Theory of Structures - I

Chapter 4. Deflection of beams [Part II of II]

Lecturer: Dr. Sanjeema Bajracharya

Contents

4.5 (Mohr's) Moment-Area Method

The integration method allows us to form general equation for slope and deflection of a beam at any desired section, anywhere along the entire structure.

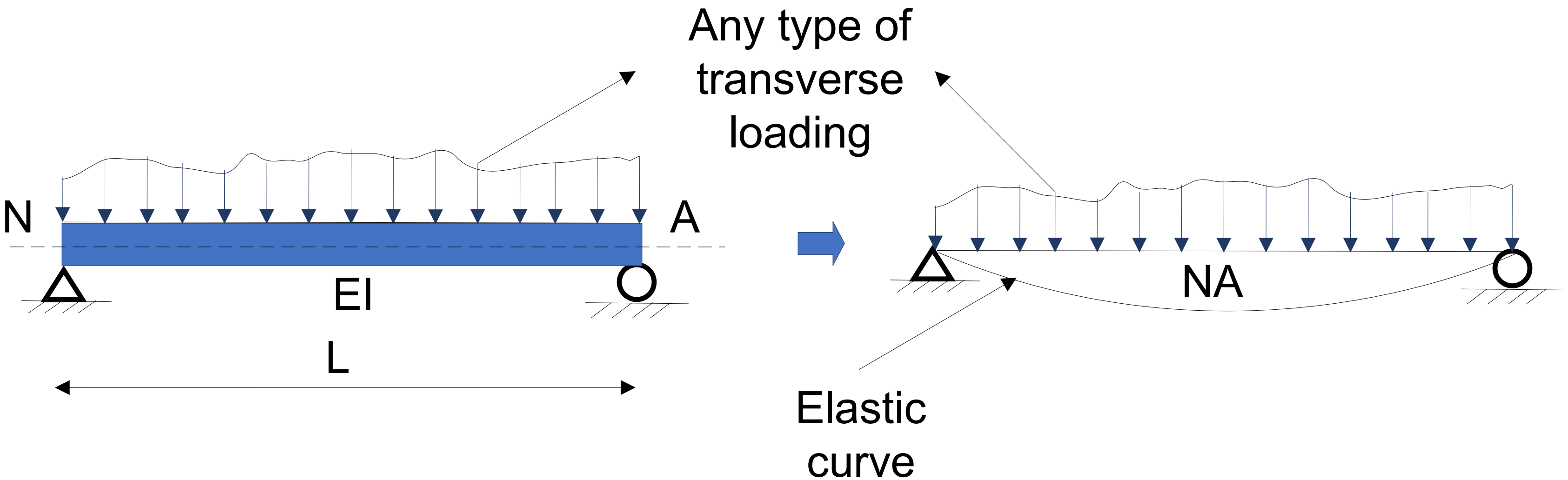
However, it may not always be necessary to find such general equation specifically when the slope and/or deflection are desired only at some specific conditions.

The **Moment-area method** provides a convenient means of determining the slope and deflections in beams and frames. It is a *semi-graphical method* that can be *conveniently used for members of varying moment of inertia and discontinuous loading*.

Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.

4.5 (Mohr's) Moment-Area Method

Moment-area theorems:

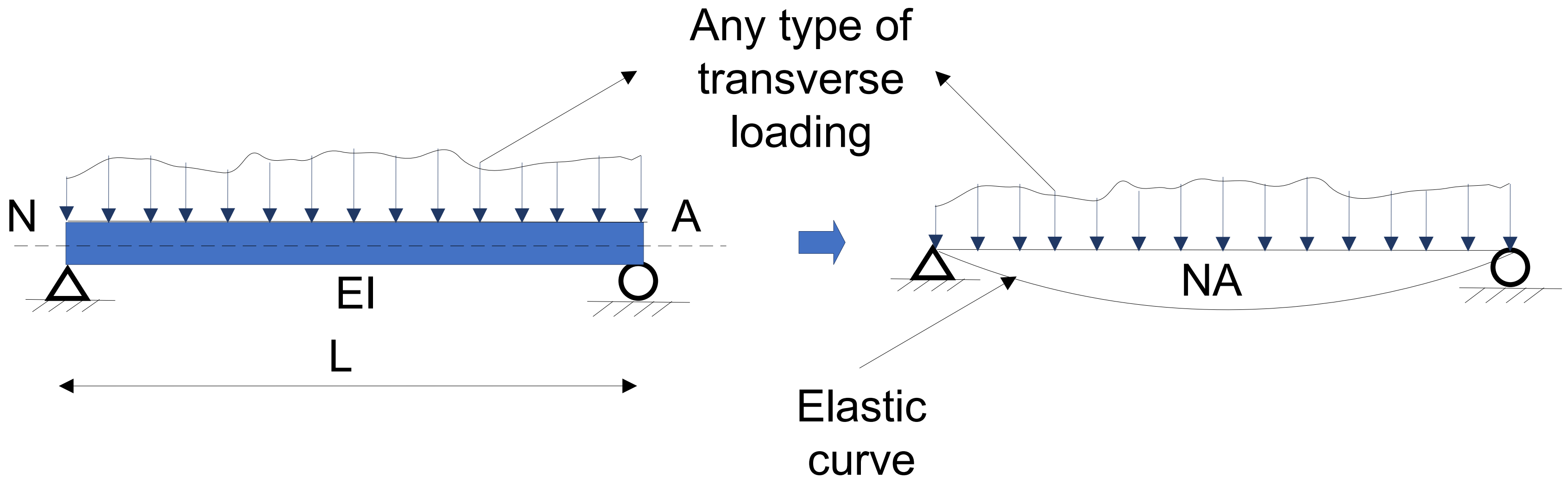


Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.

Let a beam of flexural rigidity EI and span L be subjected to any transverse loading such that its deflected elastic curve is as shown in the figure. Here, NA represents the Neutral Axis.

4.5 (Mohr's) Moment-Area Method

Moment-area theorems:

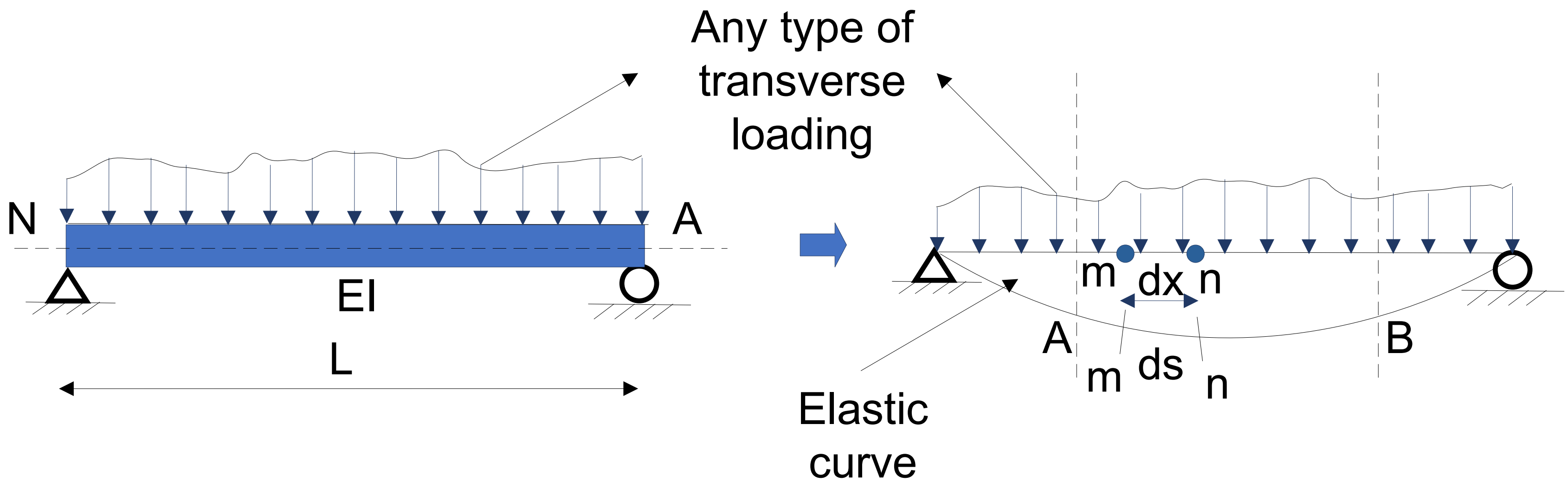


Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.

Let a beam of flexural rigidity EI and span L be subjected to any transverse loading such that its deflected elastic curve is as shown in the figure. Here, NA represents the Neutral Axis.

4.5 (Mohr's) Moment-Area Method

Moment-area theorems:

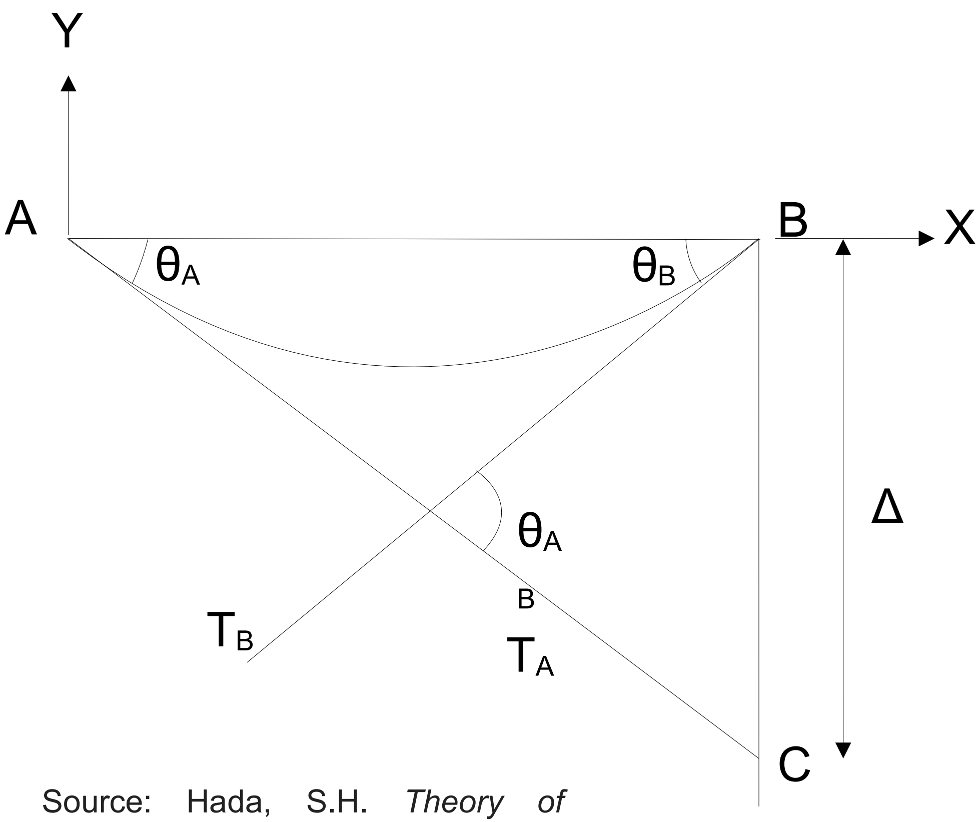


Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.

Let us consider a segment AB of the beam whose elastic curve after the application of load is represented by AmnB.

4.5 (Mohr's) Moment-Area Method

Moment-area theorems:



Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.

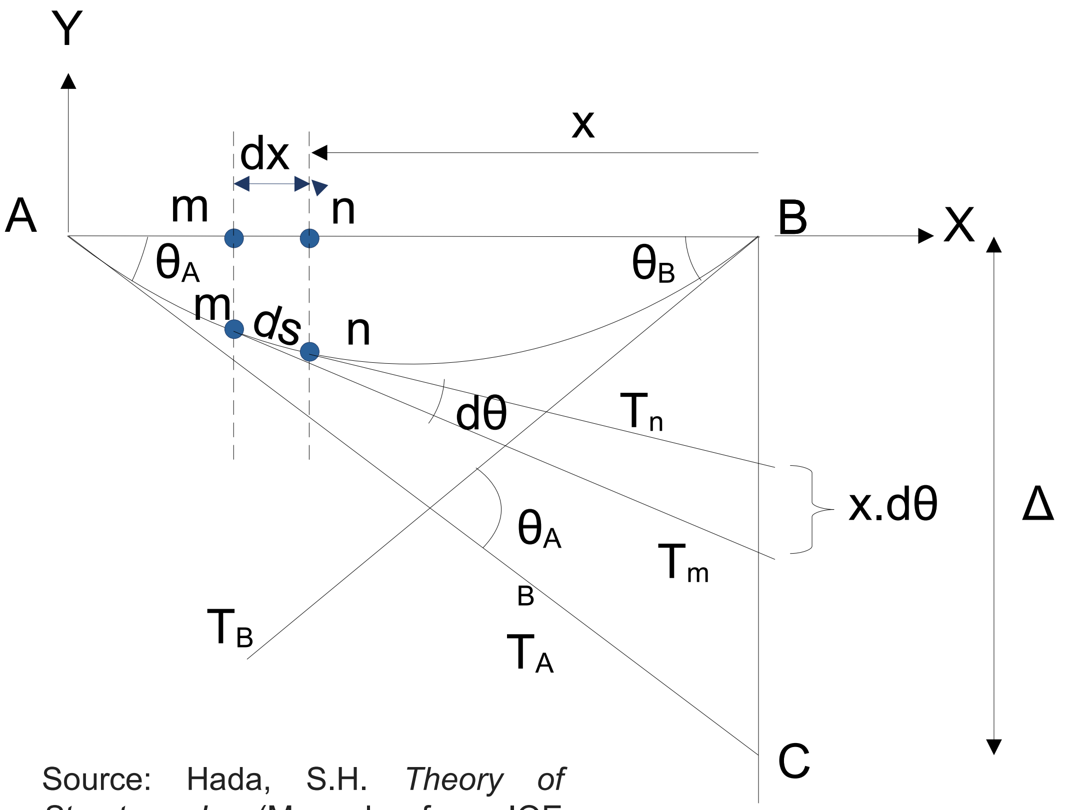
The slopes at A and B are θ_A and θ_B and θ_{AB} represents the change in angle between tangents T_A and T_B drawn at A and B, respectively.

The tangent T_A intersects the vertical line through B at C, where

BC= Δ = Deflection

4.5 (Mohr's) Moment-Area Method

Moment-area theorems:



Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.

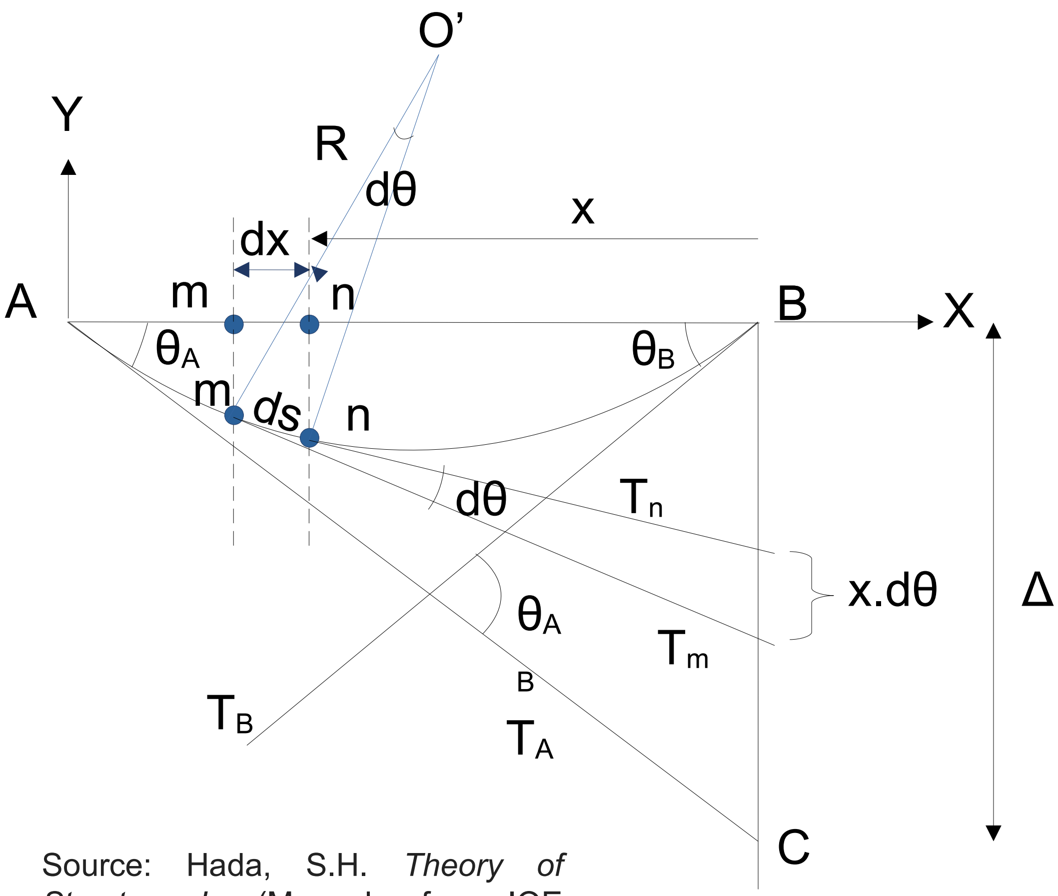
Now, consider the differential element mn of length dx at a distance x from B. Its length becomes ds in the elastic curve on application of load.

The change in the slope between these two points is given by dθ, which is the angle between tangents T_m and T_n drawn at m and n, respectively.

The deflection between these points is given by x.dθ.

4.5 (Mohr's) Moment-Area Method

Moment-area theorems:



Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.

From geometry,

$$R\,d\theta = ds$$

Considering negligible axial deformation,

$$R\,d\theta = ds = dx$$

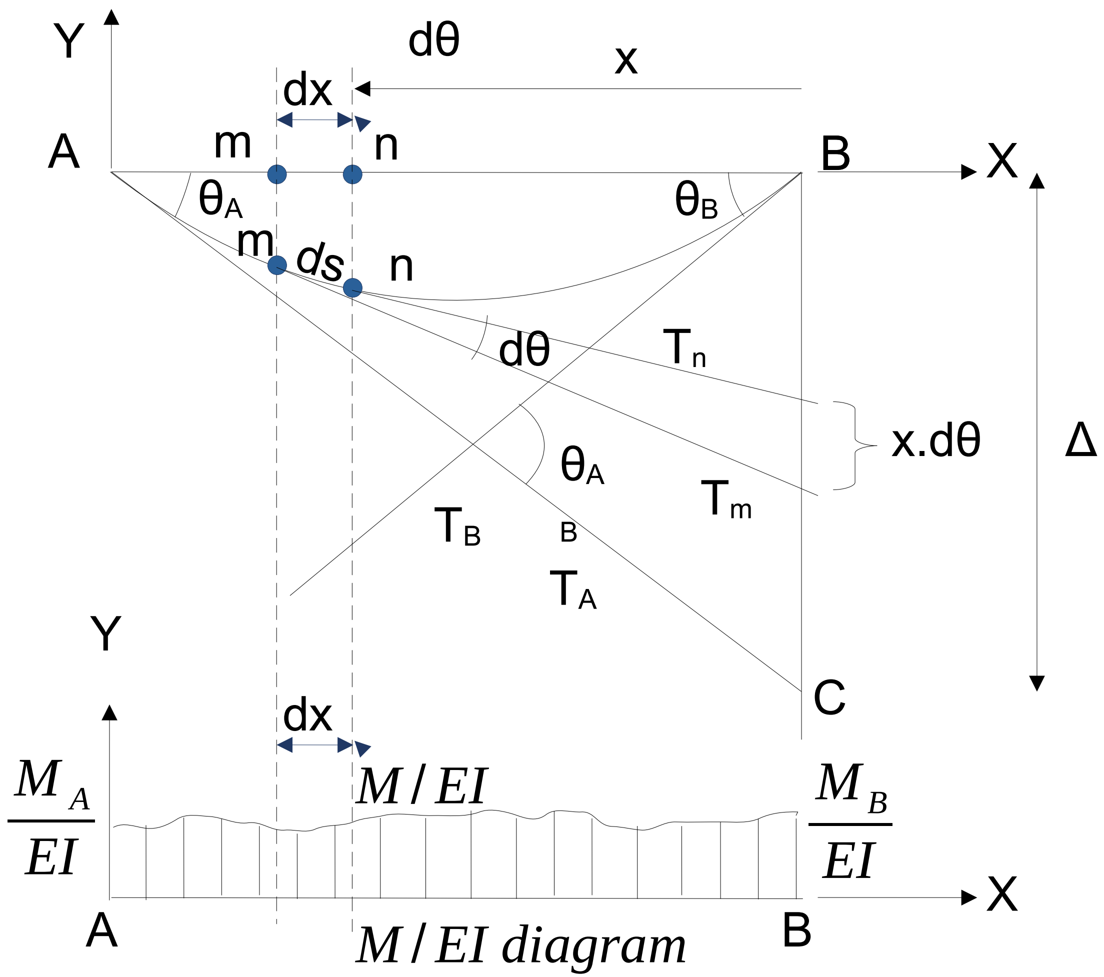
Or, $$d\theta = \frac{1}{R}dx \quad ---(1)$$

From equation of flexure, we have,

$$\frac{1}{R} = \frac{M}{EI} \quad ---(2)$$

4.5 (Mohr's) Moment-Area Method

Moment-area theorems:



$$d\theta = \frac{1}{R}dx - - - (1)$$

$$\frac{1}{R} = \frac{M}{EI} - - - (2)$$

From equations (1) and (2), we get

$$d\theta = \frac{M}{EI}dx - - - (3)$$

From above equation, we can see that the value of change in slope between points m and n is equal to the area of M/EI diagram between those points.

Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.

4.5 (Mohr's) Moment-Area Method

Moment-area theorems:

Now, the change in slope between A and B is obtained by integrating equating (3),

$$\theta_{AB} = \int_A^B d\theta = \int_A^B \frac{M}{EI} dx --- (4)$$

Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.

Theorem 1: The change in slope between two points on a straight member under flexure is equal to the area of M/EI diagram between those two points.

Where,

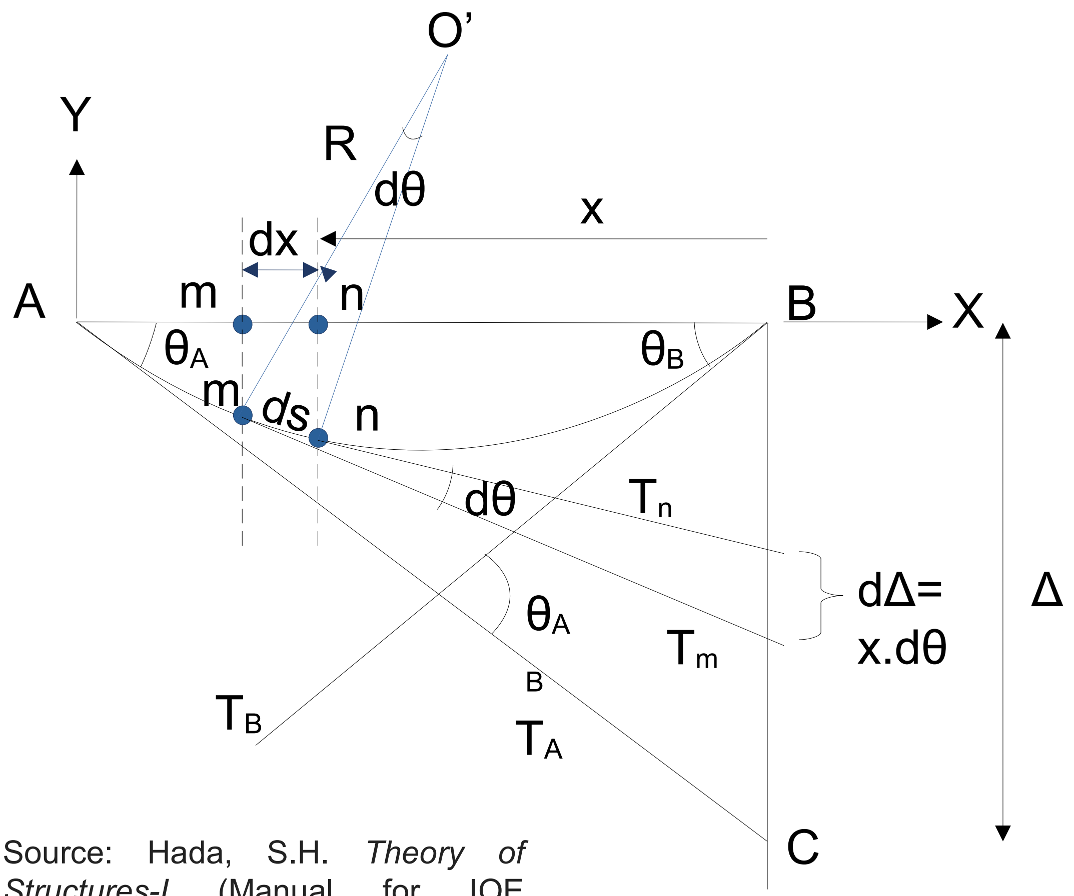
M = Bending Moment

E = Young's Modulus, and

I = Moment of Inertia

4.5 (Mohr's) Moment-Area Method

Moment-area theorems:



Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.

Again, referring to figure,

The intercept of tangents T_m and T_n on BC can be written as:

$$d\Delta = x\,d\theta = x\left(\frac{M}{EI}\right)dx$$

---(5)

4.5 (Mohr's) Moment-Area Method

Moment-area theorems:

Then, integrating equation (5), we have,

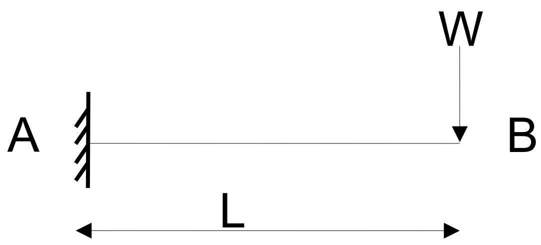
$$\Delta = \int_A^B d\Delta = \int_A^B \left(\frac{M}{EI}\right) x\,dx \quad ---(6)$$

Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.

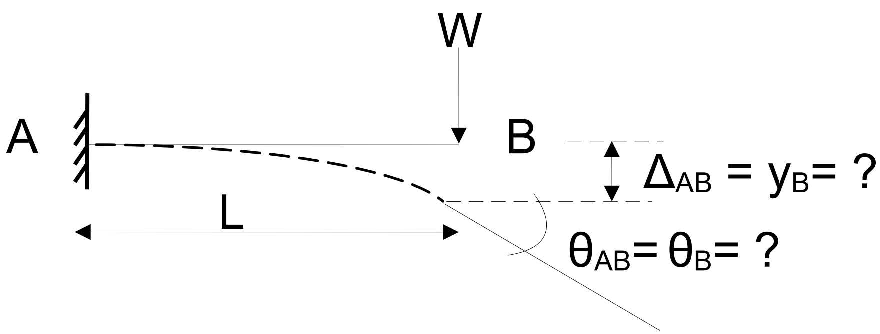
Theorem 2: Deflection at a point in a beam in the direction perpendicular to its original straight line position measured from the tangent to the elastic curve at another point is given by the moment of (M/EI) diagram about the point, where deflection is required.

4.5 (Mohr's) Moment-Area Method

Numerical#1. Using moment-area method, calculate the maximum slope and deflection of a cantilever beam loaded with point load W at its free end.



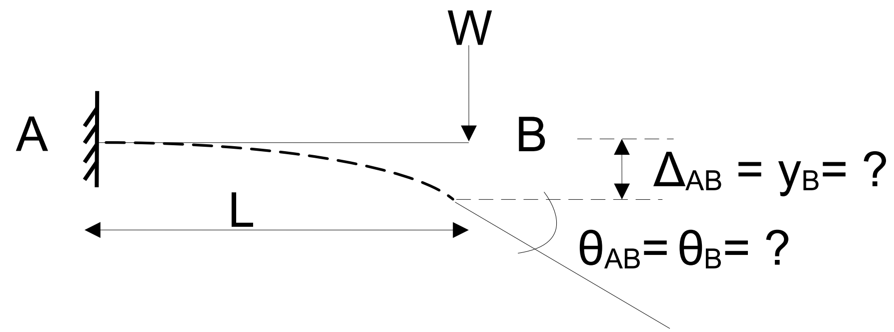
Solution:



Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.

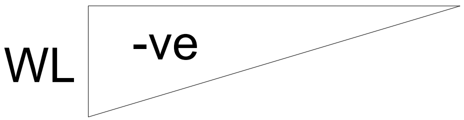
4.5 (Mohr's) Moment-Area Method

Numerical#1.



Solution:

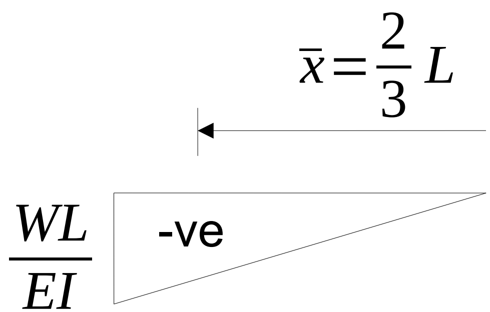
Step 1: Draw BMD for the given beam and loading condition



BMD

Step 2: Draw M/EI diagram using the BMD in step 1, where

$\bar{x} = \frac{2}{3}L$ is the c.g. of the M/EI diagram



M/EI diagram

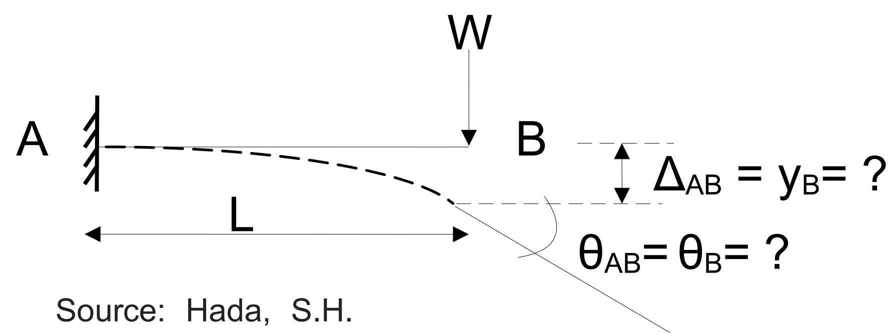
Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.

4.5 (Mohr's) Moment-Area Method

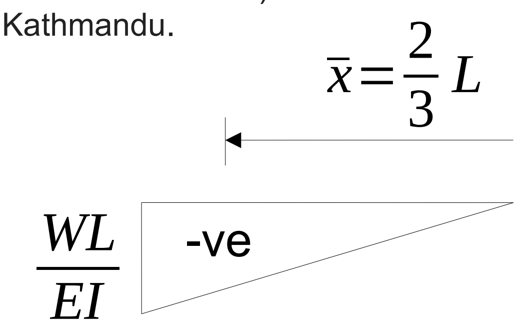
Numerical#1.

Solution:

Step 3: Calculate the slope change between points A and B, using Moment-area theorem 1, θ_{AB}. Since slope at A = 0 [Fixed support], $\theta_{AB} = \theta_B$



Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.



M/EI diagram

$$\theta_{AB} = \theta_B = \int_A^B \frac{M}{EI}\,dx$$

$$\theta_B = \int_A^B \frac{M}{EI}\,dx = Area\ of\ M/EI\ diagram\ between\ points\ A\ \text{and}\ B$$

$$\theta_B = -\frac{1}{2} * L * \frac{WL}{EI}$$

[Area of triangle = 1/2*b*h;
Negative sign because -ve M/EI diagram]

4.5 (Mohr's) Moment-Area Method

Numerical#1.

Solution:

Step 3:

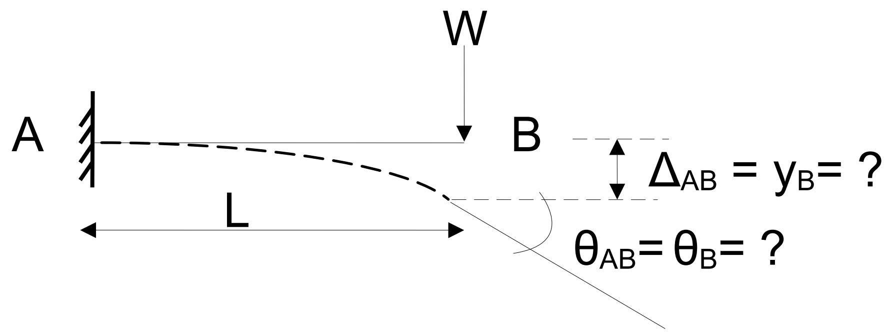
$$\theta_B = -\frac{1}{2} * L * \frac{WL}{EI}$$

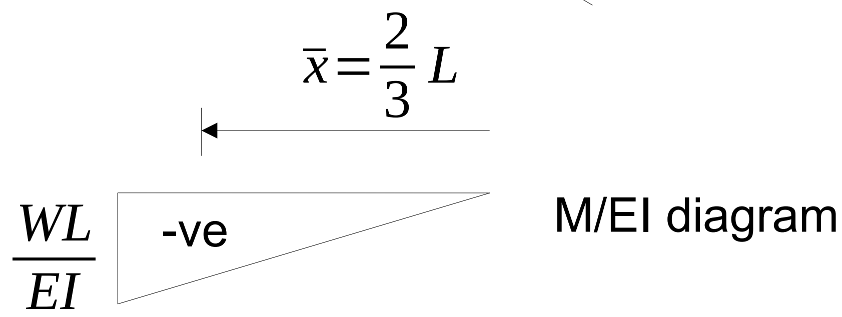
[Area of triangle = 1/2*b*h]

$$\theta_B = \frac{-WL^2}{2EI}$$

$$\theta_B = \frac{WL^2}{2EI} radians \quad \text{(Clockwise)}$$

Ans.





Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.

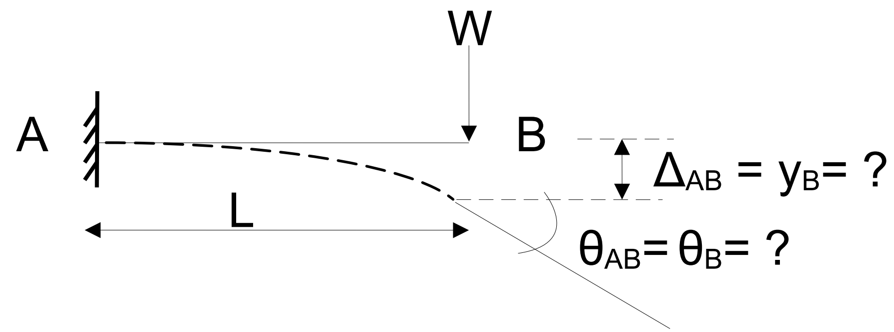
4.5 (Mohr’s) Moment-Area Method

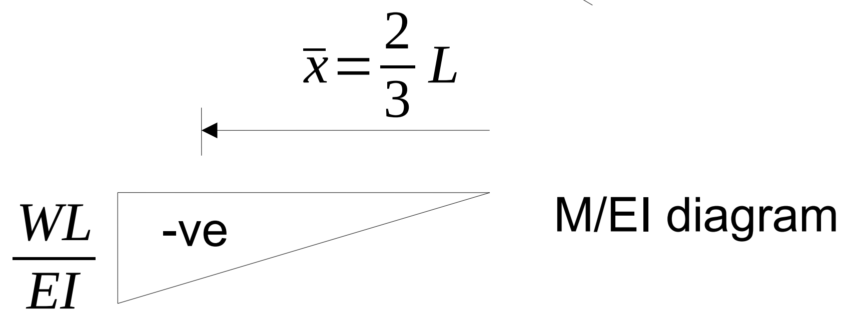
Numerical#1.

Solution:

Step 4: Calculate the deflection between points A and B, using Moment-area theorem 2, Δ_{AB}. Since deflection at A = 0 [Fixed support], $\Delta_{AB} = \Delta_B$.

$$\Delta_{AB} = \Delta_B = \int_A^B \frac{M}{EI} x\, dx$$





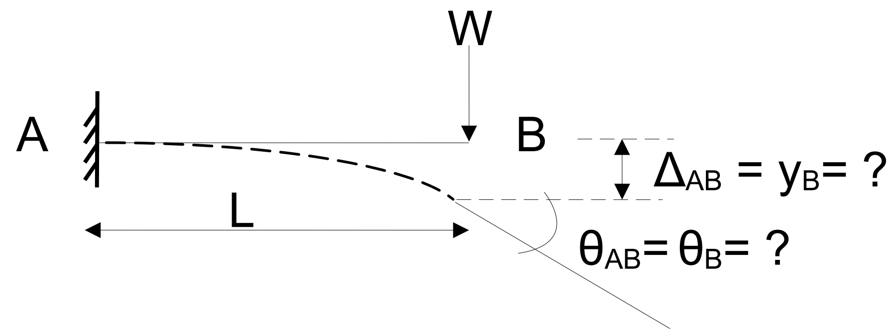
Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.

$$\Delta_B = \int_A^B \frac{M}{EI} x\, dx$$

= Moment of area of M/EI diagram between points A "and" B about point B

4.5 (Mohr's) Moment-Area Method

Numerical#1.

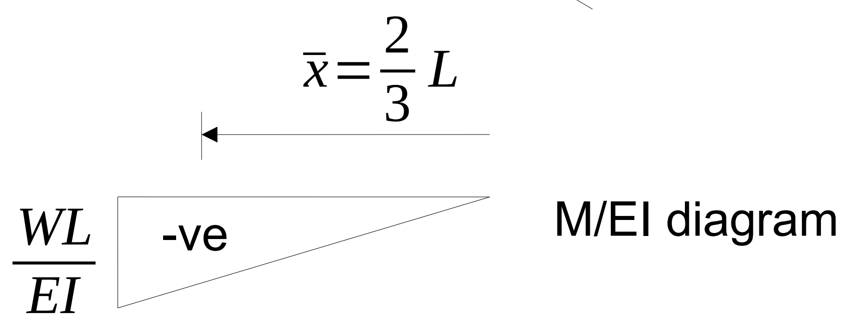


Solution:

Step 4:

$$\Delta_B = \int_A^B \frac{M}{EI}\, x\, dx$$

= Moment of area of M/EI diagram between points A "and" B about point B



$$\Delta_B = -\left(\frac{1}{2} * L * \frac{WL}{EI}\right) * \frac{3}{2} L$$

$$\Delta_B = \frac{-WL^3}{3EI} = \frac{WL^3}{3EI}\,(\;)$$

Ans.

Source: Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.

4.6 Conjugate Beam Method

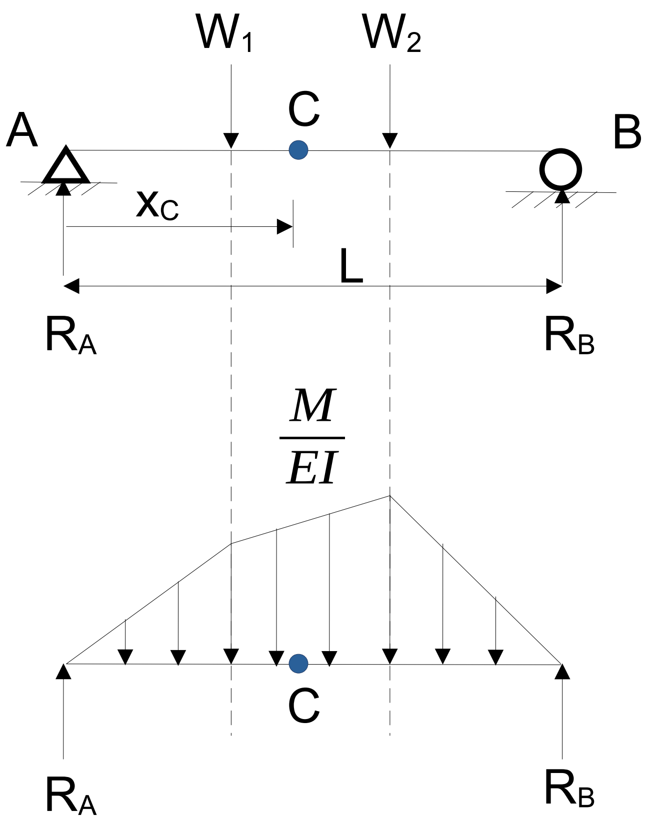
Conjugate beam theorems:
- Derived from moment-area theorems
- Very useful in finding deflection even if there is no point in the beam where the slope is zero.

In Moment-area method, both the equations for slope and deflections are for between two points. So, if we need to find the slope and/or deflection at a point, we need to carefully select the other point where the slope and/or deflection is zero. The conjugate beam method overcomes this limitation of the Moment-area method.

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

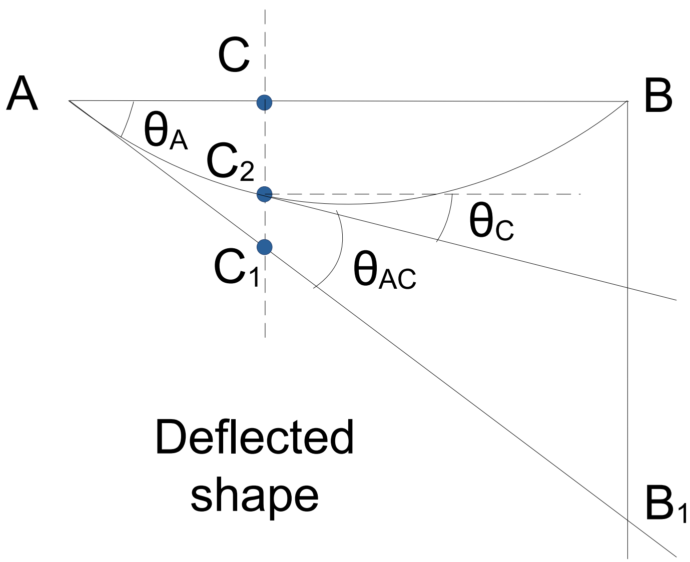
4.6 Conjugate Beam Method

Conjugate beam theorems:



$\frac{M}{EI}$ *diagram*

Let us consider a simply supported beam as shown in the adjacent figure, whose M/EI diagram and deflected shape are shown below.

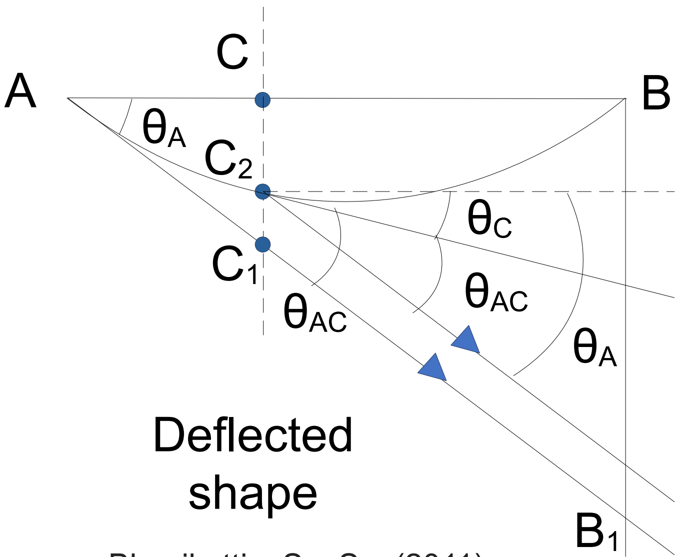


Deflected shape

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

4.6 Conjugate Beam Method

Conjugate beam theorems:



Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

Then, from geometry,

$$\theta_{AC} = \theta_A - \theta_C$$

Or, $\theta_C = \theta_A - \theta_{AC}$

From Theorem 1 of Moment-area theorems,

$$\theta_C = \theta_A - Area\ of\ M/EI\ diagram\ between\ A\ \text{and}\ C$$

From above figure,

$$\theta_A = \frac{BB_1}{AB} = \frac{Deflection\ of\ B\ w.r.t.\ tangent\ at\ A}{L}$$

4.6 Conjugate Beam Method

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

Conjugate beam theorems:

$$\theta_A = \frac{BB_1}{AB} = \frac{Deflection\, of\; B\, w.r.t.\, tangent\; at\; A}{L}$$

From Theorem 2 of Moment-area theorems,

$$\theta_A = \frac{BB_1}{AB} = \frac{1}{L}\left[Moment\; of\; area\; of\; M/EI\; diagram\; between\; A \text{ and } B\; about\; B\right]$$

Then,

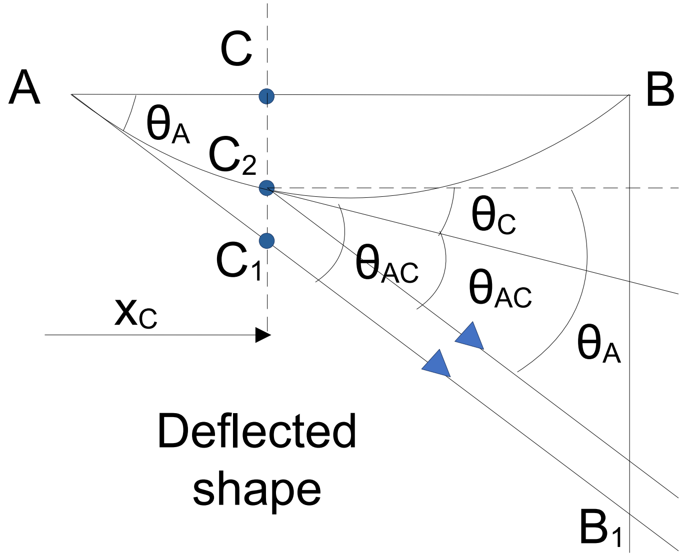
$$\theta_C = \theta_A - Area\; of\; M/EI\; diagram\; between\; A \text{ and } C$$

Becomes,

$$\theta_C = \frac{1}{L}\left\{\text{Moment of area of M/EI diagram between A and B about B}\right\} - \left\{\text{Area of M/EI diagram between A and C}\right\} \quad ---(A)$$

4.6 Conjugate Beam Method

Conjugate beam theorems:



Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

Now,
Deflection at C = $CC_2 = CC_1 - C_2C_1$

From geometry,
$CC_1 = x_C.\theta_A$

And,
C_2C_1 = Deflection of C w.r.t. tangent at A

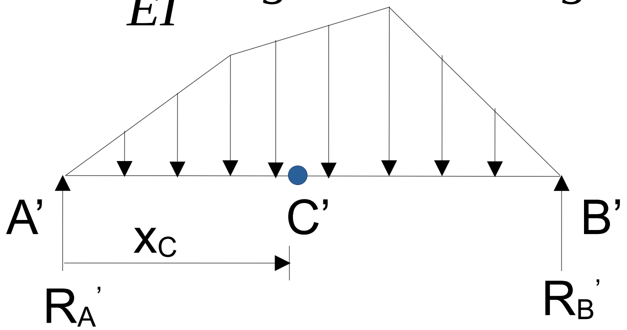
$$Deflection\ at\ C = x_C \frac{1}{L}\left(\begin{array}{c}\text{Moment of area of}\\ \text{M/EI diagram between}\\ \text{A and B about B}\end{array}\right) - \left(\begin{array}{c}\text{Moment of Area}\\ \text{of M/EI diagram}\\ \text{between A and C}\\ \text{about C}\end{array}\right) - - - (B)$$

4.6 Conjugate Beam Method

Conjugate beam theorems:

Now,
Consider an imaginary beam of the same span, loaded with M/EI diagram as shown below:

$$\frac{M}{EI}\,diagram = loading$$



Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

Then,
Reaction at A,

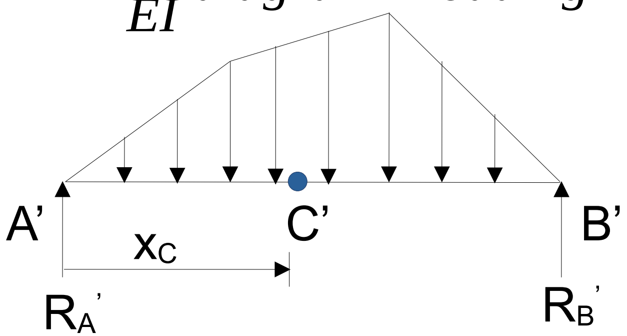
$$R_A' = \frac{Moment\ of\ loading\ about\ B}{L}$$

$$R_A' = \frac{Moment\ of\ area\ of\ M/EI\ diagram\ between\ A\ and\ B\ about\ B}{L}$$

4.6 Conjugate Beam Method

Conjugate beam theorems:

$$\frac{M}{EI}\,diagram = loading$$



Imaginary beam

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

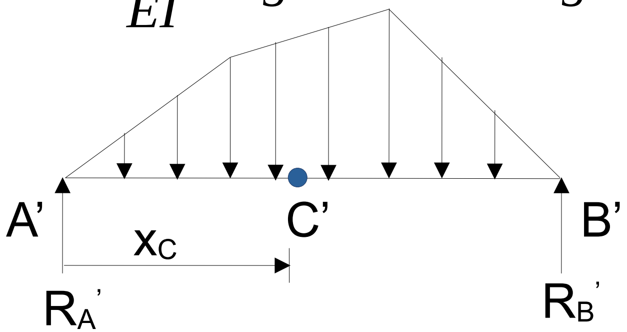
Then, Shear force at C = R_A'- Load between A and C

Shear force at C = $\frac{1}{L}$ (Moment of area of M/EI diagram between A and B about B) - (Area of M/EI diagram between A and C) ---- (C)

4.6 Conjugate Beam Method

Conjugate beam theorems:

$$\frac{M}{EI}\, diagram = loading$$



Imaginary beam

Source: Bhavikatti, S. S. (2011). *Structural Analysis – I* (4th ed.).New Delhi: Vikas Publishing House.

Then, Bending moment at C = $R_A' * x_C$ - Moment of Load between A and C about C

B.M. at C = $x_C * \frac{1}{L}$ (Moment of area of M/EI diagram between A and B about B) - (Moment of Area of M/EI diagram between A and C about C) ---- (D)

4.6 Conjugate Beam Method

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

Conjugate beam theorems:

Now, comparing equations (A) and (C)

θ_C in the given beam	=	Shear Force in the imaginary beam loaded with M/EI diagram	: Theorem 1

And, comparing equations (B) and (D)

Deflection at C in the given beam	=	Bending Moment in the imaginary beam loaded with M/EI diagram	: Theorem 2

This imaginary beam is called **conjugate beam**.

4.6 Conjugate Beam Method

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

Conjugate beam theorems:

Theorem 1

θ_C in the given beam	=	Shear Force in the imaginary beam loaded with M/EI diagram

The rotation at a point in a beam is equal to the shear force at the corresponding point in the conjugate beam.

Theorem 2

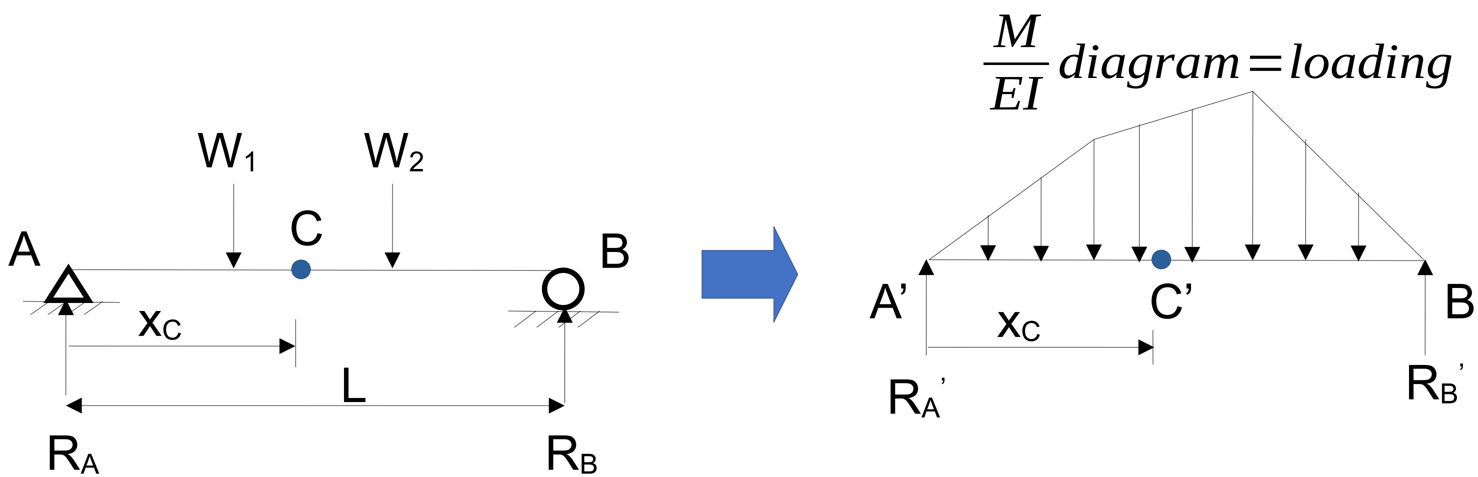
Deflection at C in the given beam	=	Bending Moment in the imaginary beam loaded with M/EI diagram

The deflection at a point in a beam is equal to the bending moment at the corresponding point in the conjugate beam.

4.6 Conjugate Beam Method

Conjugate beam:

It is an imaginary beam of the same span as the original beam, but loaded with M/EI diagram of the original beam, with suitable changes in the boundary conditions, such that the shear force and bending moment at a section will represent the rotation and deflection at that section in the original beam.



Original beam

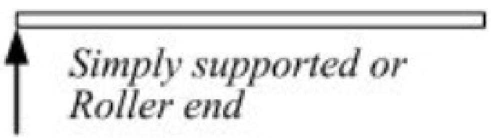
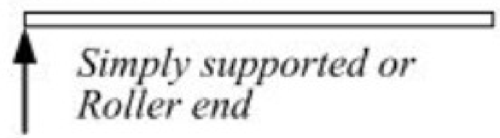
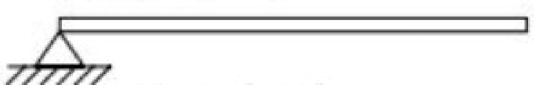
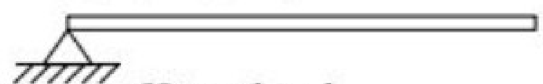
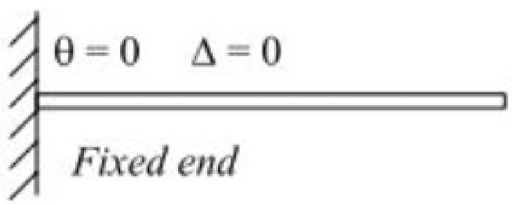
Conjugate beam

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

4.6 Conjugate Beam Method

Conjugate beam:

Table 1: Support/end conditions in original and corresponding conjugate beams

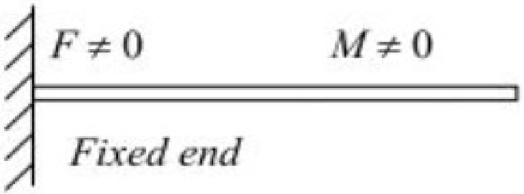
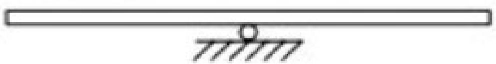
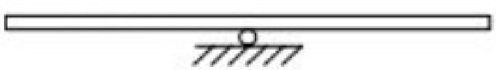
S.No.	Original beams	Conjugate beams
1.	$\theta \neq 0 \quad \Delta = 0$  *Simply supported or Roller end*	$F \neq 0 \quad M = 0$  *Simply supported or Roller end*
2.	$\theta \neq 0 \quad \Delta = 0$  *Hinged end*	$F \neq 0 \quad M = 0$  *Hinged end*
3.	$\theta = 0 \quad \Delta = 0$  *Fixed end*	$F = 0 \quad M = 0$  *Free end*

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

4.6 Conjugate Beam Method

Conjugate beam:

Table 1: Support/end conditions in original and corresponding conjugate beams

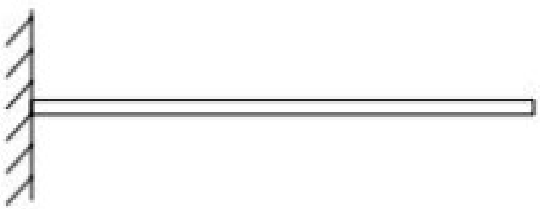
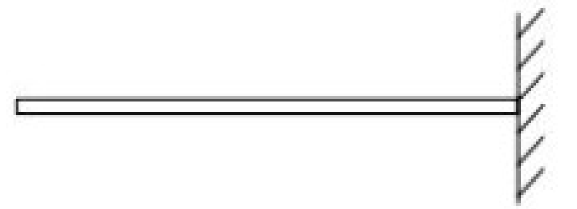
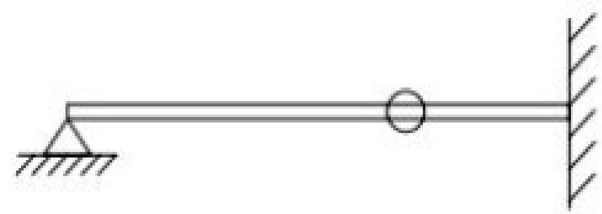
S.No.	*Original beams*	*Conjugate beams*
4.	$\theta \neq 0$ $\Delta \neq 0$  *Free end*	$F \neq 0$ $M \neq 0$  *Fixed end*
5.	$\theta \neq 0$ and continuous $\Delta = 0$  *Interior support*	$F \neq 0$ and continuous $M = 0$  *Interior hinge*
6.	$\theta \neq 0$ and discontinuous $\Delta \neq 0$  *Interior hinge*	$F \neq 0$ and discontinuous $M \neq 0$  *Interior support*

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

4.6 Conjugate Beam Method

Conjugate beam:

Table 2: Original and corresponding conjugate beams

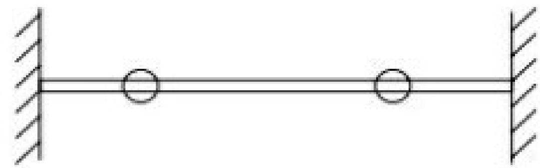
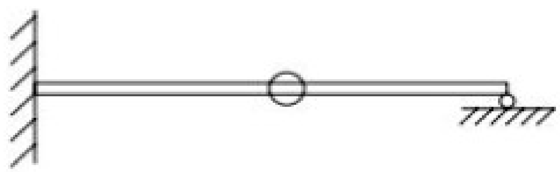
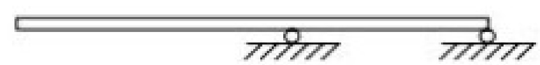
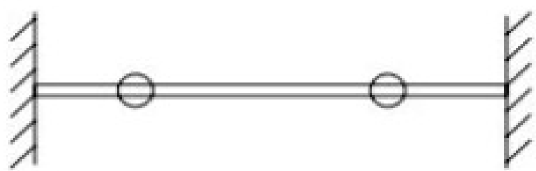
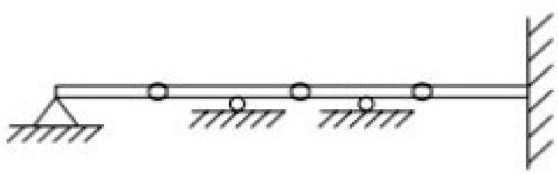
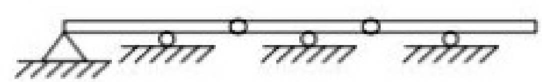
S.No.	Original beams	Conjugate beams
1.		
2.		
3.		
4.		

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

4.6 Conjugate Beam Method

Conjugate beam:

Table 2: Original and corresponding conjugate beams

S.No.	*Original beams*	*Conjugate beams*
5.		
6.		
7.		
8.		

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

4.6 Conjugate Beam Method

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

Sign conventions:

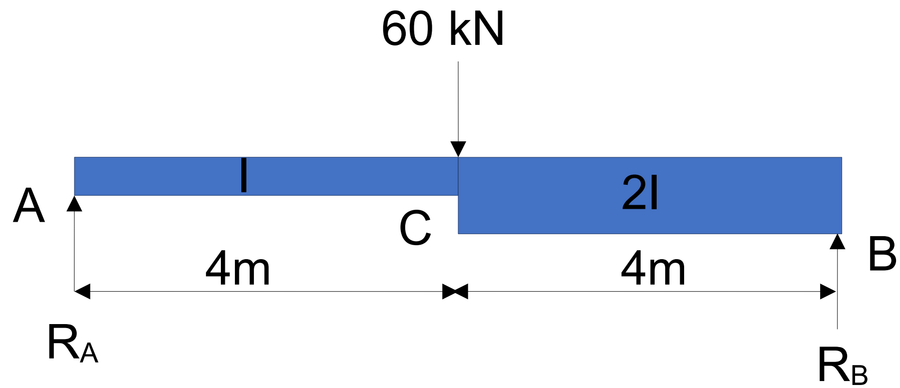
1. Sagging moment is positive moment and it gives downward deflection.
2. Left-side upward force or right-side downward force gives positive shear, which gives clockwise rotation.

Relationship between real beam and its conjugate beam

1. Span of the real and the conjugate beams are equal.
2. The M/EI diagram of the real beam becomes the load diagram of the conjugate beam
3. The shear at any section of the conjugate beam is equal to the slope of the real beam.
4. The bending moment at any section of the conjugate beam is equal to the deflection of the real beam.

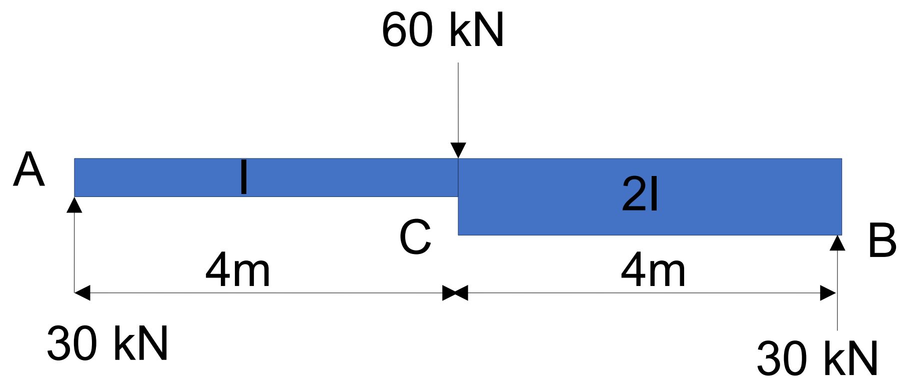
4.6 Conjugate Beam Method

Numerical#2. Determine θ_A, θ_B, θ_C, and deflection Δ_C in the given simply supported beam.

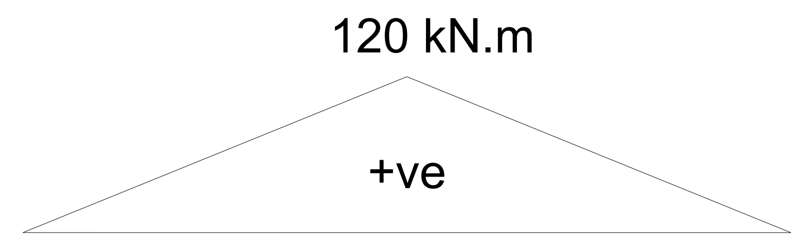


Solution:

Step 1: Calculate the reaction forces at the supports



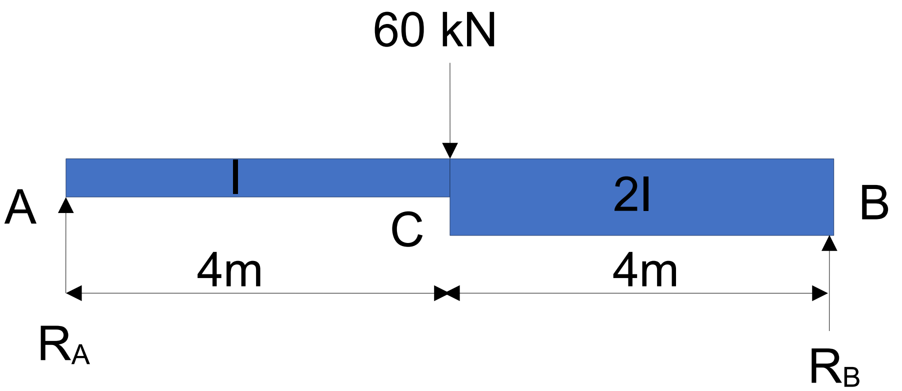
Step 2: Determine BMD of the given beam



Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

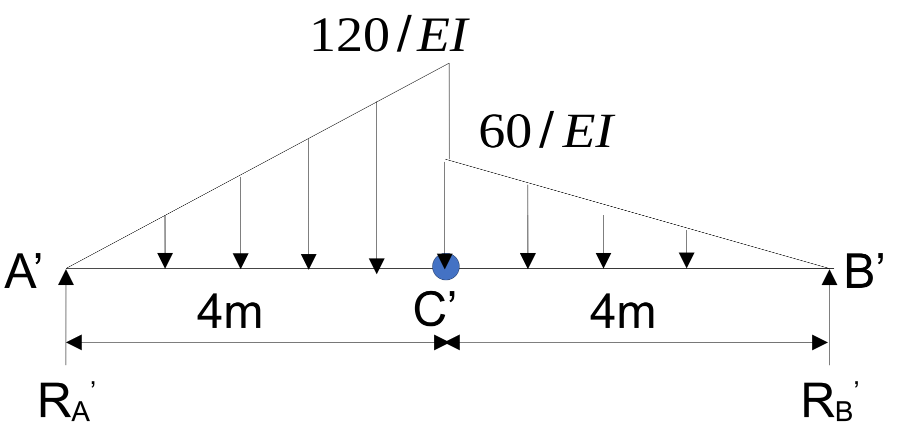
4.6 Conjugate Beam Method

Numerical#2. Determine θ_A, θ_B, θ_C, and deflection Δ_C in the given simply supported beam.



Solution:

Step 3: Construct the conjugate beam for the original beam using M/EI as loading and corresponding support conditions.



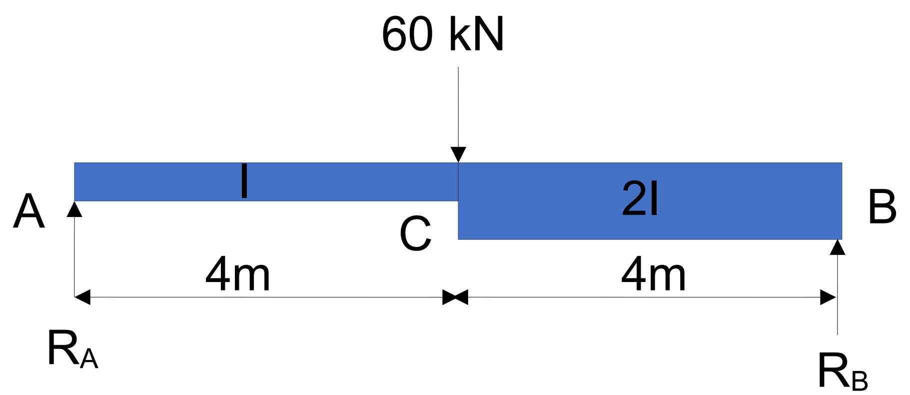
Conjugate beam

Note: For span A'C', the MOI of the section is I. Hence, M/EI =120/EI. For span C'B', the MOI of the section is 2I. Hence, M/2EI = 120/2EI = 60/EI.

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

4.6 Conjugate Beam Method

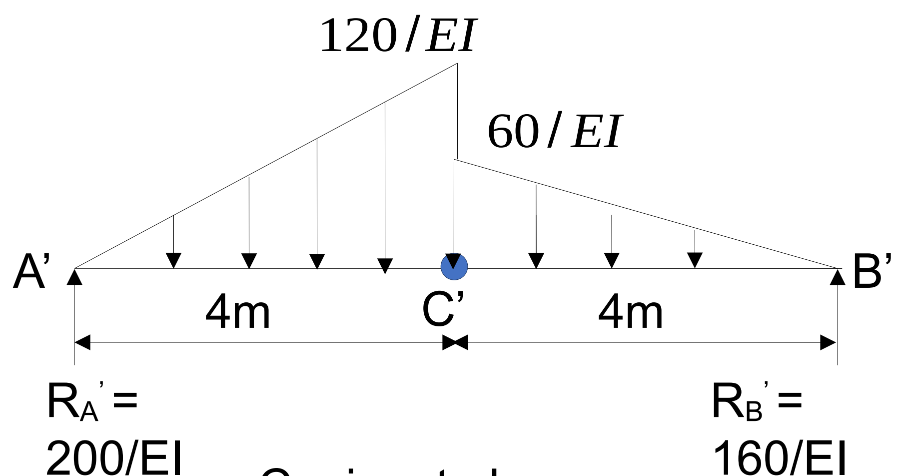
Numerical#2. Determine θ_A, θ_B, θ_C, and deflection Δ_C in the given simply supported beam.



Solution:

Step 4: Solve the conjugate beam for support reactions, shear force at required points A,B, and C, and bending moment at C.

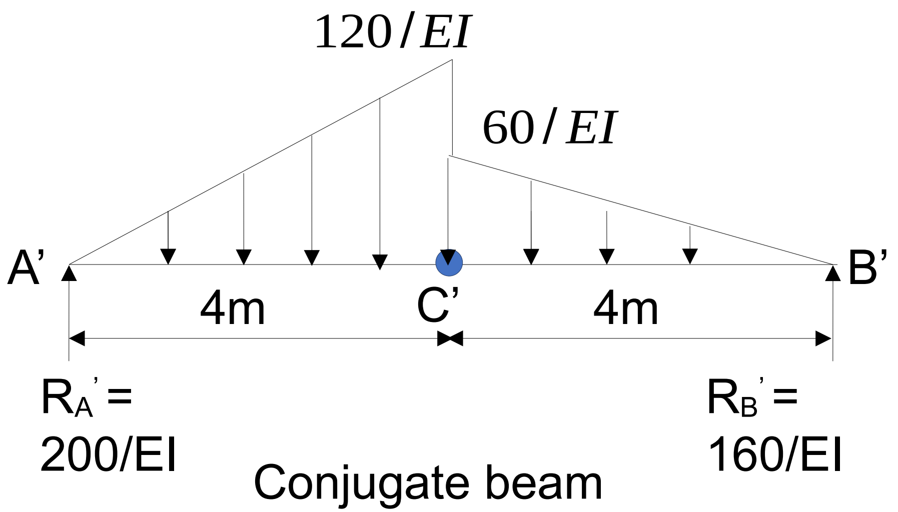
4.1 Compute support reactions R_A' and R_B' using equations of static equilibrium.



Conjugate beam

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

4.6 Conjugate Beam Method



Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

Numerical#2.

Solution:

Step 4:

4.2 Compute shear forces at A, B, and C of the conjugate beam.

$$\theta_A = Shear\ Force\ at\ A = R_A' = \frac{200}{EI} radians\ (clockwise)$$ Ans.

$$\theta_B = Shear\ Force\ at\ B = R_A' - \frac{1}{2} * 4 * \frac{120}{EI} - \frac{1}{2} * 4 * \frac{60}{EI}$$

$$\theta_B = \frac{-160}{EI} = \frac{160}{EI} radians\ (anti - clockwise)$$ Ans.

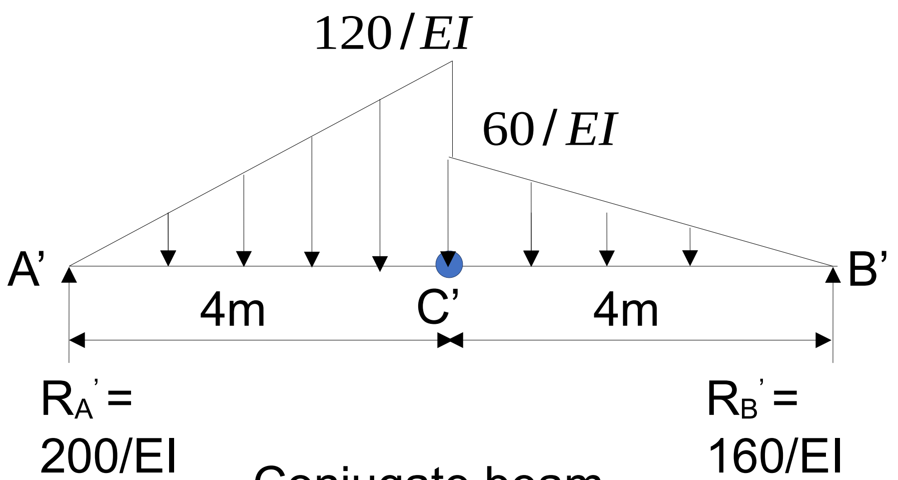
4.6 Conjugate Beam Method

Numerical#2.

Solution:

Step 4:

4.2 Compute shear forces at A, B, and C of the conjugate beam.



Conjugate beam

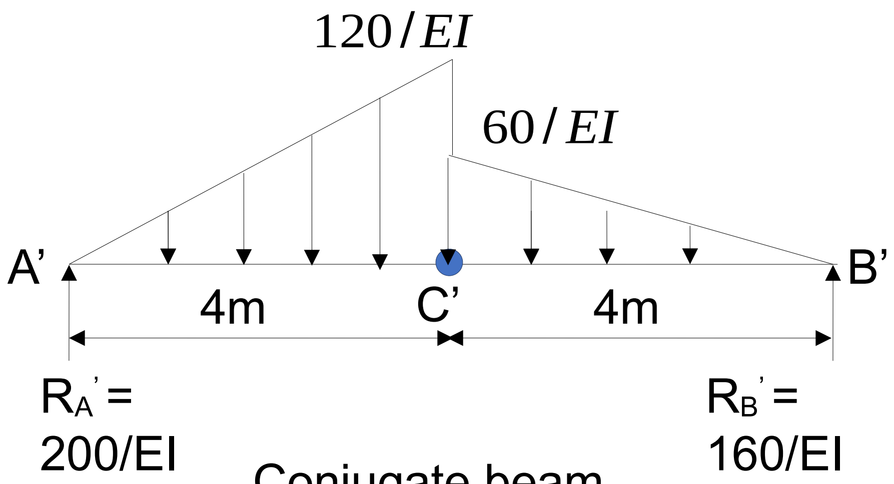
Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

$$\theta_C = Shear\ Force\ at\ C = R'_A - \frac{1}{2} * 4 * \frac{120}{EI}$$

$$\theta_C = \frac{-40}{EI} = \frac{40}{EI}\ radians\,(anti-clockwise)$$

Ans.

4.6 Conjugate Beam Method



Conjugate beam

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

Numerical#2.

Solution:

Step 4:

4.3 Compute bending moment at A C of the conjugate beam.

$$\Delta_C = Bending\ Moment\ at\ C = R'_A * 4 - \frac{1}{2} * 4 * \frac{120}{EI} * \left(\frac{1}{3} * 4\right)$$

$$\Delta_C = \frac{200}{EI} * 4 - \frac{1}{2} * 4 * \frac{120}{EI} * \left(\frac{1}{3} * 4\right)$$

$$\Delta_C = \frac{480}{EI}\left(downward\right)$$

Ans.

------------End of Lecture#6-----------------

------------End of Part II of II for Chapter4------------

References

[1] Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.
[2] Hada, S.H. *Theory of Structures-I* (Manual for IOE students) . Kathmandu.