

# Theory of Structures - I

Chapter 4. Deflection of beams [Part I  
of II]

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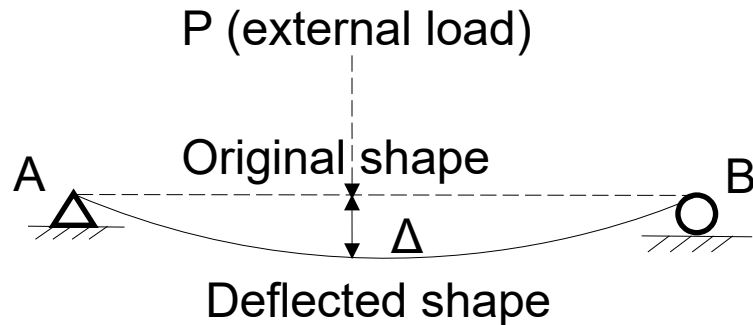
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4.2 Differential Equation of Flexure

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# 4.1 Introduction



$\Delta$  = Deflection

When external load acts, the beam deflects from its original shape. Once the external load has been removed, it will revert back to its original shape.

Other causes of deflection are:

- Change in temperature
- Lack of fit
- Creep
- Settlement of supports

Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students). Kathmandu.

# 4.1 Introduction

Q# Why is the computation of deflection necessary?

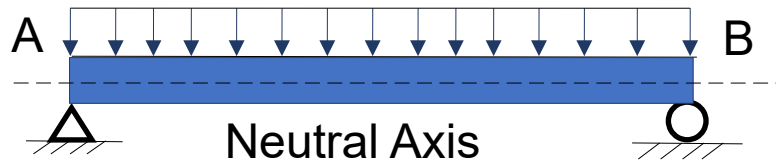
Beam → Primarily designed for strength  
→ But, must also fulfill stiffness criteria

i.e. in addition to strength, the beam should be stiff enough not to deflect more than the given limit under the load → Deflection limit

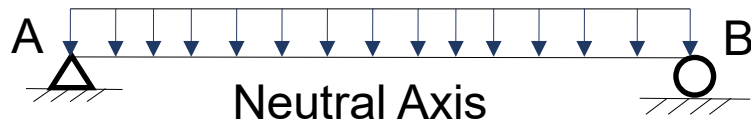
Under limit state method of design, the beam should not only satisfy the limit state of strength but also limit state of serviceability. For example, deflection in any structural member should not exceed the deflection limits so as not to affect its serviceability requirements such as comfort of the occupants.

Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students). Kathmandu.

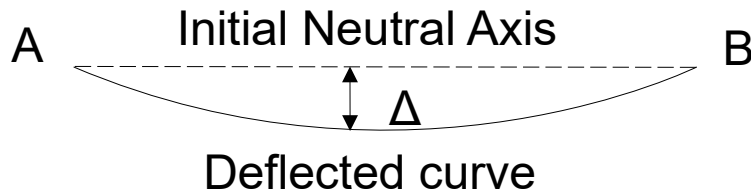
## 4.1 Introduction



Actual structural member



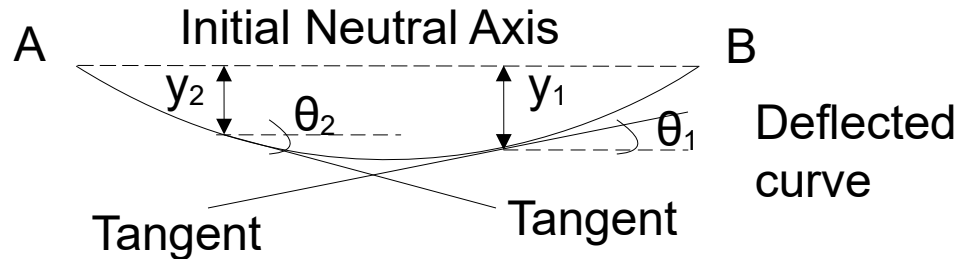
Idealized structural member



$\Delta = y =$  Deflection of the beam at the section  
 = Vertical ordinate between the deflected curve and the initial Neutral Axis

Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students). Kathmandu.

# 4.1 Introduction

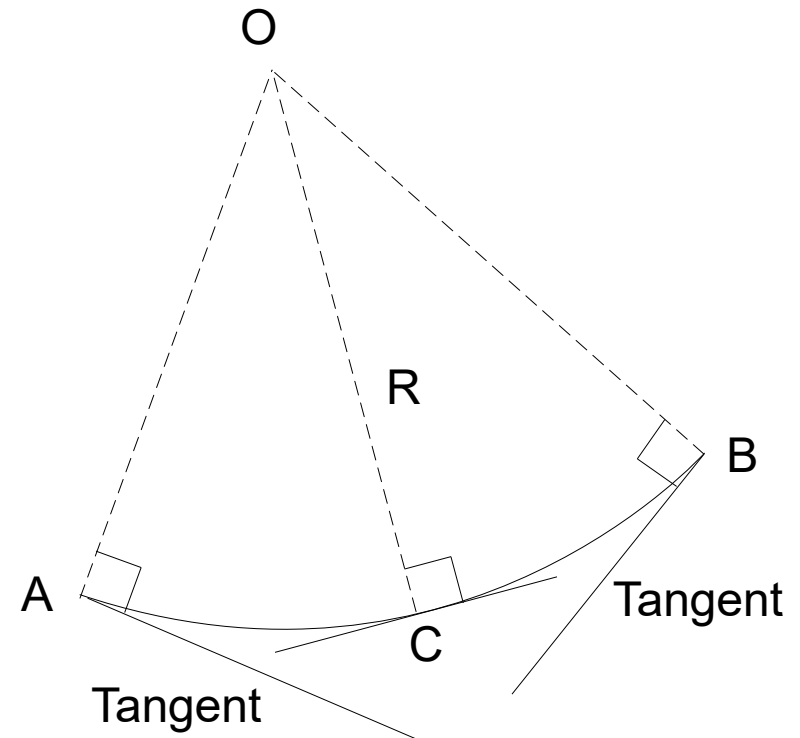


Slope of a beam at a point  
= Angle made by a tangent  
to the reference axis at that  
point

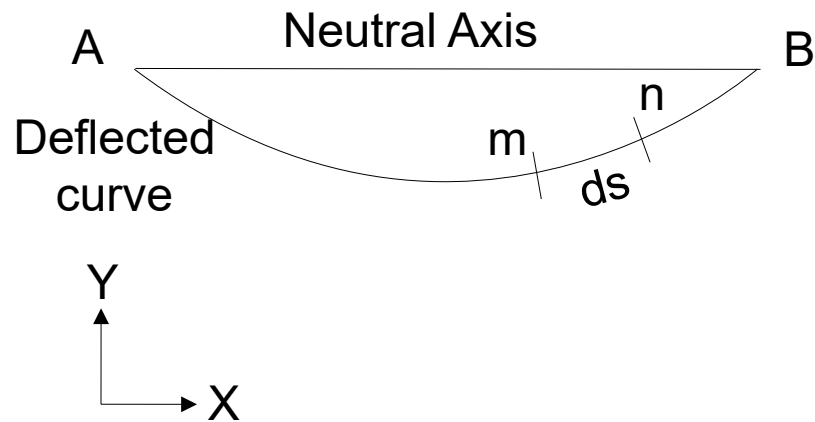
Here,  
 $OC = OA = OB = R$ ; Radius  
of curvature

$$R^{-1} = 1/R = \text{curvature}$$

Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students). Kathmandu.



## 4.2 Differential equation of flexure: Moment-curvature relationship

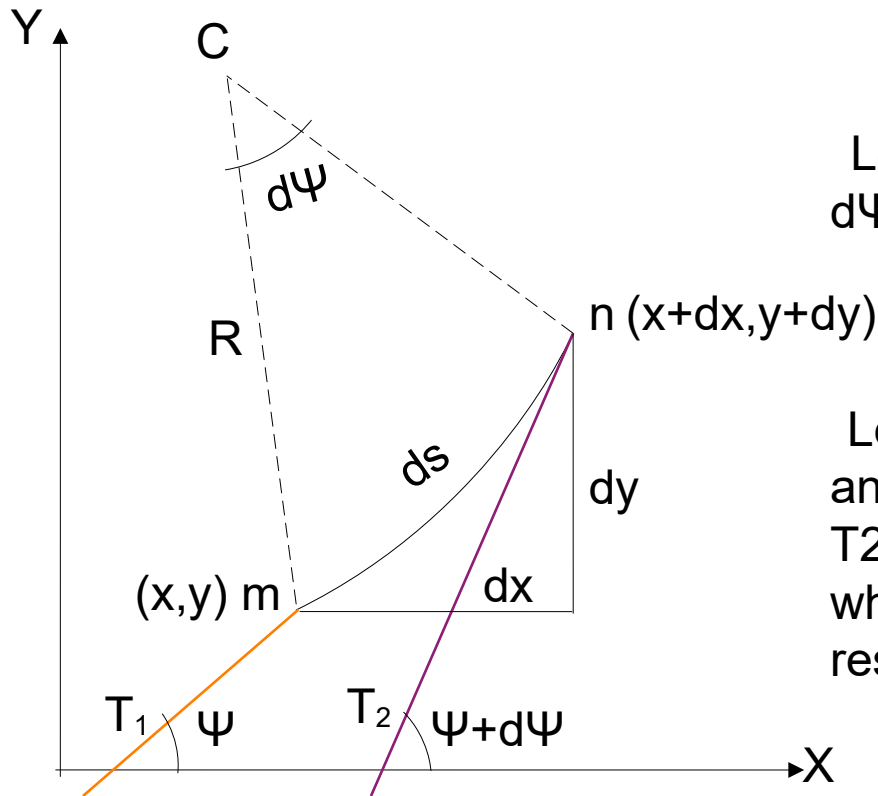


Let us consider a segment **mn** of a bent beam of length **ds**, as shown in the adjacent figure.

Here, AB is the neutral axis of the considered beam, and deflected curve under loading is shown.

Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students). Kathmandu.

## 4.2 Differential equation of flexure: Moment-curvature relationship



Let the segment  $mn$  subtend an angle  $d\Psi$  at the centre of curvature,  $C$ .

Let the coordinates at  $m$  and  $n$  be  $(x, y)$  and  $(x+dx, y+dy)$ , respectively.  $T_1$  and  $T_2$  are the tangents drawn at  $m$  and  $n$ , which subtends angles  $\Psi$  and  $\Psi+d\Psi$ , respectively at the X-axis.



## 4.2 Differential equation of flexure: Moment-curvature relationship

$$\tan\Psi = \frac{dy}{dx} \quad \text{--- (3)}$$

Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students). Kathmandu.

Differentiating equation (3) w.r.t x,

$$\frac{d(\tan\Psi)}{dx} = \frac{d^2 y}{dx^2}$$

$$\text{Or, } \frac{d(\tan\Psi)}{d\Psi} \frac{d\Psi}{dx} = \frac{d^2 y}{dx^2}$$

$$\text{Or, } \sec^2\Psi d\Psi = \frac{d^2 y}{dx^2} dx$$

$$\text{Or, } (1 + \tan^2\Psi) d\Psi = \frac{d^2 y}{dx^2} dx$$

## 4.2 Differential equation of flexure: Moment-curvature relationship

$$(1 + \tan^2 \Psi) d\Psi = \frac{d^2 y}{dx^2} dx$$

Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students). Kathmandu.

From equation (3), we have:  $\tan^2 \Psi = \left(\frac{dy}{dx}\right)^2$

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right] d\Psi = \frac{d^2 y}{dx^2} dx$$

$$\text{Or, } d\Psi = \frac{\frac{d^2 y}{dx^2} dx}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]} \text{ --- (4)}$$

## 4.2 Differential equation of flexure: Moment-curvature relationship

Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students). Kathmandu.

From equations (1), (2), and (4), we have:

$$\cancel{dx} \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{\frac{1}{2}} = R \frac{\cancel{\frac{d^2 y}{dx^2} dx}}{\left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]}$$

$$\text{Or, } \frac{1}{R} = \frac{\frac{d^2 y}{dx^2}}{\left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{\frac{3}{2}}}$$

$$\text{As } dy/dx \text{ is small, } \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{\frac{3}{2}} \approx 1 \quad \rightarrow \quad \frac{1}{R} = \frac{d^2 y}{dx^2} \quad \text{--- (5)}$$

$$ds = R \cdot d\Psi \quad \text{--- (1)}$$

$$ds = dx \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{\frac{1}{2}} \quad \text{--- (2)}$$

$$d\Psi = \frac{\frac{d^2 y}{dx^2} dx}{\left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]} \quad \text{--- (4)}$$

## 4.2 Differential equation of flexure: Moment-curvature relationship

Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students). Kathmandu.

Also, we have:

$$\frac{M}{I} = \frac{E}{R} \quad \rightarrow \quad \frac{1}{R} = \frac{M}{EI} \quad \text{--- (6)} \quad \left\{ \frac{1}{R} = \frac{d^2 y}{dx^2} \quad \text{--- (5)} \right\}$$

From equation (5) and (6),

$$\frac{d^2 y}{dx^2} = \frac{M}{EI}$$

Or,  $EI \frac{d^2 y}{dx^2} = M \quad \text{--- (7)}$

}

M = Moment at the section

$\frac{d^2 y}{dx^2}$  = Curvature at the section

EI = Flexural rigidity

This is the Moment-Curvature relationship (Differential equation of flexure)

## 4.2 Differential equation of flexure: Moment-curvature relationship

Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students). Kathmandu.

$$EI \frac{d^2 y}{dx^2} = M \quad \text{--- (7)}$$

Integrating equation (7), we get

$$EI \frac{dy}{dx} = \int M \quad \text{--- (8)}$$

This equation is used to find the **slope of the beam**,  $\theta = dy/dx$ .

Again, Integrating equation (8), we get

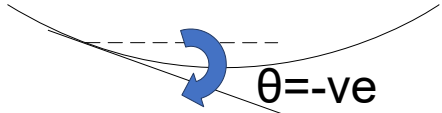
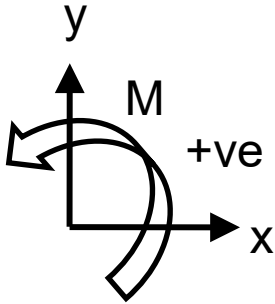
$$EI y = \int \int M \quad \text{--- (9)}$$

This equation is used to find the **deflection of the beam**,  $y$ .

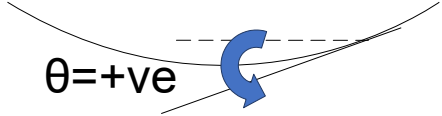
# 4.2 Differential equation of flexure: Moment-curvature relationship

**Sign Conventions:**

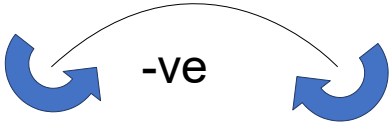
- $x=+ve$ , when measured towards the right
- $y=+ve$ , when measured upwards
- $M=$  Bending Moment =  $+ve$ , when sagging moment
- $\theta=$ slope= $+ve$ , when rotation is anticlockwise
- $y=-ve$ , when deflection is downwards



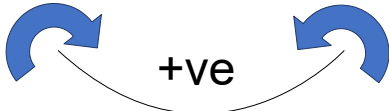
clockwise



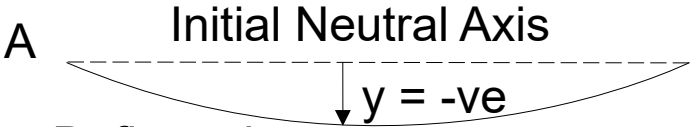
anticlockwise



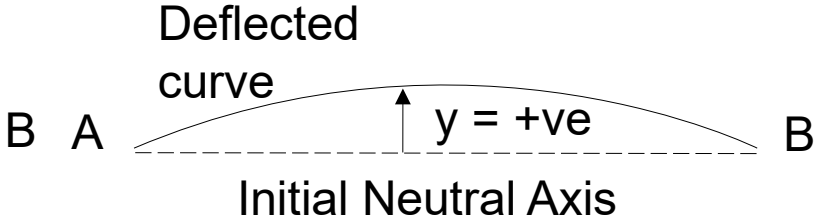
hogging moment



sagging moment



Deflected curve



Deflected curve

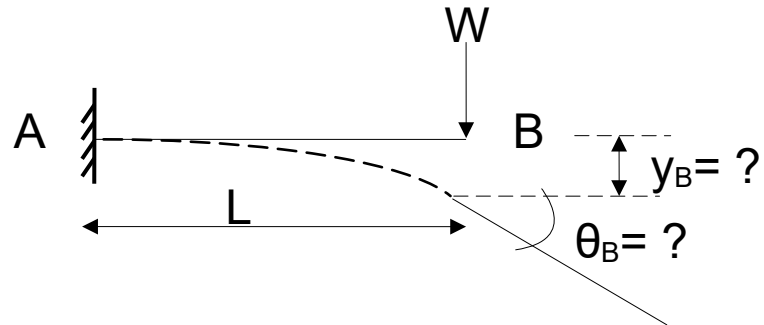
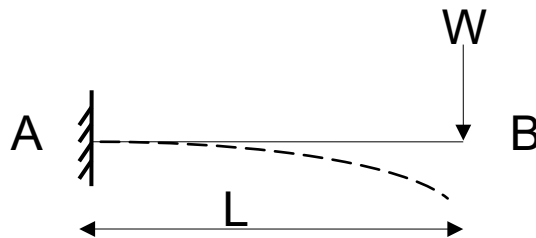
Initial Neutral Axis

Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students). Kathmandu.

## 4.3 Double Integration Method

In this method, the moment curvature relationship is integrated for finding slope and deflection at any point in a beam.

**Numerical #1.** Find maximum slope and deflection of given loaded cantilever beam.



Solution:

We have,

$$EI \frac{d^2 y}{dx^2} = M \quad \dots (1)$$

Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students). Kathmandu.

## 4.3 Double Integration Method

**Numerical #1.** Find maximum slope and deflection of given loaded cantilever beam.

Solution:

Bending moment at a section I-I of the beam, at a distance  $x$  from the free end is:

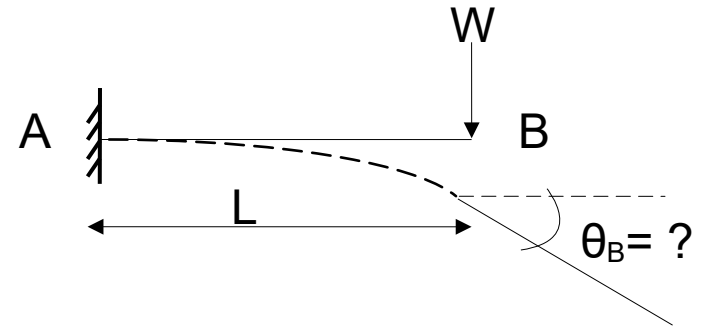
$$M_x = -W \cdot x$$

Substituting in equation (1) becomes,

$$EI \frac{d^2 y}{dx^2} = -W \cdot x$$

Integrating w.r.t.  $x$ , we get

$$EI \frac{dy}{dx} = -W \frac{x^2}{2} + c_1 \quad \text{--- (2)} \quad \text{Where, } c_1 \text{ is the integration constant}$$



Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students). Kathmandu.

## 4.3 Double Integration Method

**Numerical #1.** Find maximum slope and deflection of given loaded cantilever beam.

Solution:

Applying the boundary conditions,

At  $x=L$ ,  $dy/dx = 0$  [Fixed end]

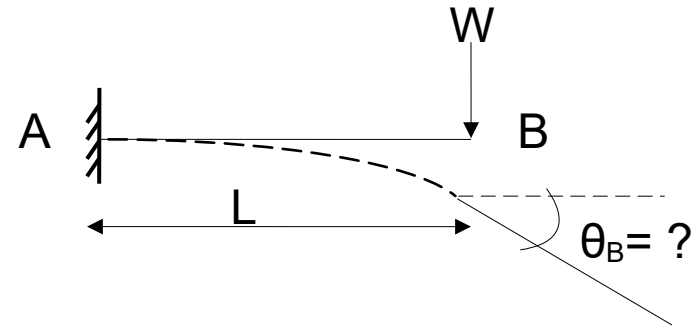
We get,

$$0 = -W \frac{L^2}{2} + c_1 \quad \Rightarrow \quad c_1 = \frac{W L^2}{2}$$

Substituting the value of  $c_1$  in equation (2), we get,

$$EI \frac{dy}{dx} = -W \frac{x^2}{2} + \frac{WL^2}{2} \quad \text{--- (3)}$$

This is the equation to find the slope of the beam at any point.



Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students). Kathmandu.

## 4.3 Double Integration Method

**Numerical #1.** Find maximum slope and deflection of given loaded cantilever beam.

Solution:

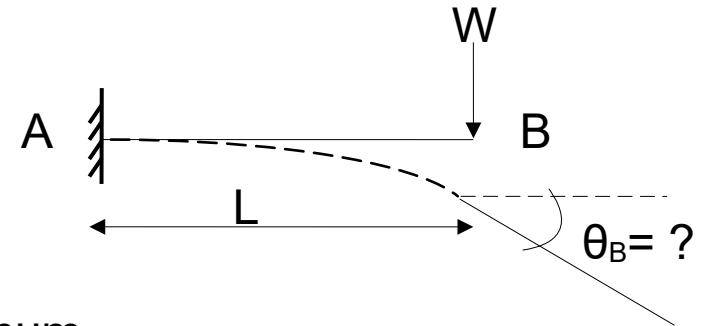
In the given beam, it is evident that maximum slope occurs at the free end B, where  $x=0$ .

Then,

$$EI \left( \frac{dy}{dx} \right)_B = 0 + \frac{WL^2}{2}$$

$$\text{Or, } EI \theta_B = \frac{WL^2}{2}$$

$$\text{Or, } \theta_B = \theta_{max} = \frac{WL^2}{2EI} \text{ radians}$$



Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students). Kathmandu.

## 4.3 Double Integration Method

**Numerical #1.** Find maximum slope and deflection of given loaded cantilever beam.

Solution:

Again, integrating equation (3), we get:

$$EI y = \frac{-W x^3}{2 * 3} + \frac{WL^2}{2} x + c_2$$

$$\text{Or, } EI y = \frac{-W x^3}{6} + \frac{WL^2}{2} x + c_2 \text{ --- (4)}$$

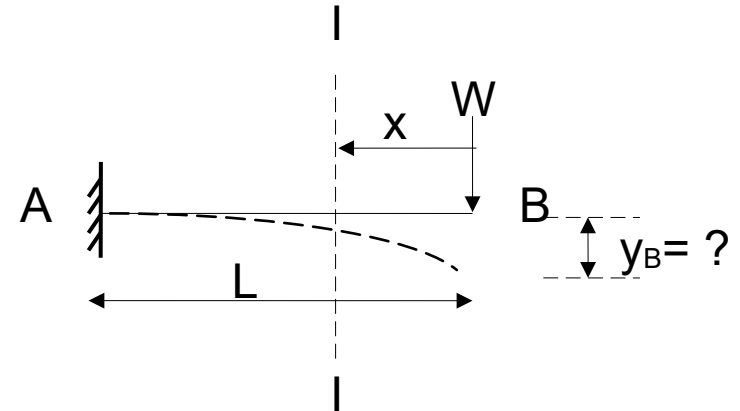
Where,  $c_2$  is the integration constant.

Applying boundary conditions,

At  $x=L$ ,  $y=0$  [Fixed end]

Then, equation (4) becomes,

$$\text{Or, } 0 = \frac{-W L^3}{6} + \frac{WL^3}{2} + c_2 \quad \rightarrow \quad c_2 = \frac{-W L^3}{3}$$

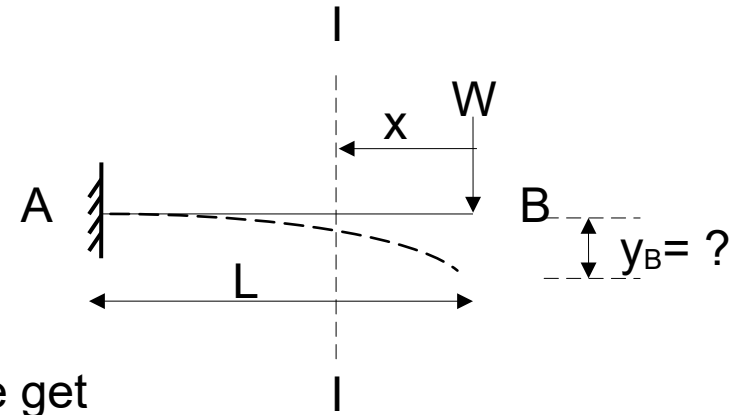


Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students). Kathmandu.

## 4.3 Double Integration Method

**Numerical #1.** Find maximum slope and deflection of given loaded cantilever beam.

Solution:



Substituting the value of  $c_2$  in equation (4), we get

$$EI y = \frac{-W x^3}{6} + \frac{WL^2}{2} x - \frac{WL^3}{3} \quad \text{--- (5)}$$

Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students). Kathmandu.

This is the equation to find deflection at any point of the beam.

Here, it is evident that maximum deflection in the beam occurs at the free end, where  $x = 0$ .

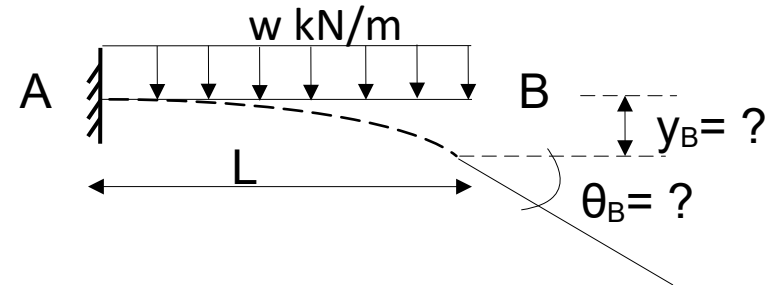
$$\text{Then, } EI y_B = 0 + 0 - \frac{WL^3}{3} \quad \Rightarrow \quad y_B = y_{max} = \frac{-WL^3}{3EI}$$

Here, -ve sign shows that the deflection is in downward direction.

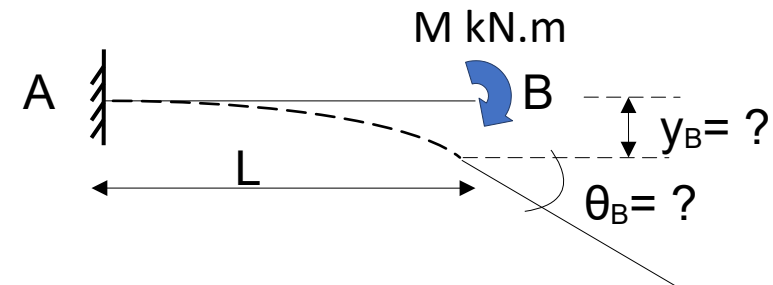
## 4.3 Double Integration Method

### Numerical for Practice

**Numerical #2.** Find maximum slope and deflection of given cantilever beam with a UDL applied throughout its length.



**Numerical #3.** Find maximum slope and deflection of given cantilever beam with a moment applied at free end.



Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students). Kathmandu.

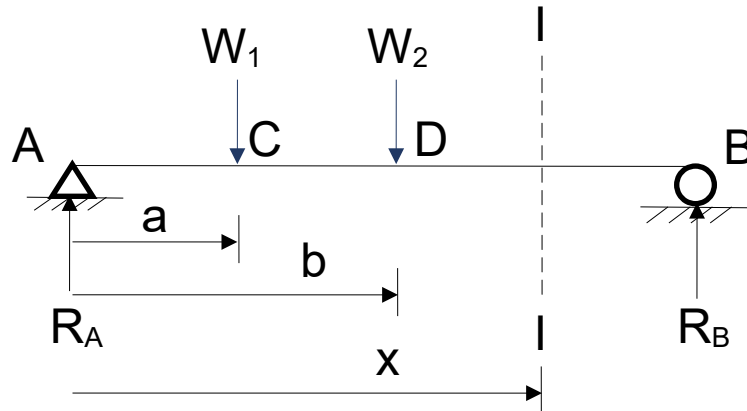
## 4.4 Macaulay's Method

In *double integration method*, when a beam carries a number of loads, each segment has to be integrated separately.

In ***Macaulay's method***, a *single equation* is formed for all the loading in a given beam. This equation is constructed in such a way that the constant of integration is applied to all the portions of the beam. This method is also called the ***Method of Singularity function***.

## 4.4 Macaulay's Method

Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students). Kathmandu.



Let AB be a beam of length L, loaded with point loads  $W_1$  and  $W_2$  at distances a and b from A, respectively.

Then, taking section I-I such that all the loads are included, the moment at any section of the beam at a distance x from point A is:

$$M_x = R_A x - \left[ \left[ W_1(x - a) \right] \right] - \left[ \left[ W_2(x - b) \right] \right] - \dots - (A)$$

Such that:

- (i) For  $x = 0$  to  $x = a$ , only the 1<sup>st</sup> term of the above equation is considered,
- (ii) For  $x = a$  to  $x = b$ , only the 1<sup>st</sup> two terms are considered, and
- (iii) For  $x = b$  to  $x = L$ , all the terms are considered.

## 4.4 Macaulay's Method

$$M_x = R_A x - \left[ \left[ W_1(x-a) \right] \right] - \left[ \left[ W_2(x-b) \right] \right] - \dots - (A)$$

Now, substituting equation (A) into the equation of flexure and integrating for slope and deflections as below:

$$EI \frac{dy}{dx} = - R_A \frac{x^2}{2} - \left[ \left[ \frac{W_1(x-a)^2}{2} \right] \right] - \left[ \left[ \frac{W_2(x-b)^2}{2} \right] \right] + c_1 - \dots - (B) \quad : \text{Slope}$$

$$EI y = - R_A \frac{x^3}{6} - \left[ \left[ \frac{W_1(x-a)^3}{6} \right] \right] - \left[ \left[ \frac{W_2(x-b)^3}{6} \right] \right] + c_1 x + c_2 - \dots - (C) \quad : \text{Deflection}$$

### Points to be noted:

- (1) The expressions (x-a) and (x-b) are integrated as a whole and not as individual terms
- (2) The constants  $c_1$  and  $c_2$  are valid for all values of x and are evaluated using boundary conditions.

## 4.4 Macaulay's Method

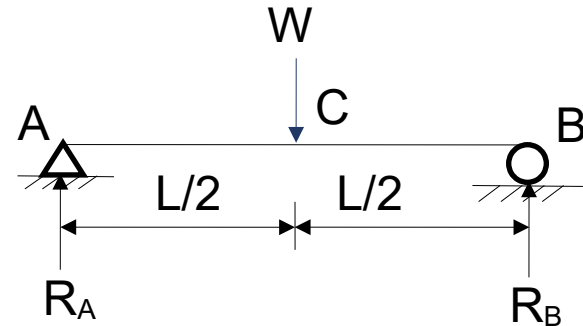
**Numerical #4.** Find the maximum slope and deflection for a simply supported beam using Macaulay's method.

Solution:

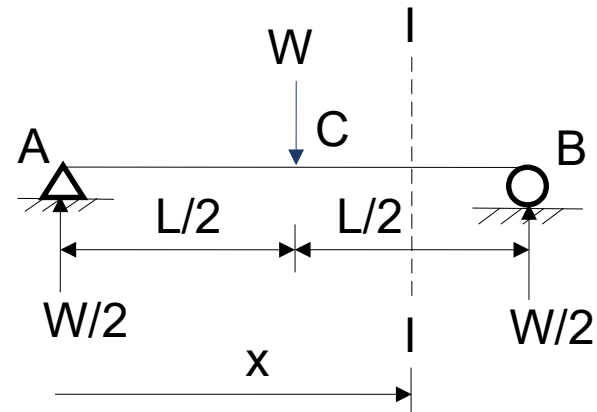
Let us take A as the origin. Then, Bending Moment at a section I-I of the beam at a distance  $x$  from A is:

$$M_x = \frac{W}{2}x - \left[ \left[ W \left( x - \frac{L}{2} \right) \right] \right]$$

Substituting in the differential equation of flexure, we get,

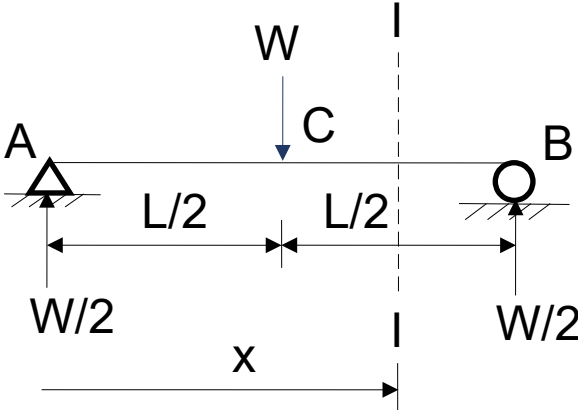


Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students). Kathmandu.



# 4.4 Macaulay's Method

**Numerical #4.** Find the maximum slope and deflection for a simply supported beam using Macaulay's method.



Solution:

$$EI \frac{d^2 y}{dx^2} = \frac{W}{2} x - \left[ \left[ W \left( x - \frac{L}{2} \right) \right] \right] \text{----- (A)}$$

Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students). Kathmandu.

Integrating above equation for slope and deflection equations,

$$EI \frac{dy}{dx} = \frac{W}{2} \frac{x^2}{2} - \left[ \left[ \frac{W \left( x - \frac{L}{2} \right)^2}{2} \right] \right] + c_1 \text{----- (B)}$$

$$EI y = \frac{W}{2} \frac{x^3}{6} - \left[ \left[ \frac{W \left( x - \frac{L}{2} \right)^3}{6} \right] \right] + c_1 x + C_2 \text{----- (C)}$$

## 4.4 Macaulay's Method

**Numerical #4.** Find the maximum slope and deflection for a simply supported beam using Macaulay's method.

Solution:

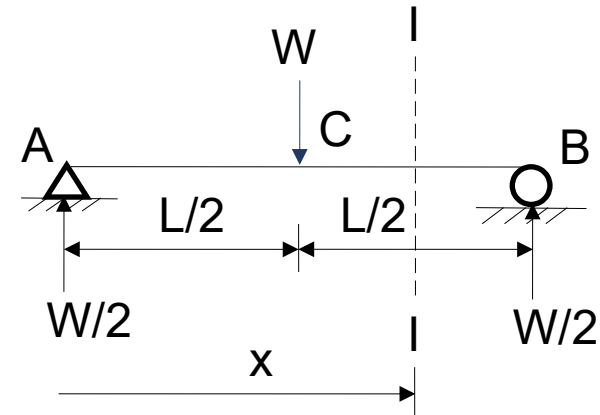
Now, applying boundary conditions,

(a) At  $x = 0$ ,  $y = 0$  {considering only the 1<sup>st</sup> term in the equation (C)}

$$EI * 0 = \frac{W}{2} * 0 + c_1 * 0 + c_2 \quad \Rightarrow \quad c_2 = 0$$

(b) At  $x = L$ ,  $y = 0$ , and substituting  $c_2 = 0$  {considering all the terms in equation (C)}

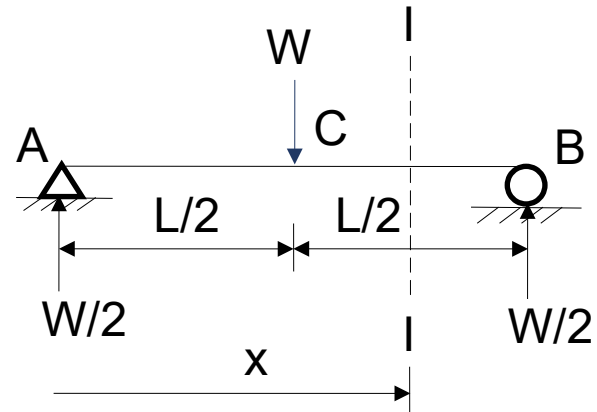
$$EI * 0 = \frac{W}{2} \frac{L^3}{6} - \left[ \left[ \frac{W \left( L - \frac{L}{2} \right)^3}{6} \right] \right] + c_1 L + 0 \quad \Rightarrow \quad c_1 = \frac{-W L^2}{16}$$



Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students). Kathmandu.

# 4.4 Macaulay's Method

**Numerical #4.** Find the maximum slope and deflection for a simply supported beam using Macaulay's method.



Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students). Kathmandu.

Solution:

Now, substituting the values of  $c_1$  and  $c_2$  in equations (B) and (C)

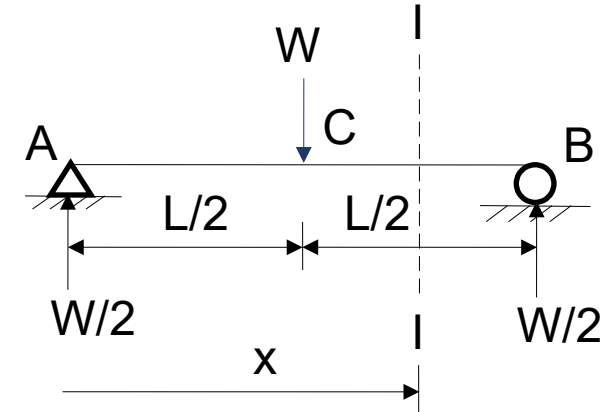
$$EI \frac{dy}{dx} = \frac{Wx^2}{4} - \left[ \left[ \frac{W \left( x - \frac{L}{2} \right)^2}{2} \right] \right] - \frac{WL^2}{16} \text{ --- (D)}$$

$$EI y = \frac{Wx^3}{12} - \left[ \left[ \frac{W \left( x - \frac{L}{2} \right)^3}{6} \right] \right] - \frac{WL^2}{16} x \text{ --- (E)}$$

We use these equations to find the slope and deflection in the beam.

## 4.4 Macaulay's Method

**Numerical #4.** Find the maximum slope and deflection for a simply supported beam using Macaulay's method.



Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students). Kathmandu.

Solution:

For maximum slope,  $x = 0$  and  $x = L$ , i.e. at supports A and B, respectively.

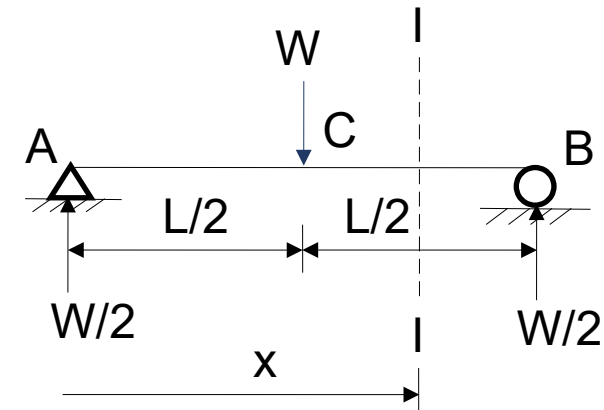
From equation (D), at  $x = 0$  (At support A) {Considering only the 1<sup>st</sup> term}

$$EI \left( \frac{dy}{dx} \right)_A = 0 - \frac{W L^2}{16}$$

$$\text{Or, } \theta_A = \frac{-W L^2}{16 EI} \quad \Rightarrow \quad \theta_A = \frac{W L^2}{16 EI} \text{ (Clockwise)}$$

## 4.4 Macaulay's Method

**Numerical #4.** Find the maximum slope and deflection for a simply supported beam using Macaulay's method.



Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students). Kathmandu.

Solution:

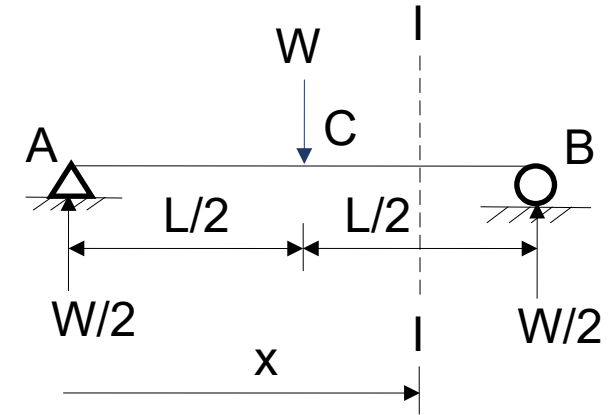
From equation (D), at  $x = L$  (At support B) {Considering all the terms}

$$EI \left( \frac{dy}{dx} \right)_B = \frac{WL^2}{4} - \left[ \left[ \frac{WL^2}{8} \right] \right] - \frac{WL^2}{16}$$

Or,  $\theta_B = \frac{WL^2}{16EI}$  (Anti-Clockwise)

## 4.4 Macaulay's Method

**Numerical #4.** Find the maximum slope and deflection for a simply supported beam using Macaulay's method.



Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students). Kathmandu.

Solution:

For maximum deflection, we know,  $x = L/2$  [since given is a simply supported beam with point load at its mid-span]

Then,

from equation (E), {Considering only the 1<sup>st</sup> term}

$$EI y_C = \frac{WL^3}{12 * 8} - \frac{WL^2}{16} \frac{L}{2}$$

Or,  $y_C = \frac{-WL^3}{48EI}$

Here, the negative sign shows that the deflection is downwards, i.e. negative y-direction.

-----End of Lecture#5-----

-----End of Part I of II for Chapter4-----

# References

[1] Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.