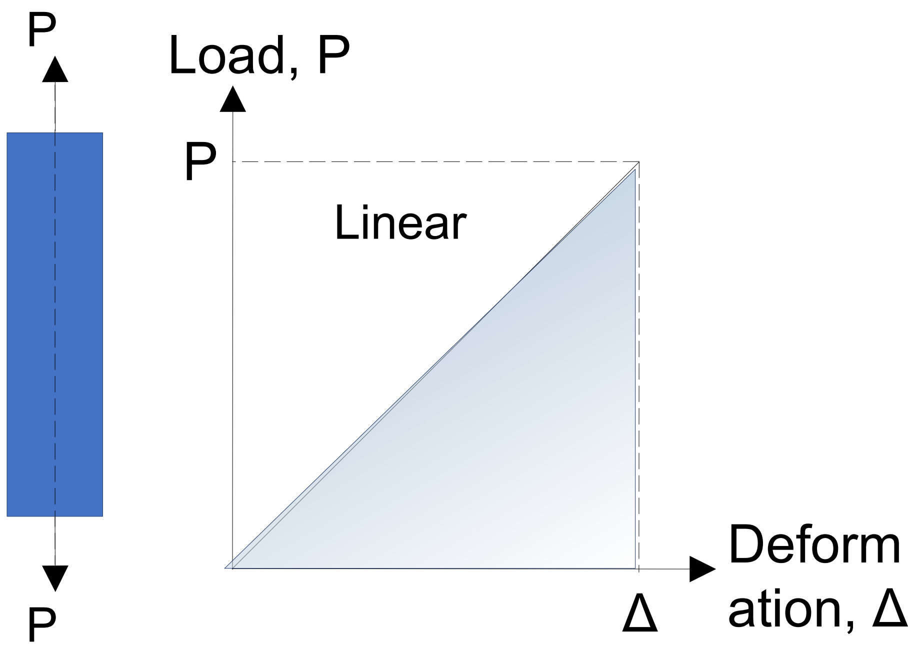
$$U_c = \int \frac{1}{2} pedv$$

For linear-elastic system,

$$U = U_c$$

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

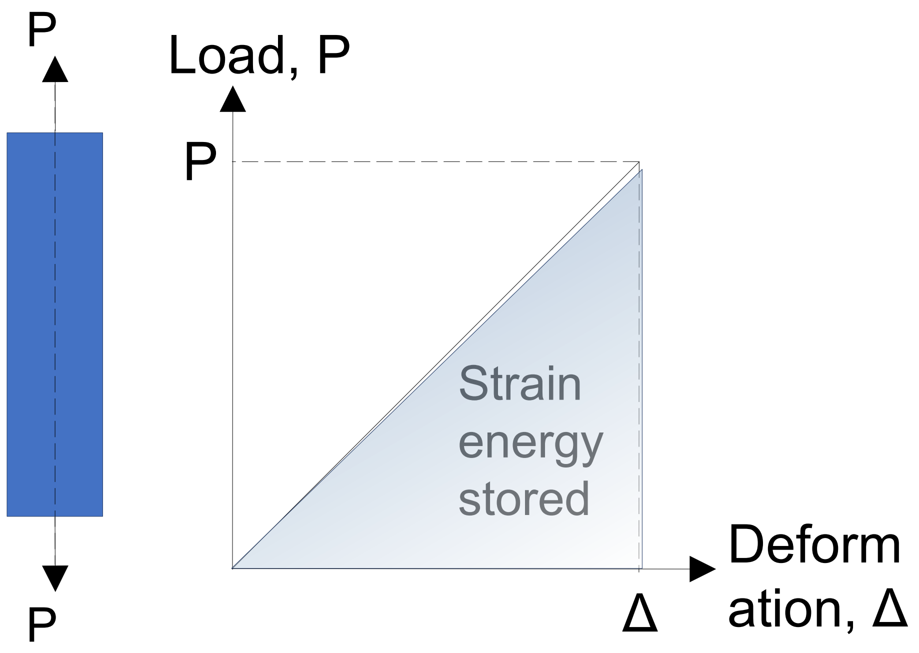
2.2 Strain energy due to gradually and suddenly applied direct load: Dynamic Multipliers



Let us consider an example of a gradually applied load: A tensile load, P, is being applied along the center-line of a prismatic cross-section such that value of load gradually increases from zero to its final value P.

Source: Thapa, K.B. (2019). *Theory of Structures-I (Determinate).* Kathmandu: Pratibha Pustak Sadan & Stationary

2.2 Strain energy due to gradually and suddenly applied direct load: Dynamic Multipliers



Strain energy,
U = Area under the curve
= ½ * P *Δ

From Strength of Materials,

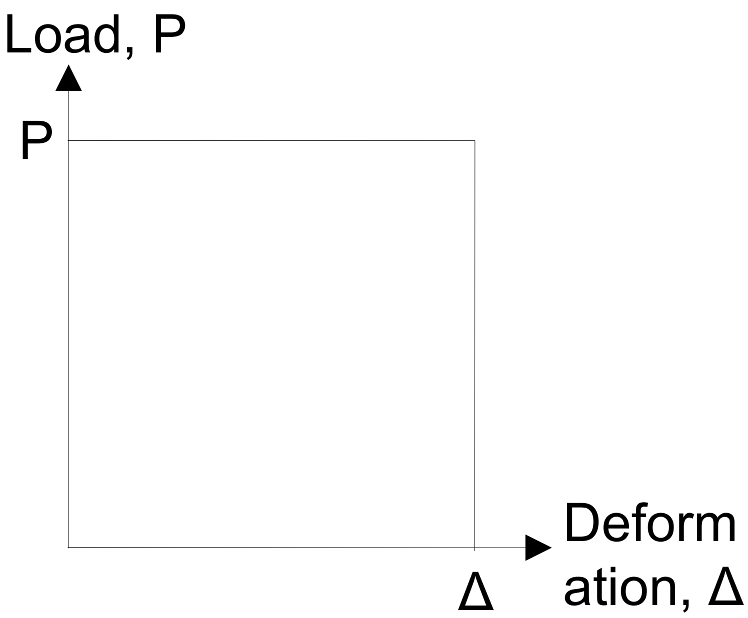
$$\Delta = \frac{PL}{AE} \quad \text{and} \quad \sigma = \frac{P}{A}$$

Then,

$$U = \frac{P^2 L}{AE} = \frac{\sigma^2 V}{2E}$$

Source: Thapa, K.B. (2019). *Theory of Structures-I (Determinate).* Kathmandu: Pratibha Pustak Sadan & Stationary

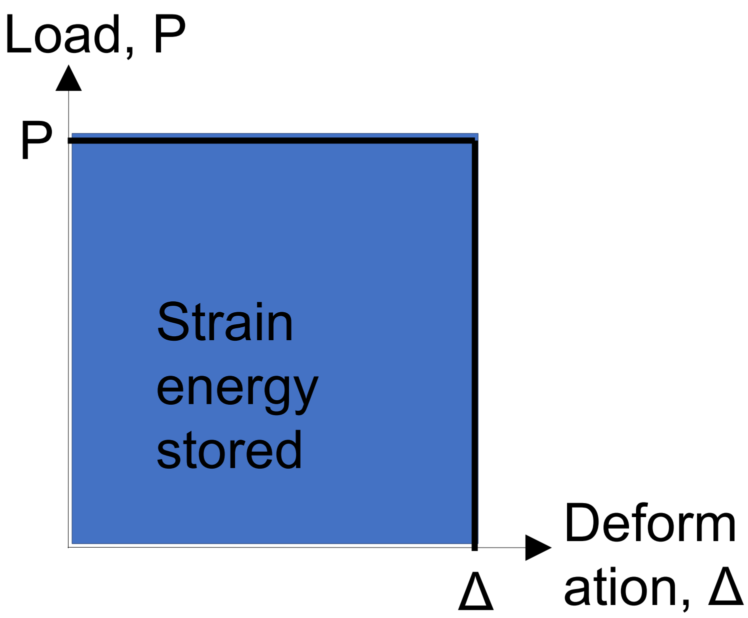
2.2 Strain energy due to gradually and suddenly applied direct load: Dynamic Multipliers



Let us consider an example of a suddenly applied load: An impact load of P acts such that the load-deformation curve as shown in the adjacent figure is obtained.

Source: Thapa, K.B. (2019). *Theory of Structures-I (Determinate).* Kathmandu: Pratibha Pustak Sadan & Stationary

2.2 Strain energy due to gradually and suddenly applied direct load: Dynamic Multipliers



Strain energy stored
= Area under the curve
= P*Δ

Then,

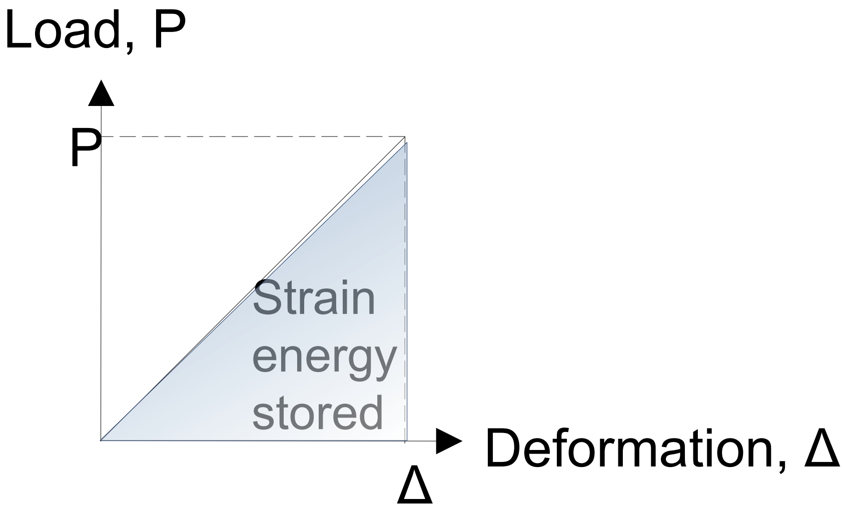
$$\frac{\sigma'^2 V}{2E} = \frac{P * PL}{AE}$$

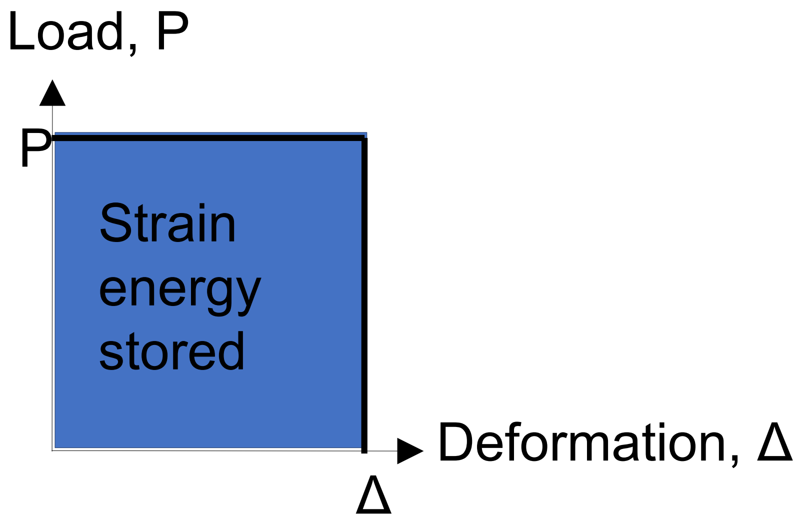
$$\frac{\sigma'^2 AL}{2E} = \frac{P * \sigma L}{E}$$

$$\sigma' = \frac{2 * P}{A}$$

Source: Thapa, K.B. (2019). *Theory of Structures-I (Determinate).* Kathmandu: Pratibha Pustak Sadan & Stationary

2.2 Strain energy due to gradually and suddenly applied direct load: Dynamic Multipliers





$$\sigma = \frac{P}{A}$$

$$U = \frac{\sigma^2 V}{2E}$$

$$\sigma' = \frac{2 * P}{A} = 2 * \sigma$$

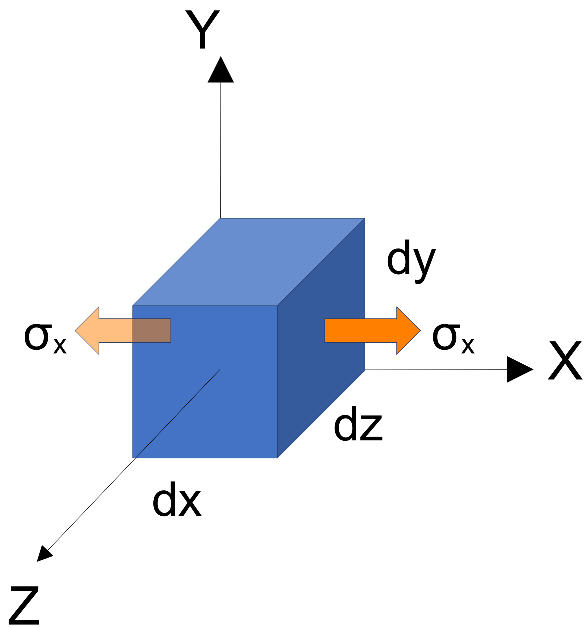
$$U' = \frac{\sigma'^2 V}{2E} = \frac{(2\sigma)^2 V}{2E} = \frac{4\sigma^2 V}{2E}$$

$$U' = 4U$$

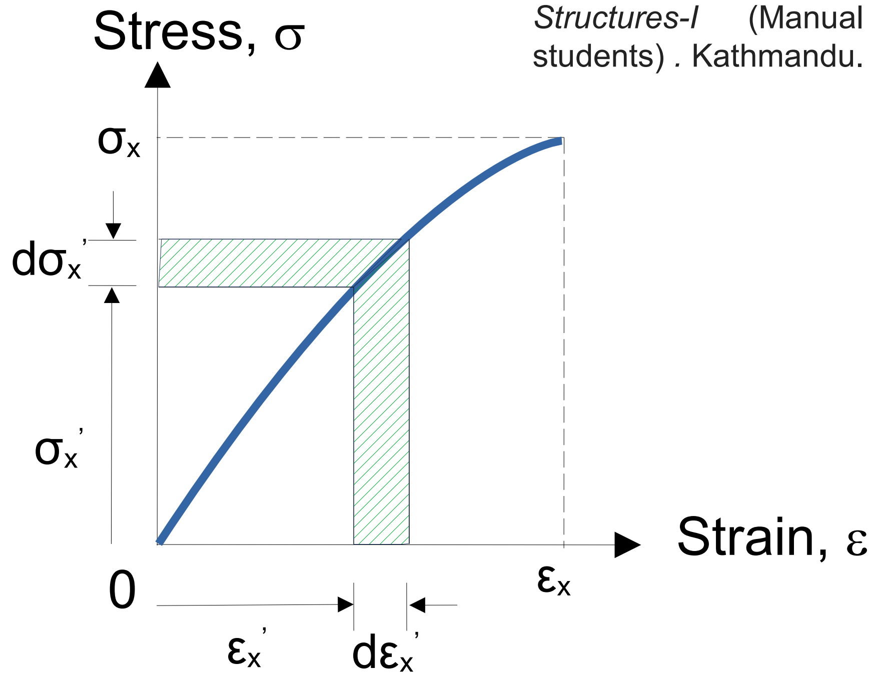
Here, 4 = Dynamic Multiplier

Source: Thapa, K.B. (2019). *Theory of Structures-I (Determinate).* Kathmandu: Pratibha Pustak Sadan & Stationary

2.3 [A] Strain energy due to axial stress



Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.

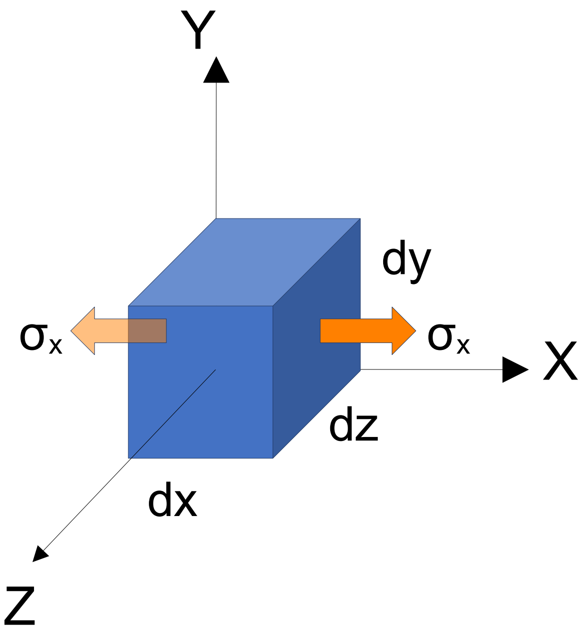


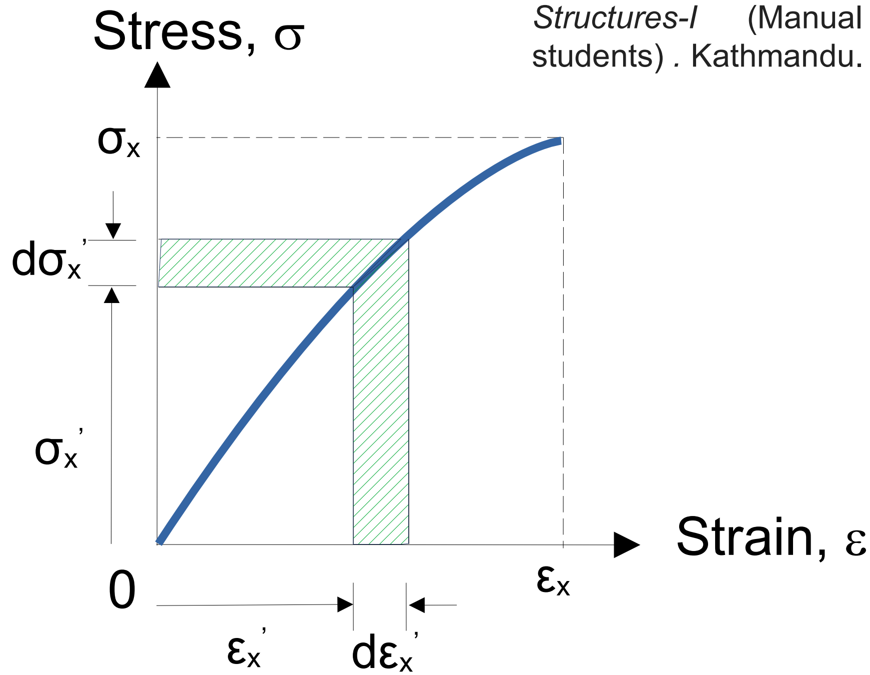
Let us consider an infinitesimal element as shown in the left figure, acted upon by normal stress, σ_x.

- Stress gradually increases from zero to its final value.
- Corresponding strain also undergoes a change from zero to its final value.

2.3 [A] Strain energy due to axial stress

Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.





Work done by external axial force on the element
= Force * Displacement
= $(\sigma_x{}'*dy*dz)*(d\varepsilon_x{}'*dx)$
= $\sigma_x{}'*\, d\varepsilon_x{}'*dV$
Where,
dV= dx*dy*dz = volume of the element

2.3 [A] Strain energy due to axial stress

Work done by external axial force on the element
= $\sigma_x{}'* d\varepsilon_x{}'*dV$

Where,
dV= dx*dy*dz = volume of the element

For linear stress-strain curve,
= $E*\varepsilon_x{}'*d\varepsilon_x{}'*dV$

Where,
$\sigma = E\varepsilon$

Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.

Strain Energy stored in the element,

$$dU = \int_0^{\varepsilon_x} E * \varepsilon_x{}' * d\varepsilon_x{}' * dV$$

$$dU = E\,dV \left.\frac{\varepsilon_x{}'^2}{2}\right|_0^{\varepsilon_x} = E\frac{\varepsilon_x^2}{2}dV = E\frac{\sigma_x^2}{2E^2}dV$$

Since,
$\sigma = E\varepsilon$

$$dU_a = \frac{\sigma_x^2}{2E}dV$$

2.3 [A] Strain energy due to axial stress

Strain Energy stored in the element,

$$dU_a = \frac{\sigma_x^2}{2E} dV$$

Total Strain Energy stored in the member,

$$U_a = \int_V \frac{\sigma_x^2}{2E} dV$$

Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.

For prismatic member (cross-section is constant),

$$U_a = \int_V \frac{\sigma^2}{2E} dV = \boxed{\int_0^L \frac{\sigma^2}{2E} A\, dx} = \frac{\sigma^2 AL}{2E}$$

$$U_a = \frac{P^2 L}{2AE} \qquad \text{[Since, } \sigma = \frac{P}{A} \text{]}$$

2.3 [B] Strain energy due to bending stress

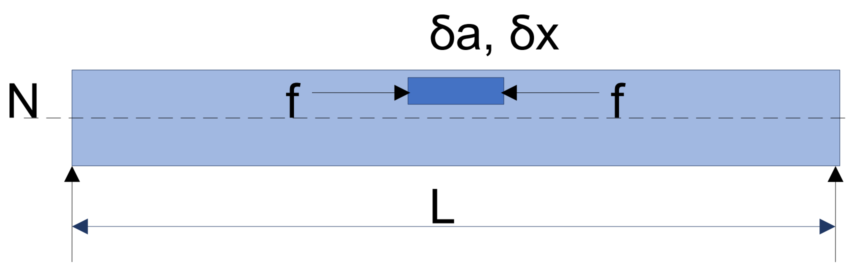


Fig. A beam under pure moment

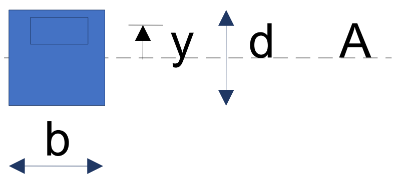


Fig. Cross-section of the beam

Let us consider a beam subjected to pure bending. Let us take an area of the element δa, its length δx, and its distance from the neutral axis y.

From flexure theorem,

Bending stress f is given by: $f = \frac{M}{I} * y$

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

Where,

M is the bending moment

I is the Moment of Inertia of the section

2.3 [B] Strain energy due to bending stress

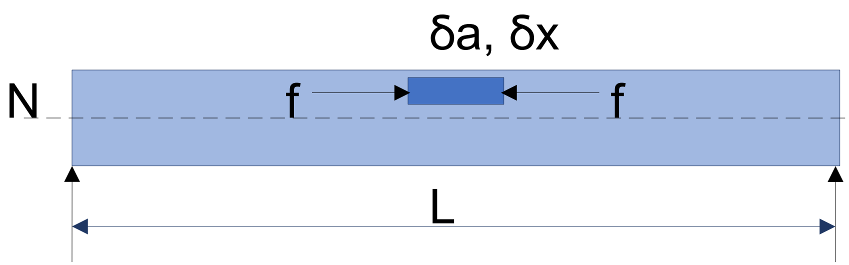


Fig. A beam under pure moment

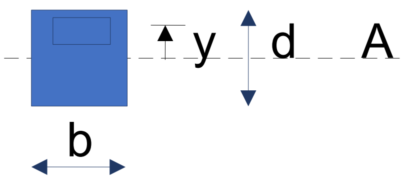


Fig. Cross-section of the beam

Bending stress, $f = \frac{M}{I} * y$

Strain, $e = \frac{f}{E} = \frac{M}{EI} * y$; E = Young's Modulus

Then,

Strain energy stored in the element

= ½ * stress*strain*dv

= $\frac{1}{2} * \frac{M}{I} y * \frac{M}{EI} y * dv$

= $\frac{1}{2} * \frac{M^2 * y^2}{EI^2} dv$

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

2.3 [B] Strain energy due to bending stress

Strain energy on the beam, U_b

$$= \int \frac{1}{2} * \frac{M^2 * y^2}{EI^2} dv$$

$$= \int_0^L \int_0^A \frac{1}{2} * \frac{M^2 * y^2}{EI^2} \delta a\, dx$$

$$= \int_0^L \frac{M^2}{2\,EI^2} \int_0^A y^2 \delta a\, dx$$

$$= \int_0^L \frac{M^2}{2\,EI^2} I\, dx$$

$$= \int_0^L \frac{M^2}{2\,EI} dx$$

$$U_b = \int_0^L \frac{M^2}{2\,EI} dx$$

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

2.3 [C] Strain energy due to shear stress

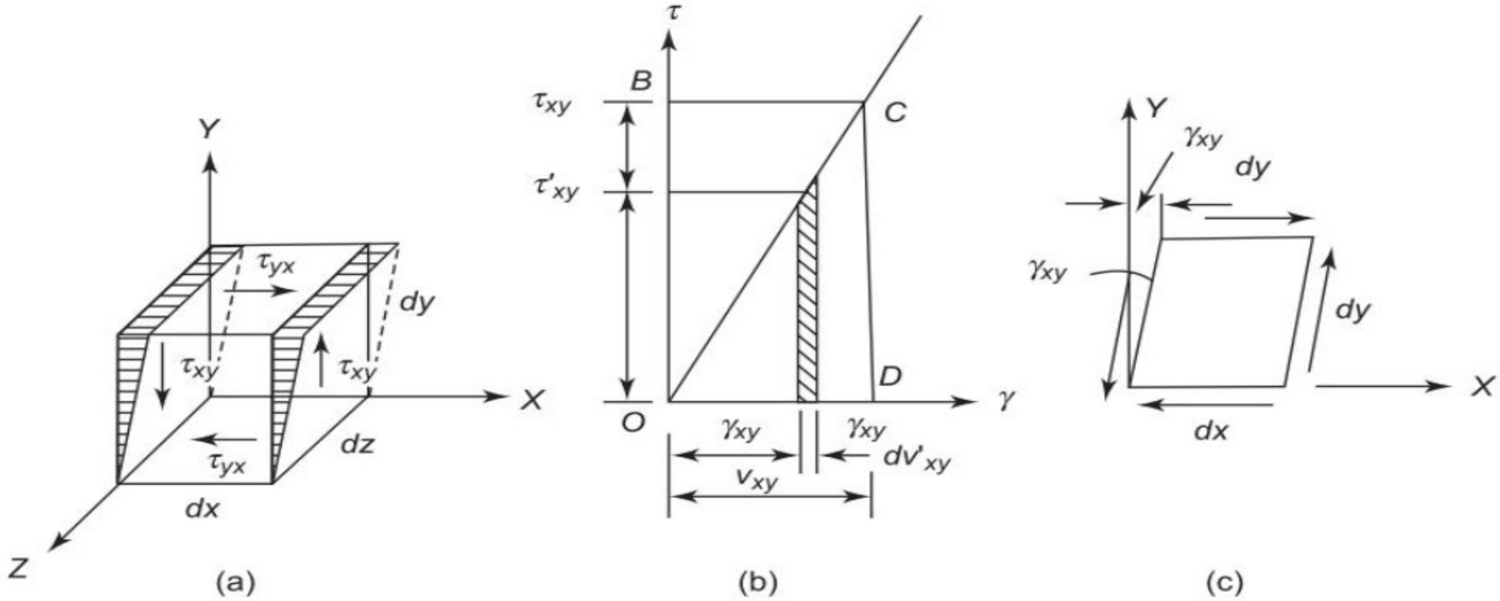


Fig. (a) Distortion of an element under pure shear stress, (b) Shear stress and shear strain relationship, and (c) Shear strains in XY plane

Let us consider an infinitesimal element under pure shear, whose bottom plane is fixed in the XZ plane, and gradual application of shear stress, $\tau_{xy} = \tau_{yx}$ will distort the element as shown in Fig (a)

Source: Reddy, C.S. (2011). *Basic Structural Analysis (3rd ed.).* New Delhi: Tata McGraw Hill..

2.3 [C] Strain energy due to shear stress

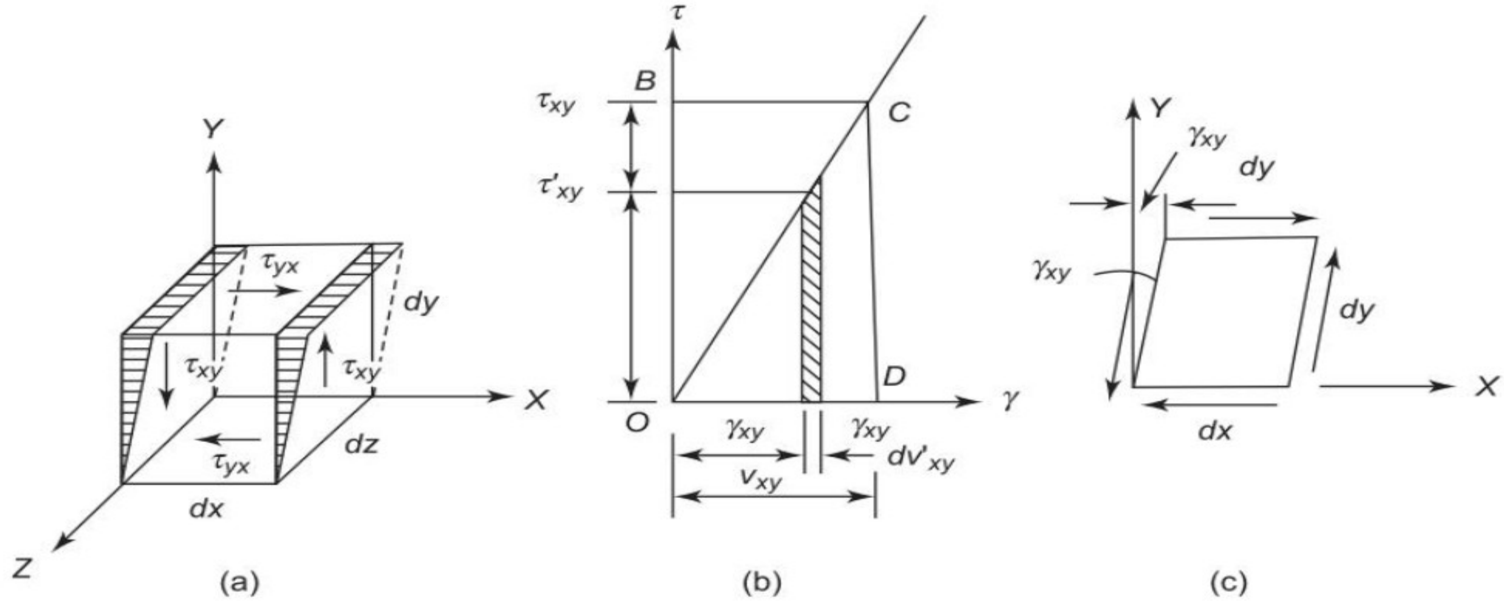


Fig. (a) Distortion of an element under pure shear stress, (b) Shear stress and shear strain relationship, and (c) Shear strains in XY plane

For infinitesimal displacement $d\gamma_{xy}'$ during deformation,

Work done = Force * Displacement

$= (\tau_{xy}' * dx * dz) * (d\gamma_{xy}' * dy)$

$= \tau_{xy}' * d\gamma_{xy}' * dV$

Where, dV is the volume of the element = dx*dy*dz

Source: Reddy, C.S. (2011). *Basic Structural Analysis (3rd ed.).* New Delhi: Tata McGraw Hill..

2.3 [C] Strain energy due to shear stress

For infinitesimal displacement $d\gamma_{xy}'$ during deformation,
Work done $= \tau_{xy}' * d\gamma_{xy}' * dV$

Then,
Strain energy stored in the element, dU_{sh}

$$= \int_0^{\gamma_{xy}} \tau_{xy}' * d\gamma_{xy}' * dV$$

$$= \int_0^{\gamma_{xy}} G\gamma_{xy}' * d\gamma_{xy}' * dV$$

From Hooke's law in Torsion, $\tau = G*\gamma$

$$= G\,dV \left.\frac{\gamma_{xy}'^2}{2}\right|_0^{\gamma_{xy}}$$

$$= G\,dV \frac{\gamma_{xy}^2}{2} = G\,dV \frac{\tau_{xy}^2}{2G^2}$$

$$= \frac{\tau_{xy}^2}{2G} dV$$

Source: Reddy, C.S. (2011). *Basic Structural Analysis (3rd ed.).* New Delhi: Tata McGraw Hill..

2.3 [C] Strain energy due to shear stress

Strain energy stored in the element, dU_{sh}

$$= \frac{\tau_{xy}^2}{2G} dV$$

Total strain energy stored due to shear,

$$U_{sh} = \int_V \frac{\tau_{xy}^2}{2G} dV$$; Shear strain γ_{xy}

$$U_{sh} = \int_V \frac{\tau_{zx}^2}{2G} dV$$; Shear strain γ_{zx}

$$U_{sh} = \int_V \frac{\tau_{yz}^2}{2G} dV$$; Shear strain γ_{yz}

Source: Reddy, C.S. (2011). *Basic Structural Analysis (3rd ed.).* New Delhi: Tata McGraw Hill..

2.3 [C] Strain energy due to shear stress

For prismatic member,

$$U_{sh} = \int_0^L \frac{\tau^2}{2G} A\, dx$$

Source: Reddy, C.S. (2011). *Basic Structural Analysis (3rd ed.).* New Delhi: Tata McGraw Hill..

We have,

$$\tau = k \frac{V}{A}$$

Where,
V = Shear force
A = Cross-sectional area of prismatic member

Substituting, we get

$$U_{sh} = \int_0^L \frac{KV^2}{2AG} dx$$

Where,
K = 1.2 for rectangular section
K = 10/9 for circular section

2.3 [D] Strain energy due to torsion

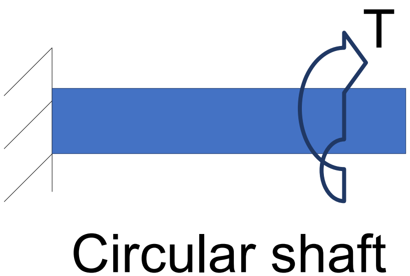


Fig (a)

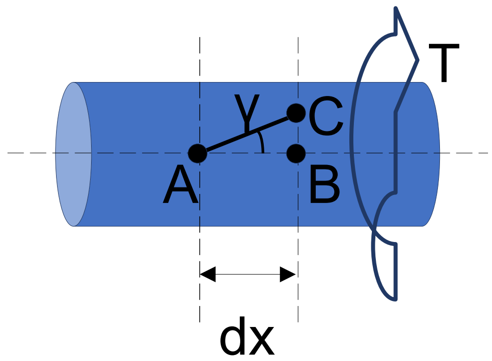


Fig (b)

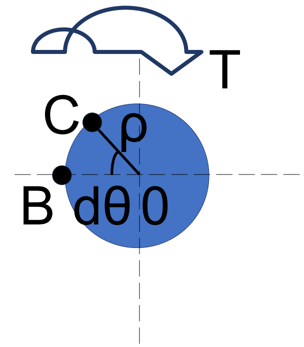


Fig (c)

Let us consider a circular shaft on which a torsional force T is acting at its free end (Fig a). Let us take a small length dx along the shaft, where due to T, point B reached point C causing a shear strain of γ (Fig b). In the cross-section (Fig c), torsion T causes a rotation dθ such that point B moves to point C. ρ is the radius of the circular shaft.

Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.

2.3 [D] Strain energy due to torsion

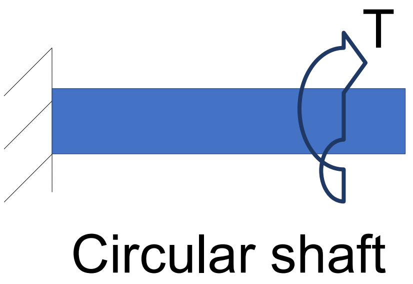


Fig (a)

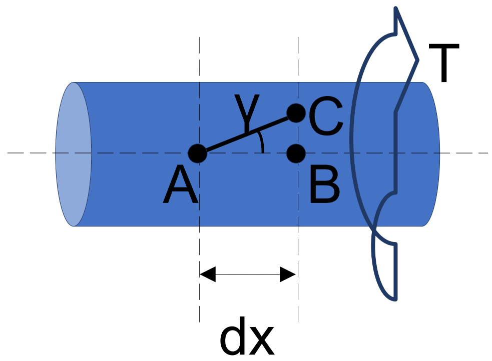


Fig (b)

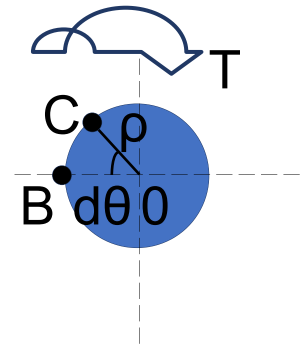


Fig (c)

External work done = ½ * Force * Deformation
= ½ * T * dθ

Shear Strain, $\gamma = \frac{BC}{AB} = \frac{\rho\, d\theta}{dx} - - - - - - - - (1)$

Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.

2.3 [D] Strain energy due to torsion

Shear Strain, $\gamma = \frac{BC}{AB} = \frac{\rho \, d\theta}{dx} - - - - - - - - (1)$

Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.

From equation of Torsion,

$$\frac{T}{J} = \frac{\tau}{\rho} \quad \Rightarrow \quad \tau = \frac{T\rho}{J}$$

Then,
Shear strain, $\gamma = \frac{\tau}{G} = \frac{T\rho / J}{G} - - - - - - - - (2)$

From equations (1) and (2),

$$\frac{\rho \, d\theta}{dx} = \frac{T\rho / J}{G} \quad \Rightarrow \quad d\theta = \frac{T}{GJ} dx - - - - - - - (3)$$

2.3 [D] Strain energy due to torsion

External work done $= \frac{1}{2} * T * d\theta$

$= \frac{1}{2} * T * \frac{T}{GJ} dx$

$= \frac{T^2}{2GJ} dx$

Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.

Total strain energy stored in the member due to Torsion is,

$$U_T = \int_0^L \frac{T^2}{2GJ} dx$$

-------------End of Chapter 2------------------

References

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[4] Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.