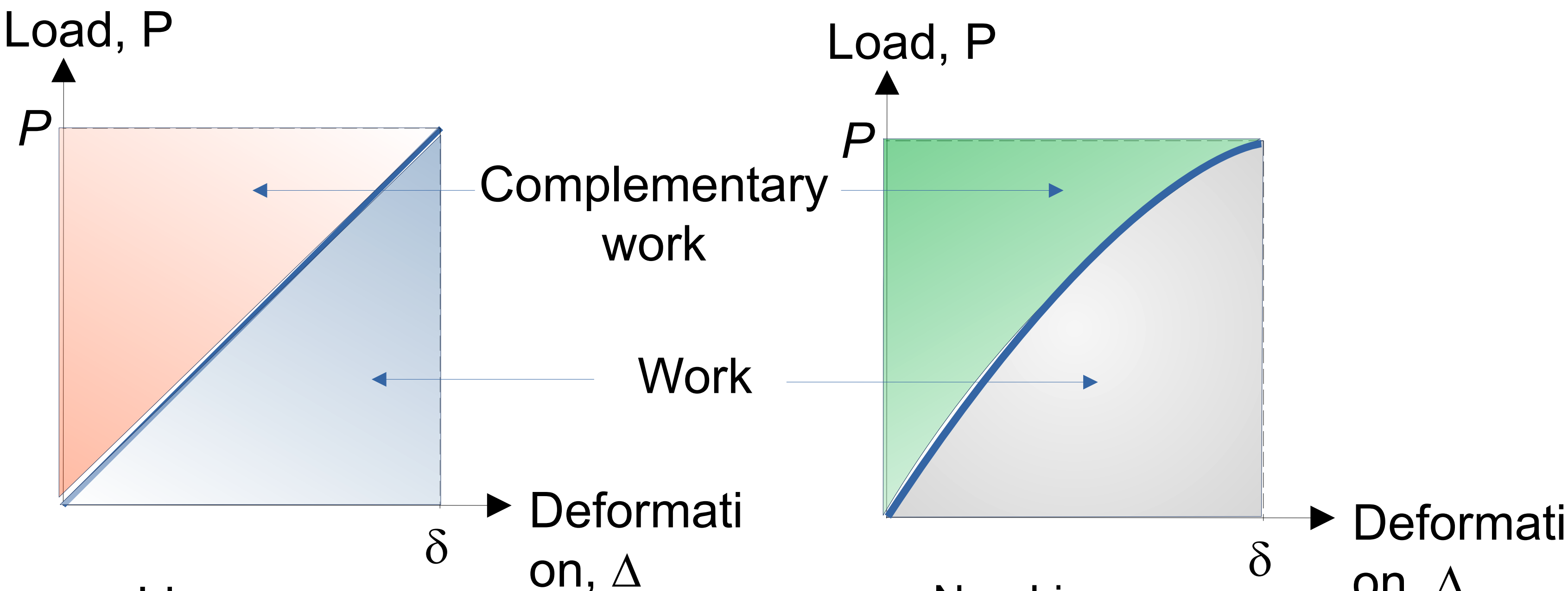
Theory of Structures - I

Chapter 3. Analysis by Virtual Work Method [Part I of II]

Lecturer: Dr. Sanjeema Bajracharya

Contents

3.1 Work and complementary work



Linear Elastic material

Non-Linear Elastic material

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

3.2 Displacement of beams and frames by real work method

From law of conservation of energy,

Internal Work Done (Strain Energy) = External Work Done

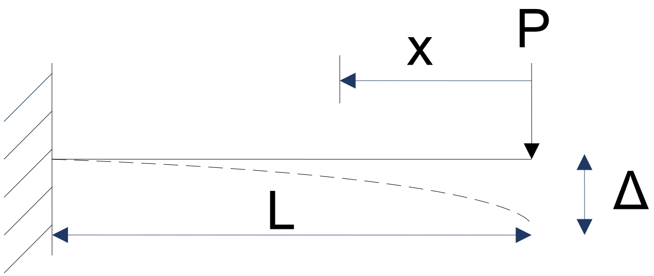
$$U = \sum_{0}^{n} \frac{1}{2} P \Delta$$

This equation is now used to find the deflection in beams and frames subject to axial, bending, shear, and torsional stress.

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

3.2 Displacement of beams and frames by real work method

Numerical#1. Using strain energy method (real work method), determine the deflection of the free end of a cantilever of length L subjected to a concentrated load P at the free end.



Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

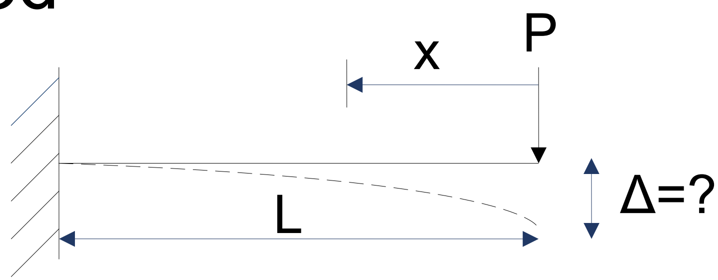
Solution:

Step 1: Calculate the bending moment at a distance ‘x’ from the free end

$M = P*x$

3.2 Displacement of beams and frames by real work method

Numerical#1.



Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

Solution:

Step 1: M = P*x

Step 2: Calculate the strain energy due to bending

$$U_b = \int_0^L \frac{M^2}{2EI} dx = \int_0^L \frac{(Px)^2}{2EI} dx$$

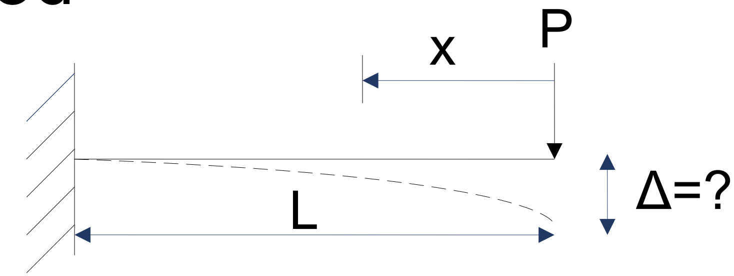
$$U_b = \frac{P^2}{2EI} \int_0^L (x)^2 dx = \frac{P^2}{2EI} \frac{x^3}{3} \Big|_0^L = \frac{P^2 L^3}{6EI} \; ---(1)$$

Step 3: Calculate the work done by load P

= ½ *P*Δ -------(2)

3.2 Displacement of beams and frames by real work method

Numerical#1.



Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

Solution:

Internal Work Done (Strain Energy) = External Work Done

Step 4: From equations (1) and (2),

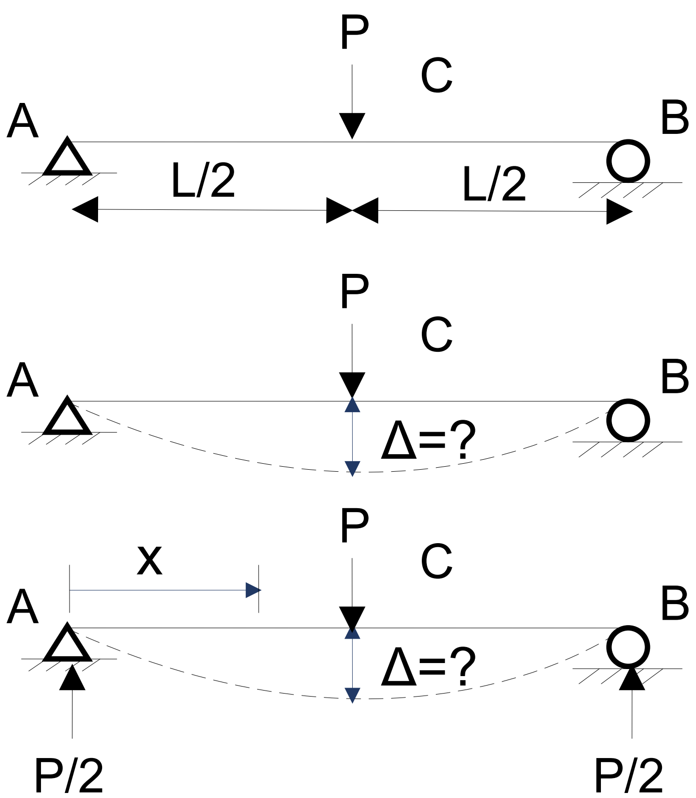
$$\frac{P^2 L^3}{6\,EI} = \frac{1}{2} P\Delta$$

$$\Delta = \frac{P\,L^3}{3\,EI}$$

Which is the required displacement at the free end of the cantilever beam.

3.2 Displacement of beams and frames by real work method

Numerical#2. Using real work method, determine the deflection of at the mid-span of a simply supported beam of length L, with a point load P acting at its mid-span.



Solution:

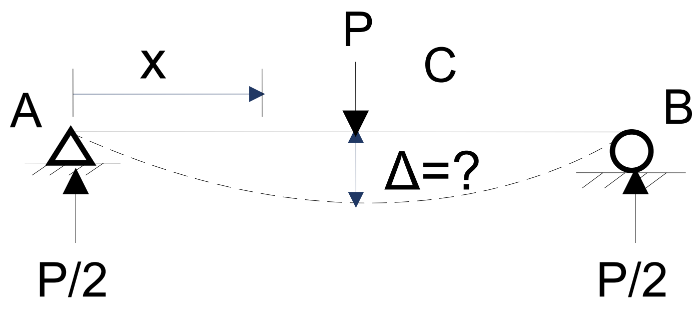
Step 1: Calculate the bending moment at a distance ‘x’ from the support A

$$M = P/2 * x$$

Source: Reddy, C.S. (2011). *Basic Structural Analysis (3rd ed.).* New Delhi: Tata McGraw Hill

3.2 Displacement of beams and frames by real work method

Numerical#2.



Solution:

Step 1: M = P/2*x

Step 2: Calculate the strain energy due to bending

$$U_b = 2\int_0^{\frac{L}{2}} \frac{M^2}{2EI}\,dx = 2\int_0^{\frac{L}{2}} \frac{\left(\frac{P}{2}x\right)^2}{2EI}\,dx$$

$$U_b = \frac{P^2}{4EI}\int_0^{L} (x)^2\,dx = \frac{P^2}{4EI}\,\frac{x^3}{3}\Big|_0^{\frac{L}{2}} = \frac{P^2L^3}{96EI} - - - (1)$$

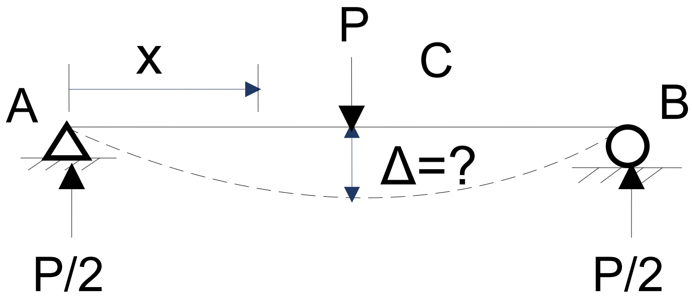
Step 3: Calculate the work done by load P

= ½ *P*Δ -------(2)

Source: Reddy, C.S. (2011). *Basic Structural Analysis (3rd ed.).* New Delhi: Tata McGraw Hill

3.2 Displacement of beams and frames by real work method

Numerical#2.



Solution:

Step 4: From equations (1) and (2),

$$\frac{P^2 L^3}{96\,EI} = \frac{1}{2} P\Delta$$

$$\Delta = \frac{P L^3}{48\,EI}$$

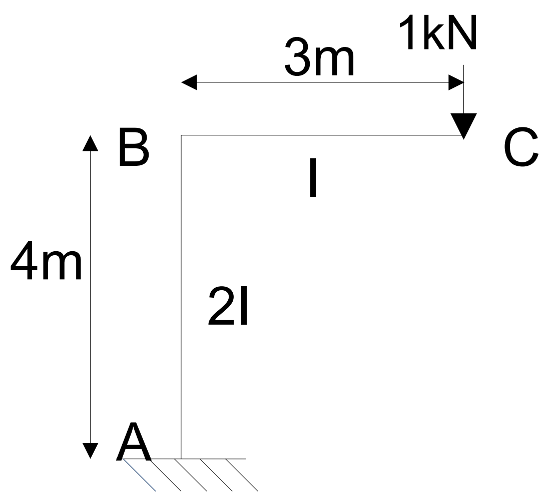
Which is the required displacement at the mid-span of the simply supported beam

Internal Work Done (Strain Energy) = External Work Done

Source: Reddy, C.S. (2011). *Basic Structural Analysis (3rd ed.).* New Delhi: Tata McGraw Hill

3.2 Displacement of beams and frames by real work method

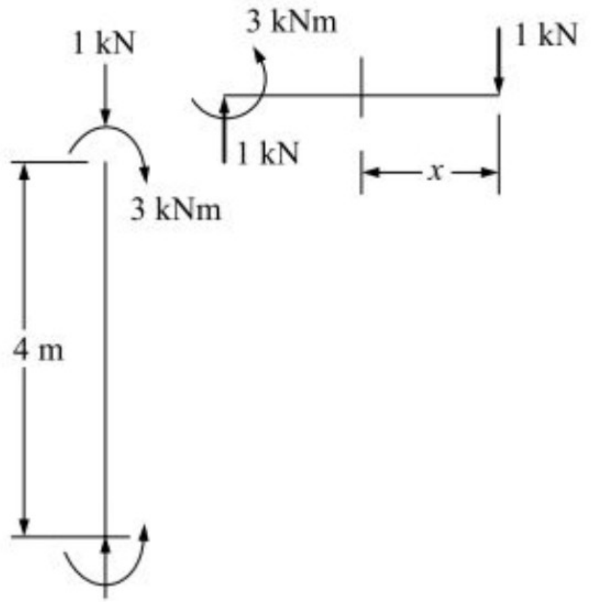
Numerical#3. Determine the vertical deflection of point C in the frame shown below. Take E = 200 kN/mm^2 and I = 30*10^6 mm^4.



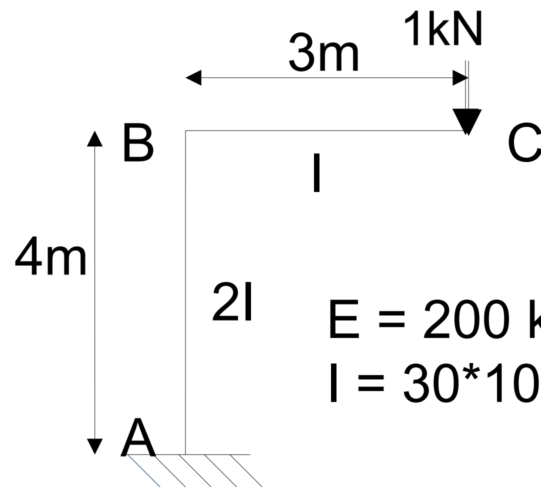
Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

Solution:

Step 1: Draw Free Body Diagram (FBD) of the frame



3.2 Displacement of beams and frames by real work method



$E = 200\ kN/mm^2$ and
$I = 30*10^6\ mm^4$

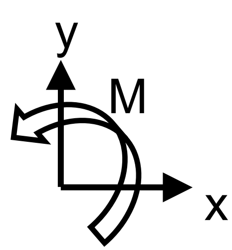
Numerical#3.

Solution:

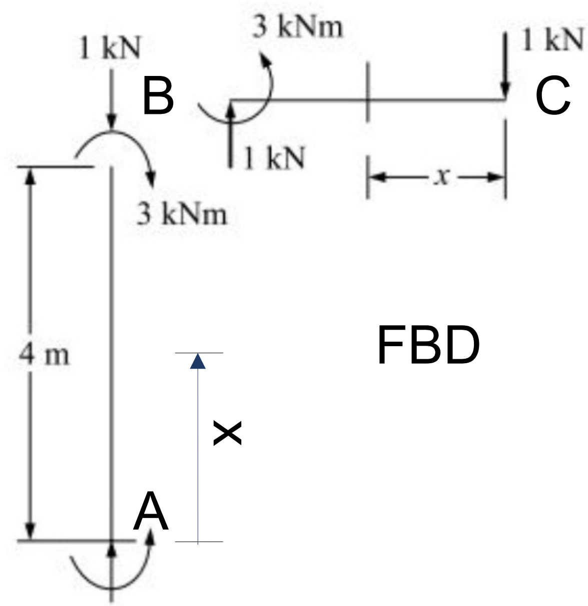
Step 2: Calculate bending moment for each portion of frame – AB and BC

Taking following positive sign conventions,

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

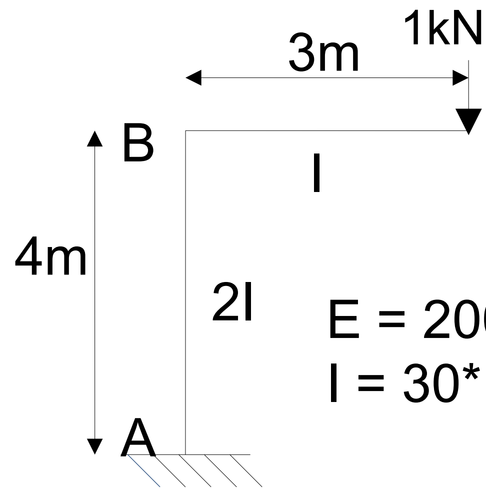


Portion	Origin	Limit	Bending Moment, M
AB	B	$0 \le x \le 4$	-3
BC	C	$0 \le x \le 3$	-1*x =-x



FBD

3.2 Displacement of beams and frames by real work method



E = 200 kN/mm^2 and
I = 30*10^6 mm^4

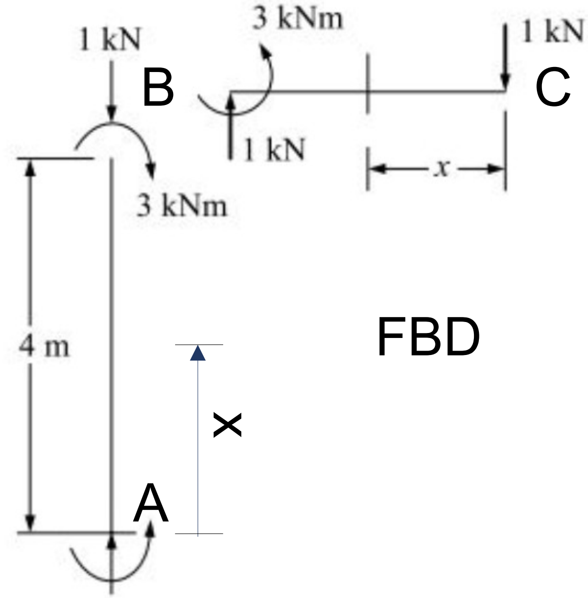
Numerical#3.

Solution:

Step 3: Calculate total strain energy stored in the frame

Portion	Origin	Limit	Bending Moment, M	I
AB	B	0 ≤ x ≤ 4	-3	2I
BC	C	0 ≤ x ≤ 3	-1*x =-x	I

$$Strain\,energy = \int_0^4 \frac{M_{AB}^2}{2E(2I)}dx + \int_0^3 \frac{M_{BC}^2}{2EI}dx$$



FBD

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

3.2 Displacement of beams and frames by real work method

Numerical#3.

Solution:

Step 3: Calculate total strain energy stored in the frame

Portion	Origin	Limit	Bending Moment, M	I
AB	B	0 ≤ x ≤ 4	-3	2I
BC	C	0 ≤ x ≤ 3	-1*x =-x	I

$$Strain\,energy=\int_0^4 \frac{M_{AB}^2}{2E(2I)}dx+\int_0^3 \frac{M_{BC}^2}{2EI}dx$$

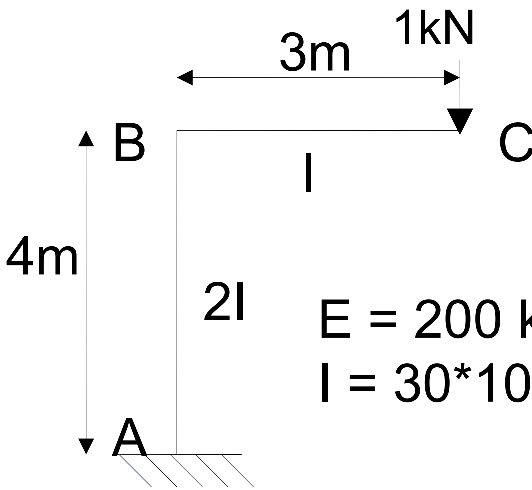
$$S.E.=\int_0^4 \frac{-3^2}{2E(2I)}dx+\int_0^3 \frac{-x^2}{2EI}dx=\frac{9}{4EI}\,x\Big|_0^4+\frac{1}{2EI}\,\frac{x^3}{3}\Big|_0^3$$

$$S.E.=\frac{9}{4EI}*4+\frac{1}{2EI}*\frac{3^3}{3}=\frac{13.5}{EI}----(1)$$

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

3.2 Displacement of beams and frames by real work method

Numerical#3.



E = 200 kN/mm^2 and
I = 30*10^6 mm^4

Solution:

Step 4: Calculate the work done by load 1 kN

= ½ *1*Δ -------(2)

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

Step 5: From equations (1) and (2),

$$\frac{13.5}{EI} = \frac{1}{2}\Delta$$

$$\Delta = \frac{27}{EI}$$

Which is the required deflection at point C of the given frame due to 1 kN load at C.

3.2 Displacement of beams and frames by real work method

Numerical#4. A square steel bar 40 mm * 40 mm in section, 3 m long is subjected to an axial pull of 128 kN. Taking E = 200 GN/m^2, calculate the elongation of the bar. Also, calculate the energy stored in the bar during extension.

Given:

F = 128 kN

A = 40*40 = 1600 mm^2

Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.

We have,

$\sigma = F/A = 128*10^3/1600 = 80 N/mm^2$

Also,

$\sigma = E\varepsilon$

Or, 80 N/mm^2 = $200*10^3$ N/mm^2 * ε

Or, $\varepsilon = 80/(200*10^9)$

3.2 Displacement of beams and frames by real work method

Numerical#4. A square steel bar 40 mm * 40 mm in section, 3 m long is subjected to an axial pull of 128 kN. Taking E = 200 GN/m^2, calculate the elongation of the bar. Also, calculate the energy stored in the bar during extension.

Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.

Solution:

$\varepsilon = 80/(200*10^3)$

Or, $\Delta L/L = 80/(200*10^3)$

Or, $\Delta L = 80/(200*10^3)*3*10^3$

Or, $\Delta L = 1.2$ mm

Which is the required elongation of the bar

Then,

$$U_a = \frac{\sigma^2 AL}{2E}$$

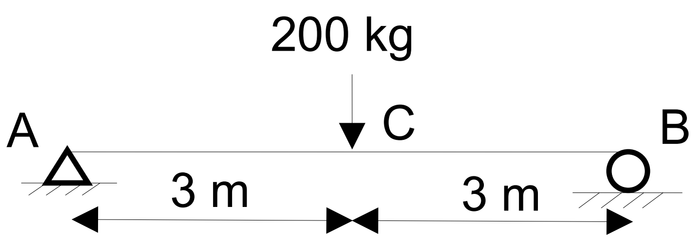
$$U_a = \frac{80^2 * 1600 * 3 * 10^3}{2 * 200 * 10^3}$$

$$U_a = 76.8\,N.m = 76.8\,J$$

Which is the required energy stored in the bar during extension.

3.2 Displacement of beams and frames by real work method

Numerical#5. A rectangular beam 20 cm * 40 cm is simply supported on a span of 6 m and carries a central load of 200 kg. Calculate the strain energy due to shear and bending. Neglect the self-weight of the beam. Take $E = 2*10^6$ kg/cm^2 and $G = 0.85*10^6$ kg/cm^2.



Given:
b= 20 cm
d = 40 cm
L = 6m
$E = 2*10^6$ kg/cm^2
$G = 0.85*10^6$ kg/cm^2

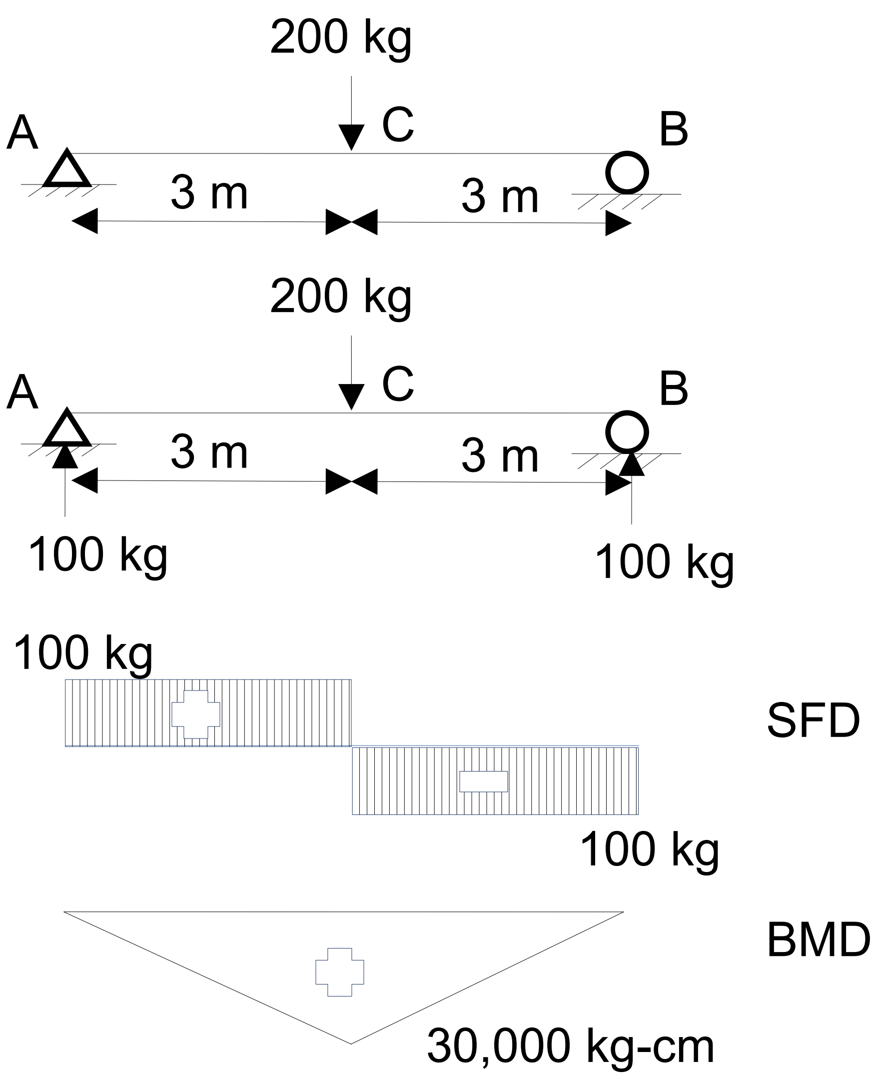
Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.

3.2 Displacement of beams and frames by real work method

Numerical#5.

Solution:

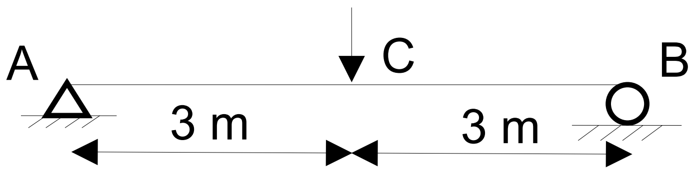
Step 1: Calculate support reactions, shear force diagram, and bending moment diagram



Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.

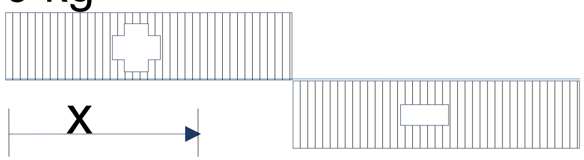
3.2 Displacement of beams and frames by real work method

Numerical#5.



Solution:

Step 2: Calculate strain energy due to shear stress



$$U_{sh} = \int_0^L \frac{KV^2}{2AG} dx$$

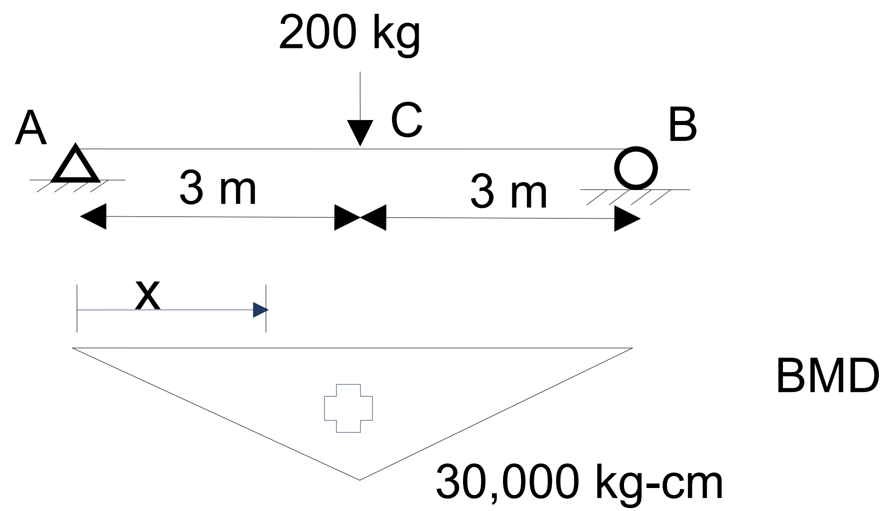
$$U_{sh} = 2\int_0^3 \frac{1.2 * V^2}{2AG} dx = \frac{1.2V^2}{AG} \int_0^3 dx = \frac{1.2V^2}{AG} x|_0^3$$

$$U_{sh} = \frac{1.2 * 100^2}{(20 * 40) * 0.85 * 10^6} (3 * 10^2) = 0.005294 \, kg - cm$$ Ans.

Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.

3.2 Displacement of beams and frames by real work method

Numerical#5.



Solution:

Step 3: Calculate strain energy due to bending stress

$$U_b = 2\int_0^3 \frac{M^2}{2\,EI}dx = \int_0^3 \frac{(100*x*100)^2}{EI}dx$$

Where,
$I = bd^3/12 = 20*40^3/12$
$= 106666.67\ cm^4$

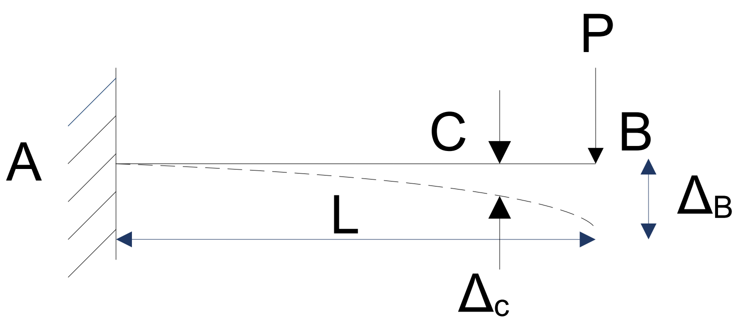
$$U_b = \int_0^3 \frac{10^8 * x^2}{2*10^6*106666.67}dx = \frac{10^8}{2*10^6*106666.67}\left.\frac{x^3}{3}\right|_0^{3*100} = 4218.75\ kg-cm$$

Ans.

Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.

3.3 Limitations of Method of Real Work

#1. Is applicable only when the structure is subjected to a single load.

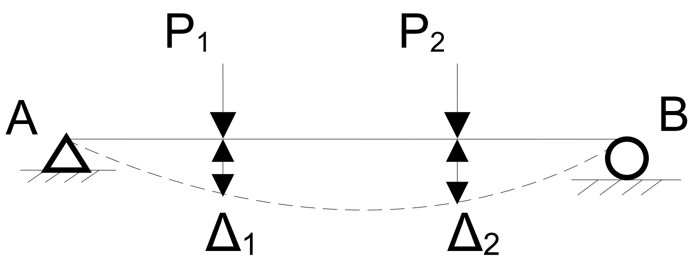


Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

Here,
Δ_B can be found using Real work method but Δ_C cannot be.

3.3 Limitations of Method of Real Work

#2. The deflection can be obtained only at the point of application of load and in the direction of load



Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

From law of conservation of energy,

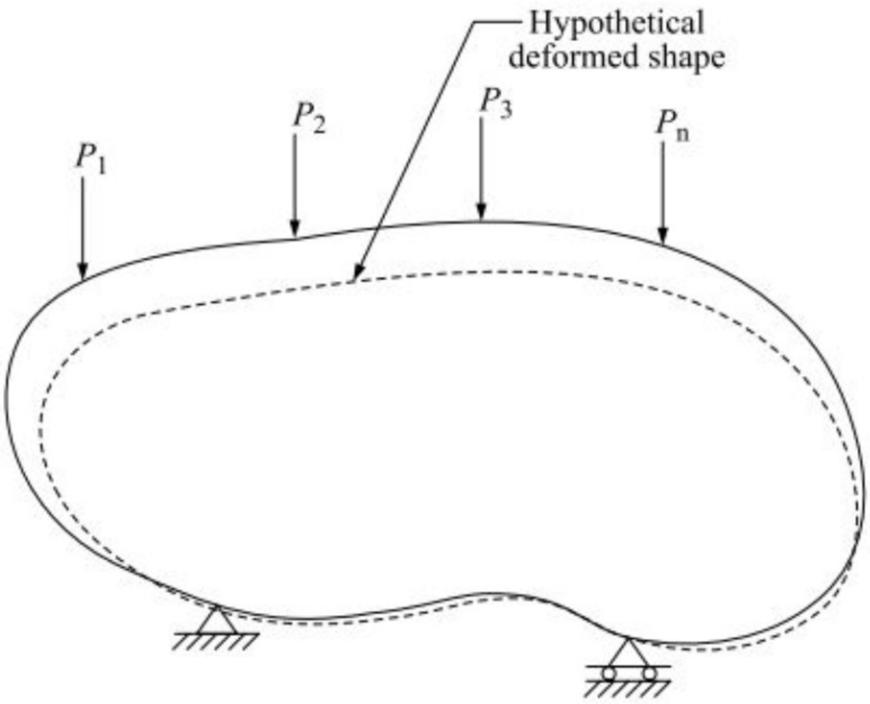
External Work Done = Strain energy stored due to bending

$$\int_0^L \frac{M^2}{2EI} dx = \frac{1}{2} P_1 \Delta_1 + \frac{1}{2} P_2 \Delta_2$$

Here, we have 2 unknowns Δ_1 and Δ_2 ,but only 1 available equation. Hence, cannot be solved for.

3.4 Displacements by Method of Virtual Work

Principle of Virtual Work



Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

Let us consider a body subjected to a set of real forces P_1, P_2, P_3,...P_n. Let the body undergo deformation as shown in the figure.

This **hypothetical** deformation is called **virtual deformation**.

3.4 Displacements by Method of Virtual Work

Work done by real forces due to virtual displacements is called virtual work

Conversely, work done by hypothetical force during real displacements is called virtual work

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

In summary,

Real work done = Real force * Real deformation

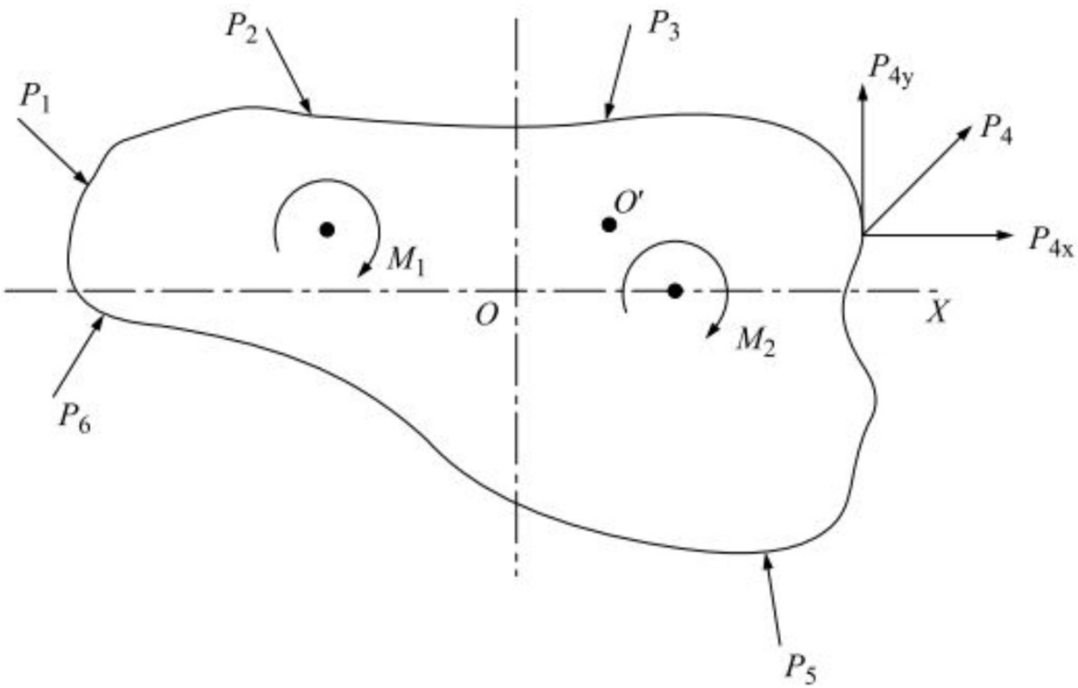
Virtual work done = Virtual force * Real deformation

OR

Virtual work done = Real force * Virtual/hypothetical deformation

3.4 Displacements by Method of Virtual Work

Bernoulli's Principle of Virtual Displacement

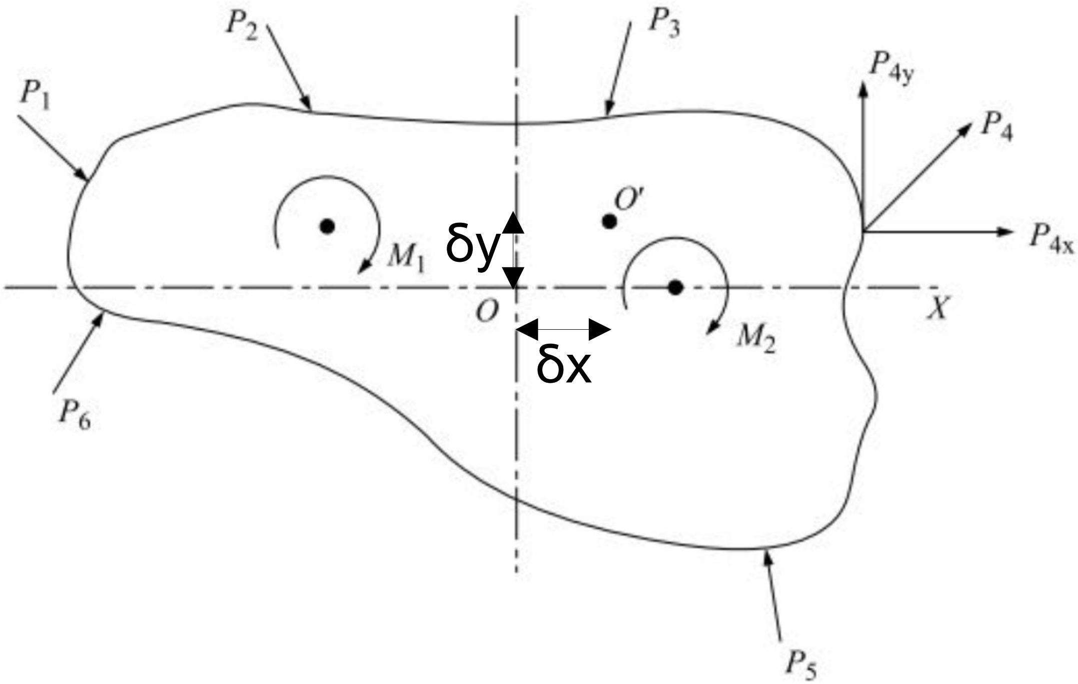


Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

Let us consider a rigid body shown in the figure subjected to P-system of forces and M-system of moments. Let P_x and P_y be the components of forces in x- and y- directions.

3.4 Displacements by Method of Virtual Work

Bernoulli's Principle of Virtual Displacement



Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

Suppose a virtual displacement OO' is given to the rigid body, whose component in x- and y- directions be δx and δy. Since it is a rigid body, all the forces will have displacements δx and δy in x- and y- directions.

3.4 Displacements by Method of Virtual Work

Bernoulli's Principle of Virtual Displacement

Virtual work done = $\Sigma P_x \, \delta x + \Sigma P_y \, \delta y$
$= \delta x \, \Sigma P_x + \delta y \, \Sigma P_y$

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

Since the body is in equilibrium,

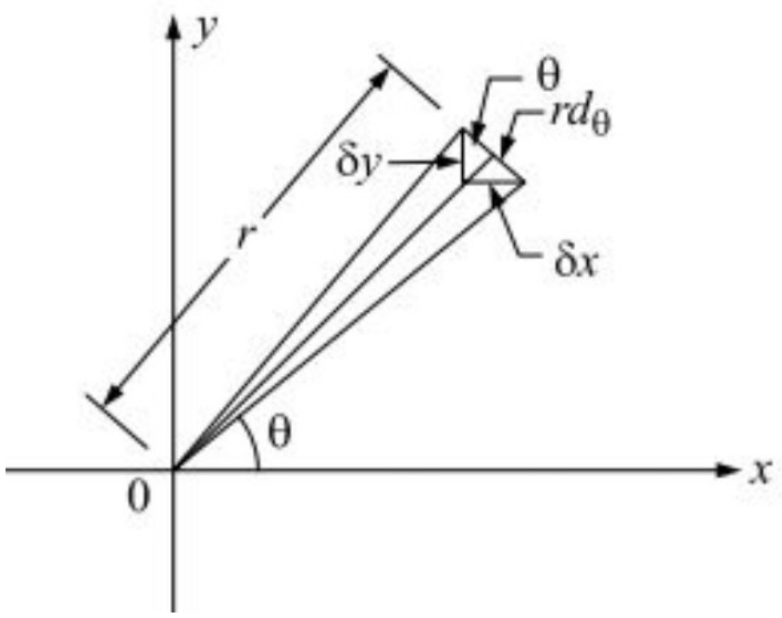
$\Sigma P_x = 0$ and $\Sigma P_y = 0$

Therefore, Virtual work done = 0 -----(1)

3.4 Displacements by Method of Virtual Work

Bernoulli's Principle of Virtual Displacement

Similarly,
Consider the rotation of the rigid body by a virtual rotation dθ. If a point Q (x,y) is at a distance r from the origin,



$$\delta x = r\,d\theta \sin\theta = r\sin\theta\, d\theta = y\, d\theta$$
$$\delta y = r\,d\theta \cos\theta = r\cos\theta\, d\theta = x\, d\theta$$

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

3.4 Displacements by Method of Virtual Work

Bernoulli's Principle of Virtual Displacement

Then,

Virtual work done by real forces due to virtual displacements

$= \Sigma M\, d\theta + \Sigma P_x\, \delta x + \Sigma P_y\, \delta y$

$= \Sigma M\, d\theta + \Sigma P_x\, y\, d\theta + \Sigma P_y\, x\, d\theta$

$= d\theta\, (\Sigma M + \Sigma P_x\, y + \Sigma P_y\, x)$

$$\left\{ \begin{array}{l} \delta x = r\, d\theta \sin\theta = r\sin\theta\, d\theta = y\, d\theta \\ \delta y = r\, d\theta \cos\theta = r\cos\theta\, d\theta = x\, d\theta \end{array} \right\}$$

Since the body is in equilibrium,

$$\Sigma M + \Sigma P_x\, y + \Sigma P_y\, x = 0$$

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

Therefore, Virtual work done = 0 -----(2)

3.4 Displacements by Method of Virtual Work

Bernoulli's Principle of Virtual Displacement

Hence, from equations (1) and (2),

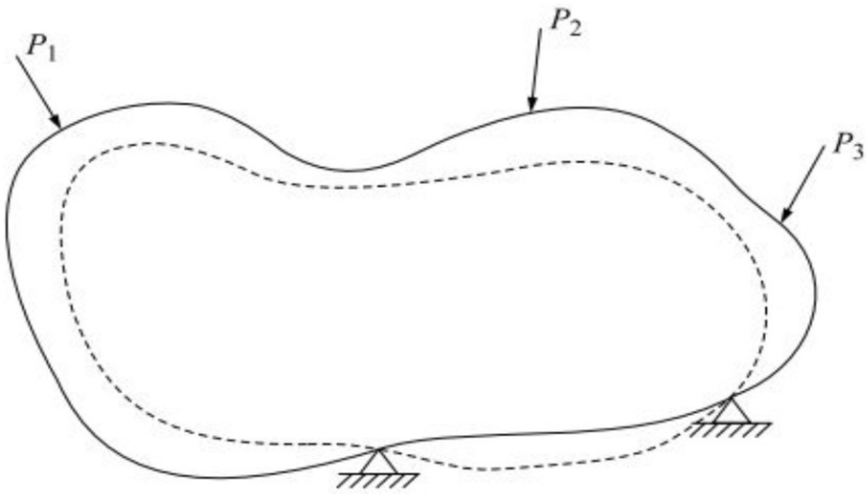
'If a rigid body is in equilibrium under a system of forces and/or moments, the virtual work done by this system of forces and/or moments during virtual displacement is zero.'

This is called 'Bernoulli's principle of virtual displacement'

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

3.4 Displacements by Method of Virtual Work

Principle of Virtual Work for Deformable bodies



Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

Let us consider an elastic body subjected to a system of forces, causing real internal stresses and every element in the body is in equilibrium under the action of external forces and internal stresses.

3.4 Displacements by Method of Virtual Work

Principle of Virtual Work for Deformable bodies

Let dw_θ be the virtual work of the external forces acting on an element.

Since the body is elastic, there will be
1. Virtual displacement and also
2. Virtual deformation (strain)

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

Hence,

External Work Done = Virtual work done as a rigid body + Virtual strain energy of the element

Or, $dw_\theta = dw_r + dw_i$

3.4 Displacements by Method of Virtual Work

Principle of Virtual Work for Deformable bodies

$$\text{Or, } dw_\theta = dw_r + dw_i$$

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

From the principle of virtual displacement,

$$dw_r = 0$$

Hence,

$$dw_\theta = dw_i$$

Now, Integrating over the entire body will give,

$$W_\theta = W_i$$

Where,

W_θ= Total virtual work done by the force system

W_i = Internal virtual strain energy of the entire body

3.4 Displacements by Method of Virtual Work

Principle of Virtual Work for Deformable bodies

“If a deformable body in equilibrium under a system of forces is given virtual deformation, the virtual work done by the system of forces is equal to the internal virtual work done by the stresses due to that system of forces ”

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

-------------End of Lecture#3------------------

------------End of Part I of II for Chapter3-------------

References

[1] Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.
[2] Reddy, C.S. (2011). *Basic Structural Analysis (3rd ed.).* New Delhi: Tata McGraw Hill.
[3] Hada, S.H. *Theory of Structures-I* (Manual for IOE students) *.* Kathmandu.