Theory of Structures - I

Chapter 6. Statically Determinate Arches [Part I of II]

Lecturer: Dr. Sanjeema Bajracharya

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6.1 Types of Arch
6.2 Three-hinged arches with supports at the same and different levels
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Part-I

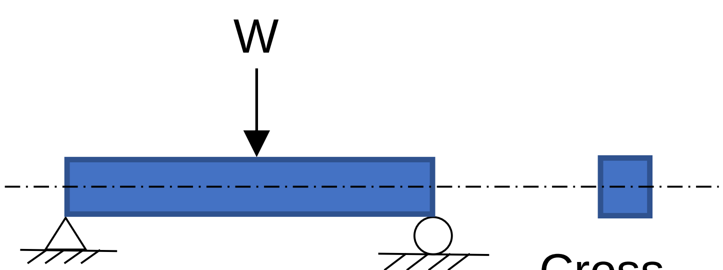
6.1 Types of Arch

Why an arch?

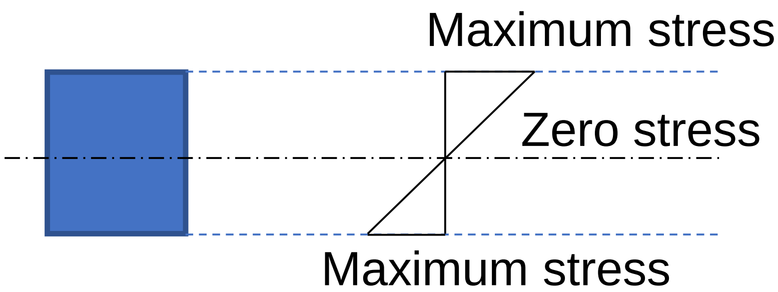
Beams → Transfer the applied load(s) to the end supports by bending and shear

→ In doing so, usually either one or two points at a particular section are subjected to maximum stress, as shown by the points of zero and maximum stress locations along the depth of the cross-section of the beam, which is usually uniform and prismatic.

→ In such cases, material in most of the portion are *under-stressed* and *under-utilized*.



Cross-section



Uniform Cross-section of a beam

Stress distribution along the depth of the cross-section

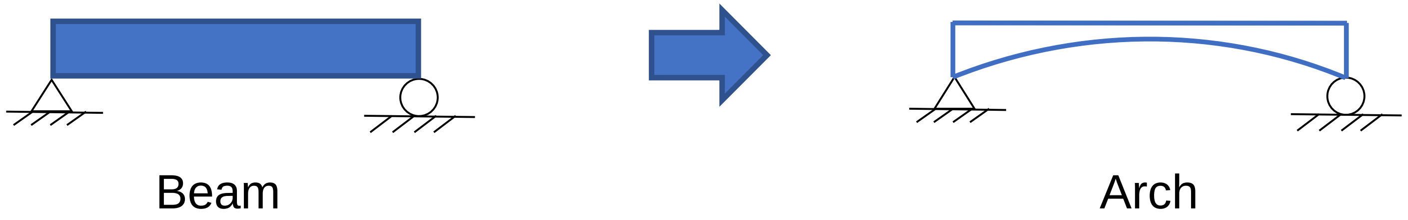
Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

6.1 Types of Arch

Why an arch?

Beams → In addition, for large spans, the self-weight of the beams contribute to the stress in such large proportions that it becomes very difficult to design beams for larger spans like in case of bridges and hence, the *beam becomes very uneconomical*.

Hence, for large spans like bridges, arches are provided instead of beams.



Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

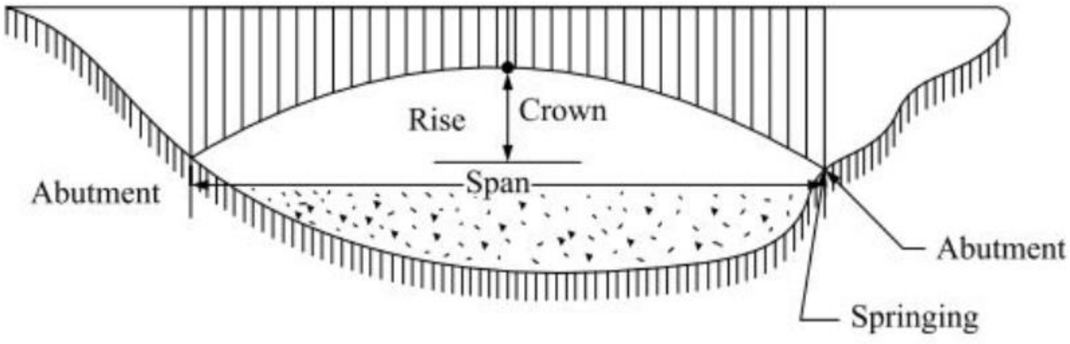
6.1 Types of Arch

What is an arch?

Arches are nothing but curved beams (usually in the vertical plane) that transfer loads to their plane.

Anatomy of an arch

1. Arches transfer loads to *abutments* at *springing points* (where, hinges may be provided).
2. The topmost part is called the *crown*, which sometimes has a hinge.

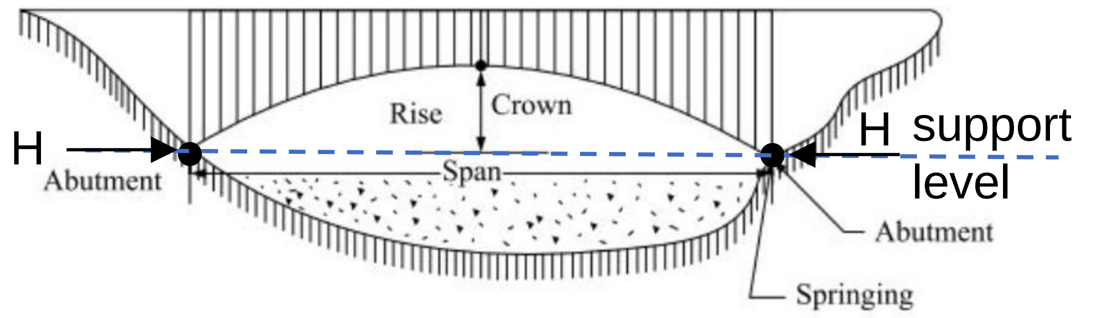


Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

6.1 Types of Arch

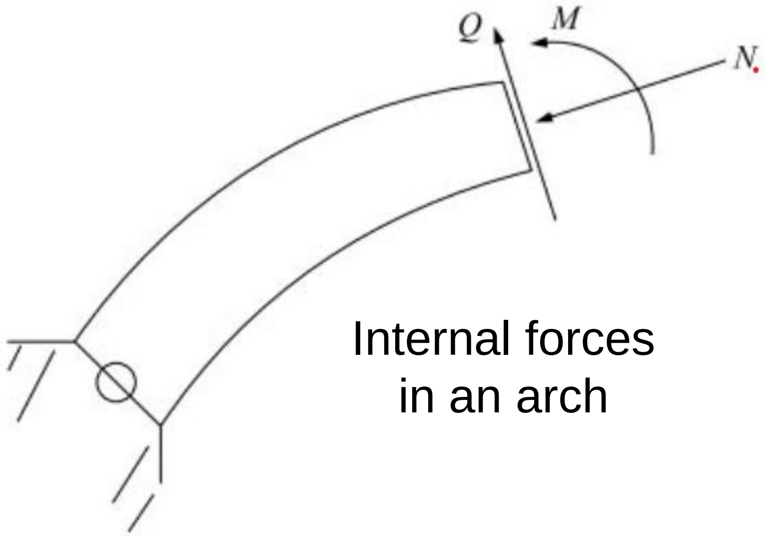
Anatomy of an arch

3. The height of the crown above the support level is a called *rise*.
4. Because of the curved nature of arches, they give rise to horizontal forces (H).



5. Any section in the arch will be subjected to
   I. Normal thrust (N),
   II. Radial shear (Q), and
   III. Bending moment (M)

   as shown in the adjacent figure.



Internal forces in an arch

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

6.1 Types of Arch

Anatomy of an arch

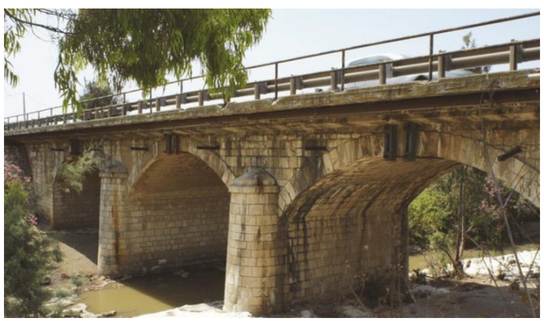
6. Loads get transferred partly by axial compression and partly by flexure.
7. In axial compression, each and every particle of the cross-section of the structure is subjected to stress equally. Hence, the material is fully utilized.
8. Reduction in the BM results in smaller sections for the arch compared to the section required fro beams to transfer the same load.

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

6.1 Types of Arch

[A] Based on materials used

1. Steel arch
2. Concrete arch
3. Masonry arch
4. Timber arch



Masonry arch bridge

Source: Kindij, A., Ivankovic, A. M., & Vasilj, M. (2013). Assessment of masonry arch bridge with concrete deck. In *7th International Conference on Arch Bridges, ARCH* (Vol. 13).

Bayonne bridge between New York and New Jersey (Steel arch bridge)

Source: https://upload.wikimedia.org/wikipedia/commons/5/56/High_BB_from_Bayonne_jeh.jpg

Bijuli bajar concrete arch bridge, Kathmandu.

Source: https://www.youtube.com/watch?v=Spe1Stgeqyg

Eagle river timber arch bridge

Source: https://en.wikipedia.org/wiki/Eagle_River_Timber_Bridge#/media/File:Eagle_River_Timber_Bridge.JPG

6.1 Types of Arch

[B] Based on geometry

1. Parabolic arch
2. Circular arch
3. Elliptical arch
4. Composite arch

Bridge 812 at Birchington-on-Sea (semi-elliptical arch)

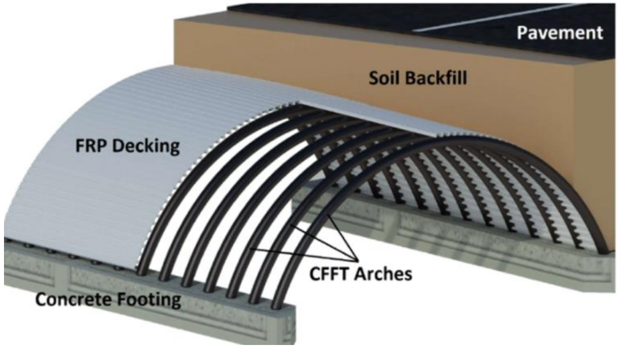
Source: https://sremg.org.uk/structures/struct_07.html

Bixby Creek bridge, California (parabolic arch)

Source: https://en.wikipedia.org/wiki/Parabolic_arch#/media/File:ML_Bixby_Creek_Bridge.JPG

São Gonçalo Bridge, Portugal (semi-circular arch)

Source: https://structurae.net/en/structures/sao-goncalo-bridge



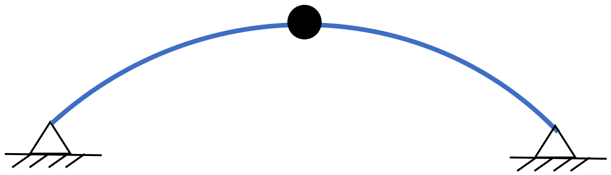
Buried, composite arch bridge

Source: Majeed, H. S. (2019). *Development of Finite Element Techniques to Simulate Concrete-Filled Fiber-Reinforced Polymer Tube Structures*. The University of Maine.

6.1 Types of Arch

[C] Based on boundary conditions

1. Three-hinged arch
2. Two-hinged arch
3. Hingeless or Fixed arch



Three-hinged arch

Hingeless/ Fixed arch

Two-hinged arch

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

6.2 Three-hinged arches with supports at the **same** and different **levels**

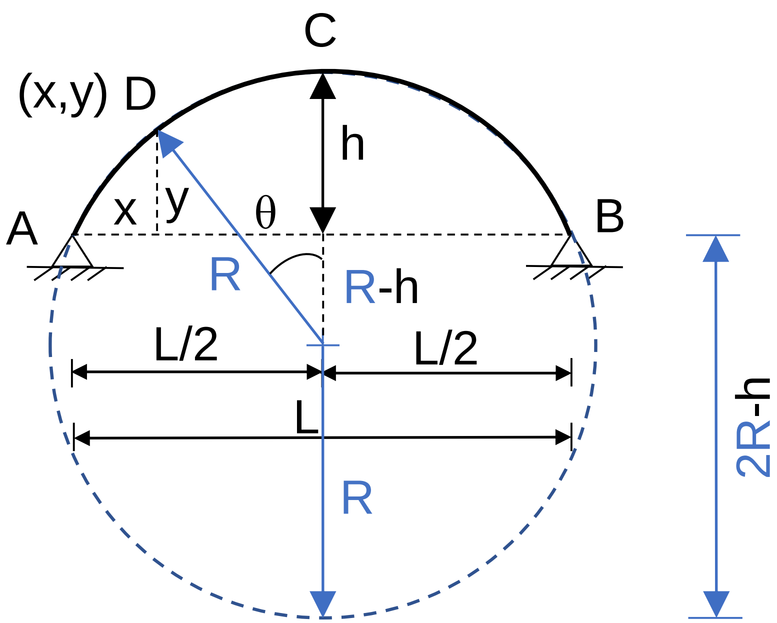
[A] Circular arch

Let us consider a circular arch of span L and rise h, with a radius R.

From the properties of a circle,

$$\frac{L}{2} * \frac{L}{2} = \mathrm{h} * (2\mathrm{R} - \mathrm{h})$$

$$Or, R = \frac{L^2}{8h} + \frac{h}{2}$$



Circular arch

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

6.2 Three-hinged arches with supports at the **same** and different **levels**

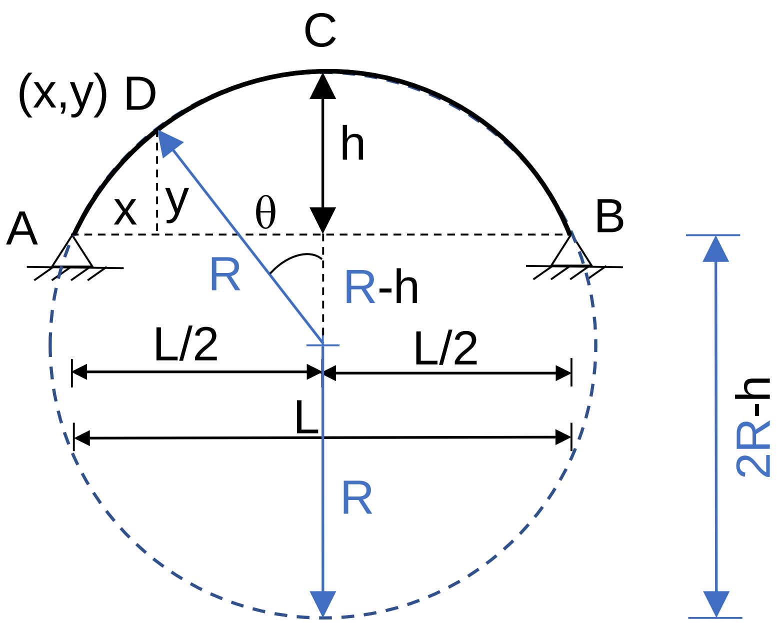
[A] Circular arch

Taking origin at support A, the coordinate of any point D on the arch is given by:

$$x = \frac{L}{2} - R\sin\theta$$

$$y = R\cos\theta - (R - h)$$

$$Or, y = h - R(1 - \cos\theta)$$



Circular arch

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4^{th} ed.).New Delhi: Vikas Publishing House.

6.2 Three-hinged arches with supports at the **same** and different **level**s

[B] Parabolic arch

The equation of the parabola is:

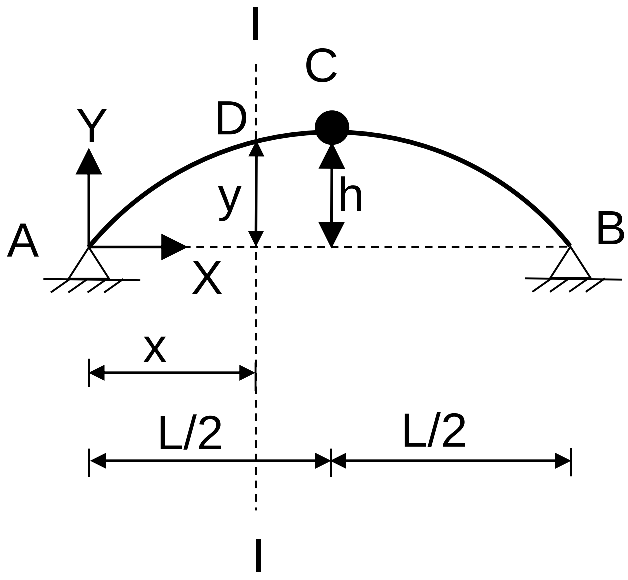
$$y = kx(L - x)$$

Taking origin at support A, we have,

At $x = \frac{L}{2}, y = h$.

Substituting in the equation of parabola, we get

$$h = k\frac{L}{2}\left(L - \frac{L}{2}\right) \Rightarrow k = \frac{4h}{L^2}$$



Parabolic arch

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

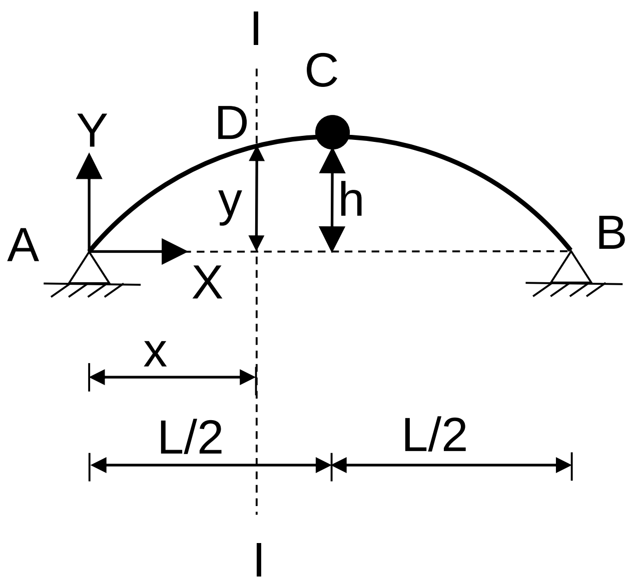
6.2 Three-hinged arches with supports at the **same** and different **level**s

[B] Parabolic arch

$$k = \frac{4h}{L^2}$$

Again, substituting the value of k in the equation of parabola, we get

$$y = \frac{4hx}{L^2}(L - x)$$



Parabolic arch

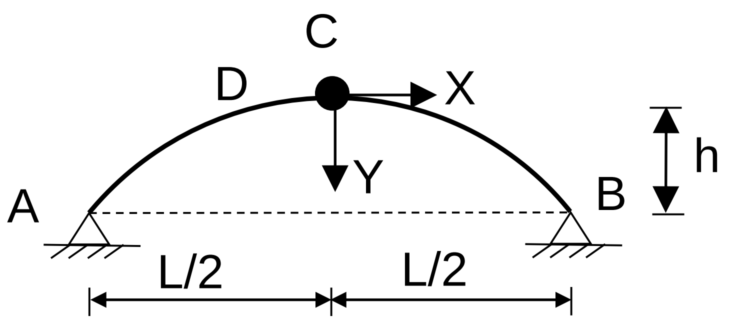
Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

6.2 Three-hinged arches with supports at the **same** and different **level**s

[B] Parabolic arch

If the crown is taken as the origin, the equation of parabolic curve becomes:

$$\frac{x^2}{y} = a = constant$$



Parabolic arch

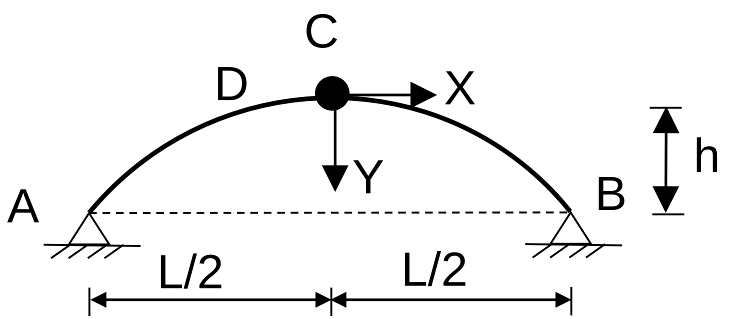
Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

6.2 Three-hinged arches with supports at the **same** and different **level**s

[B] Parabolic arch

(a) Springing points at the same level

If the crown is taken as the origin, the equation of parabolic curve becomes:



Parabolic arch

At $x = \frac{L}{2}, y = h$.

$$\frac{(\frac{L}{2})^2}{h} = a \Rightarrow a = \frac{L^2}{4\text{h}}$$

Substituting in the equation,

$$\frac{x^2}{y} = \frac{L^2}{4\text{h}}$$

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

6.2 Three-hinged arches with supports at the same and **different levels**

[B] Parabolic arch
(b) Springing points at different levels

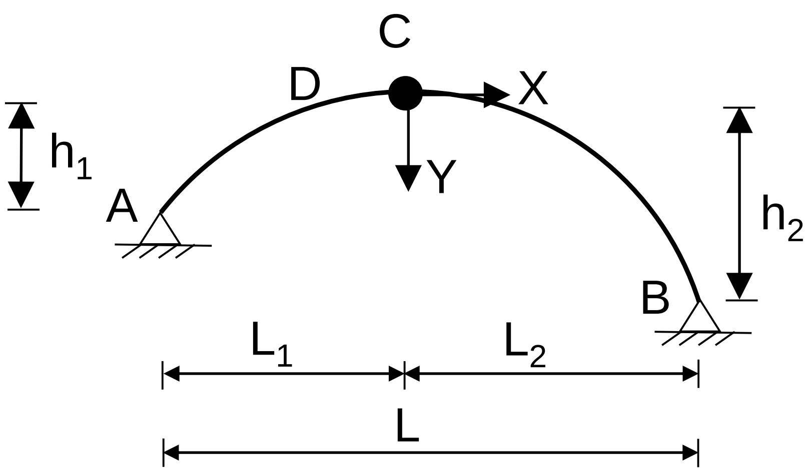
$$\frac{x^2}{y} = constant$$

Or, $\frac{x}{\sqrt{y}} = constant$

Applying this equation to points A and B, we get,

$$\frac{L_1}{\sqrt{h_1}} = \frac{L_2}{\sqrt{h_2}} = constant$$

$$\Rightarrow \frac{L_1+L_2}{\sqrt{h_1}+\sqrt{h_2}} = constant$$



Parabolic arch

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

6.2 Three-hinged arches with supports at the same and **different levels**

[B] Parabolic arch
(b) Springing points at different levels

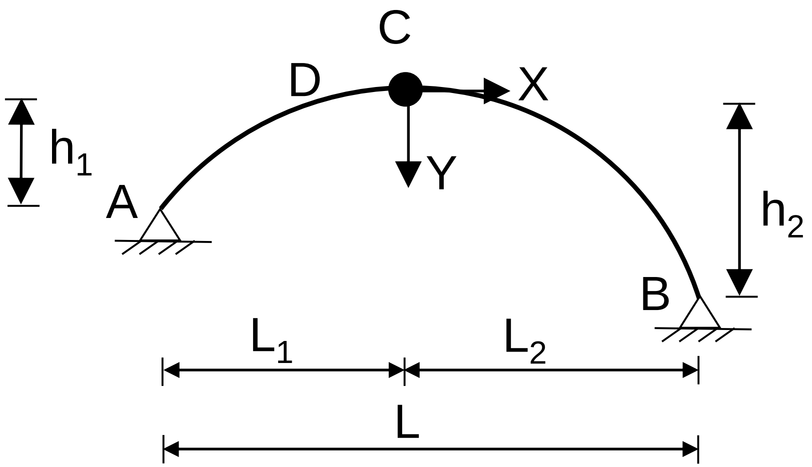
$$\Rightarrow \frac{L_1+L_2}{\sqrt{h_1}+\sqrt{h_2}} = constant$$

$$\Rightarrow \frac{L}{\sqrt{h_1}+\sqrt{h_2}} = constant$$

$$\Rightarrow \frac{L_1}{\sqrt{h_1}} = \frac{L_2}{\sqrt{h_2}} = \frac{L}{\sqrt{h_1}+\sqrt{h_2}} = constant$$

$$\Rightarrow L_1 = \frac{L\sqrt{h_1}}{\sqrt{h_1}+\sqrt{h_2}}$$

$$\Rightarrow L_2 = \frac{L\sqrt{h_2}}{\sqrt{h_1}+\sqrt{h_2}}$$



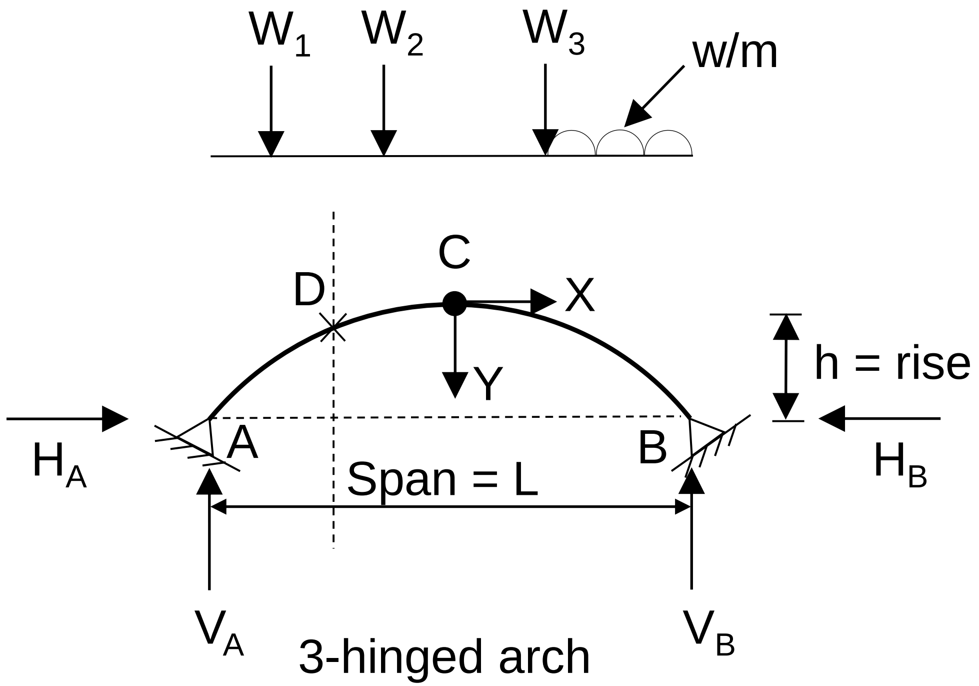
Parabolic arch

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

6.2 Three-hinged arches with supports at the **same** and different **levels**

Analysis for static loads

Let us consider a 3-hinged arch subjected to loads as shown in the figure.



We have,
the equations of equilibrium:

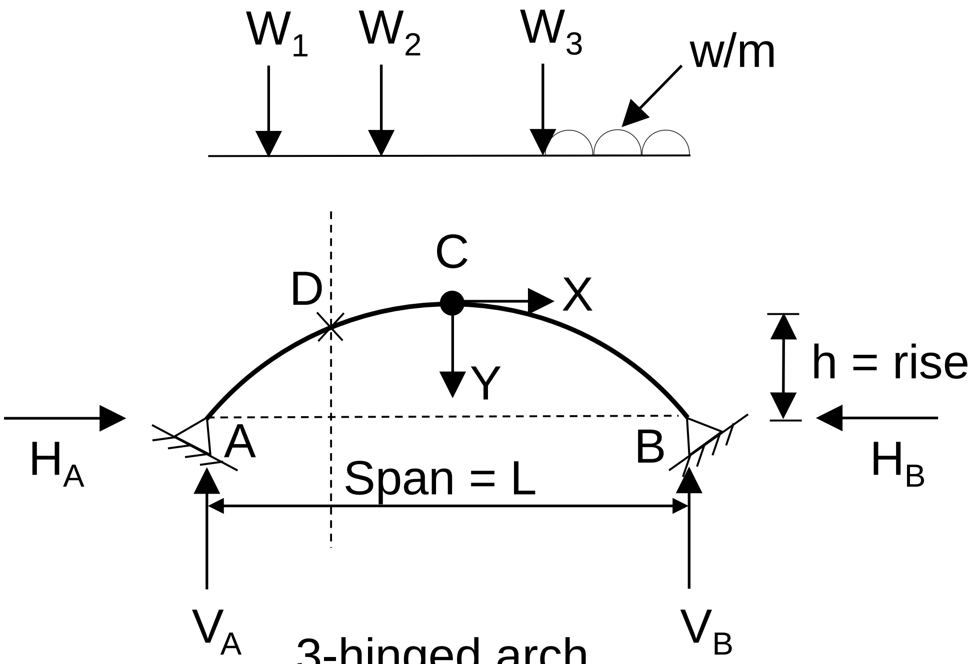
$\Sigma F_H = 0;\ \Sigma F_V = 0$
$\Sigma M_A\ or\ \Sigma M_B = 0$

Here, one additional equation is available, i.e.
$\Sigma M_C = 0$ [Since C is a hinge]

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

6.2 Three-hinged arches with supports at the **same** and different **levels**

Analysis for static loads



3-hinged arch

If no horizontal load is acting,

$$H_A = H_B$$

Since the loads tend to spread the arch, the horizontal thrust is in the inward direction.

Let us consider a section D in the arch, where the following internal forces are developed:

V= Vertical shear

Q=Radial shear

N=Normal thrust

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

6.2 Three-hinged arches with supports at the **same** and different **level**s

Analysis for static loads

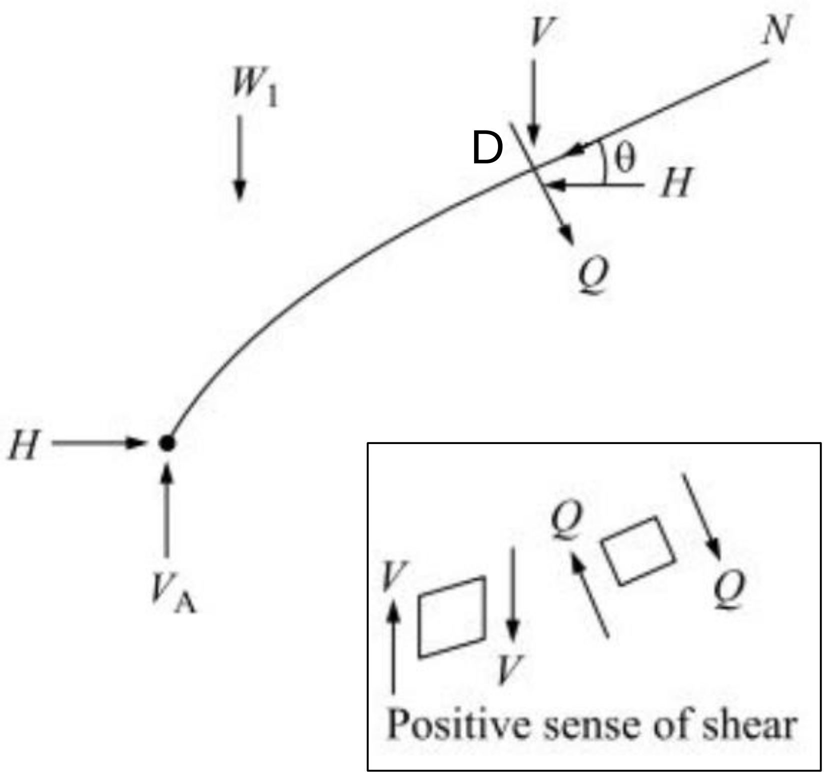
Let us consider a section D in the arch, where the following internal forces are developed:

V= Vertical shear

Q=Radial shear

N=Normal thrust

$H_A=H_B=H$



Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

6.2 Three-hinged arches with supports at the **same** and different **level**s

Analysis for static loads

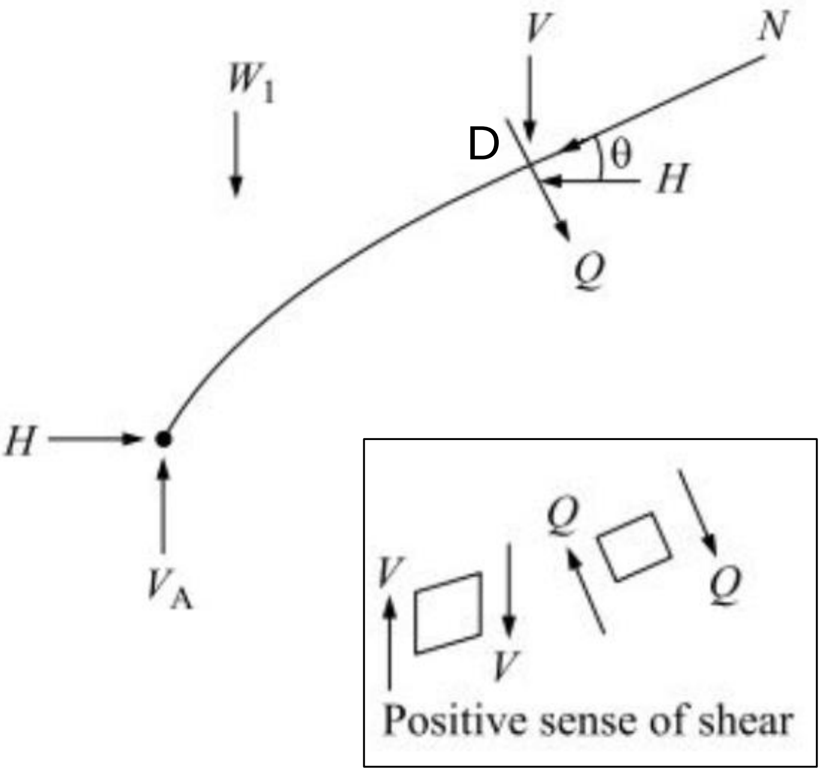
Let the normal to the section make an angle θ with the horizontal.
Then,

Normal thrust:
N = H cosθ + V sinθ

Radial shear:
Q = Vcosθ – Hsinθ

Vertical shear:
V = same as in simply supported beam

Bending moment:
M = M for beam – H*y

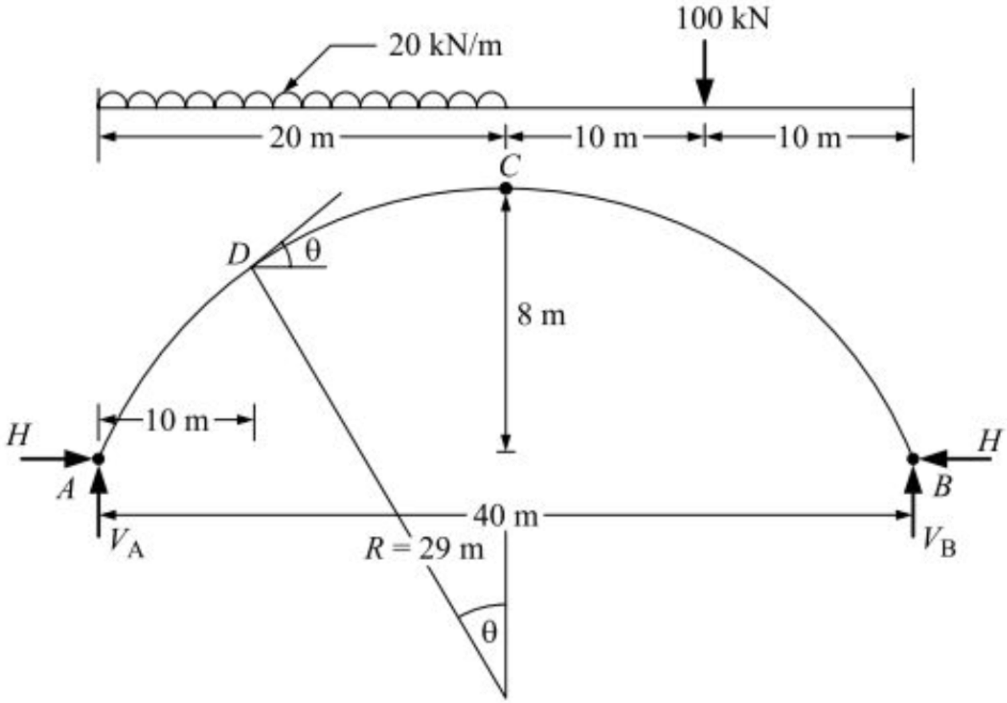


Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

6.3 Determination of support reactions, shear forces, normal forces, and bending moments

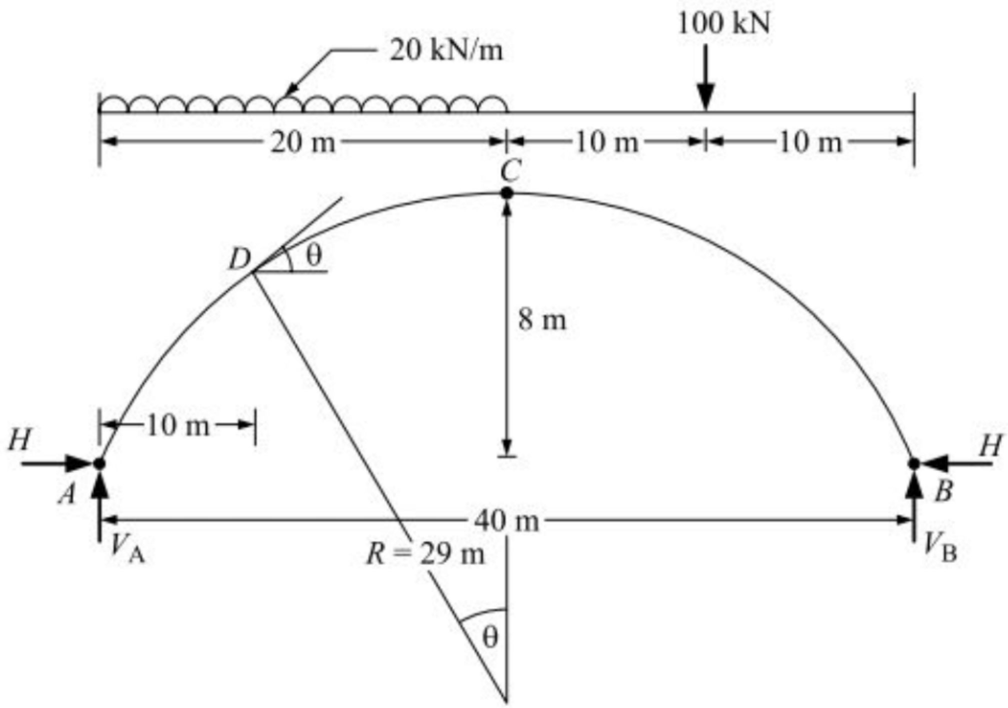
Numerical #1. A 3-hinged circular arch having supports at same level, has a span of 40 m and a central rise of 8 m. It carries a UDL of 20 kN/m over the left half of the span together with a concentrated load of 100 kN at the right quarter span point. Find the reaction at the supports, normal thrust, and radial shear developed at a section 10 m from the left support.

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.



6.3 Determination of support reactions, shear forces, normal forces, and bending moments

Numerical #1.



Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

Solution:

We have,
$\Sigma M_B = 0$

Or, $V_A * 40 - 20 * 20 * \left(20 + \frac{20}{2}\right) - 100 * 10 = 0$

Or, $V_A = 325\ kN$

Then,
$\Sigma F_V = 0$
=> $V_A + V_B = 20 * 20 + 100$
$\Rightarrow V_B$ = 500 - V_A = 500 – 325
$\Rightarrow V_B = 175\ kN$

6.3 Determination of support reactions, shear forces, normal forces, and bending moments

Numerical #1.

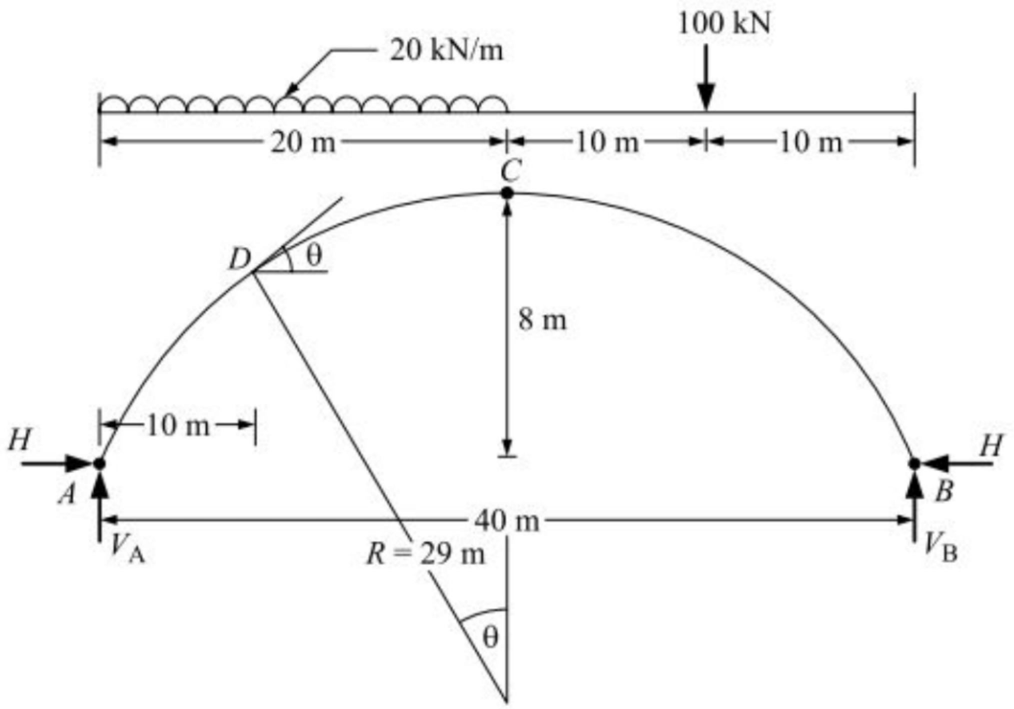
Solution:

Since C is hinged,
$\Sigma M_{C(left)} = 0$

Or, $V_B * 20 - 100 * 10 - H * 8 = 0$

Or, $175 * 20 - 100 * 10 - H * 8 = 0$

$\Rightarrow H = 312.5 \text{ kN}$



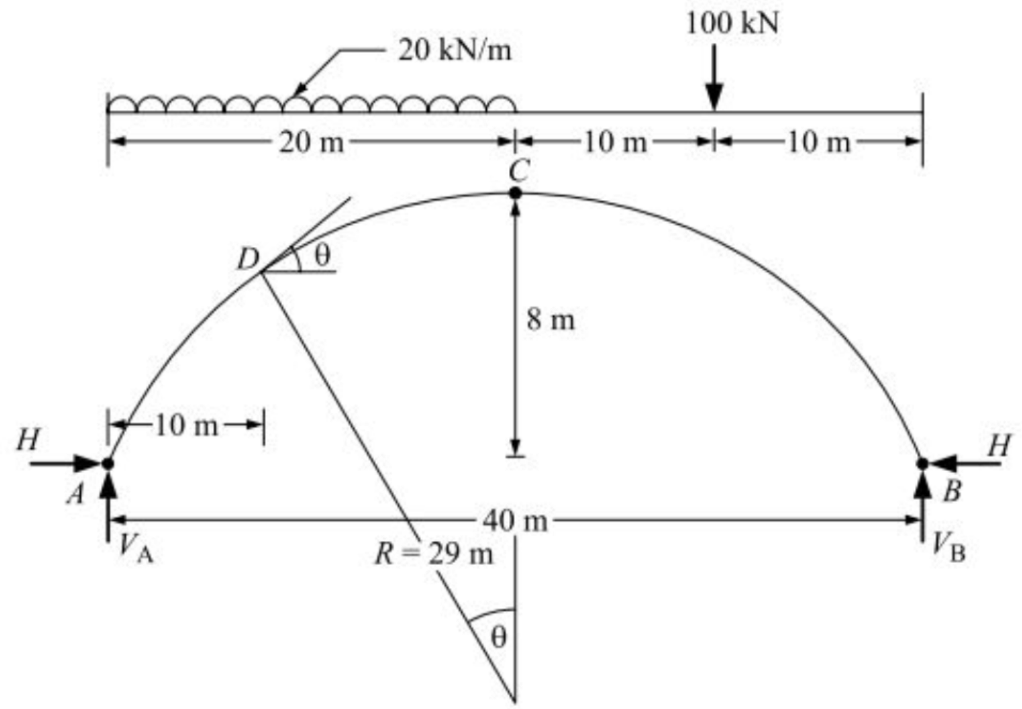
Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

6.3 Determination of support reactions, shear forces, normal forces, and bending moments

Numerical #1.

Solution:

Let D be the point 10 m from the left support, where normal thrust and radial shear are to be found.



We have, for circular arch,

$$R = \frac{L^2}{8h} + \frac{h}{2}$$

$$\Rightarrow R = \frac{40^2}{8*8} + \frac{8}{2}$$

$$\Rightarrow R = 29 \text{ m}$$

Slope at D = $\theta = sin^{-1}\left(\frac{10}{R}\right) = sin^{-1}\left(\frac{10}{29}\right)$

$$\Rightarrow \theta = 20.171^{\circ}$$

Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4^{th} ed.).New Delhi: Vikas Publishing House.

6.3 Determination of support reactions, shear forces, normal forces, and bending moments

Numerical #1.

Solution:

Vertical shear at D,

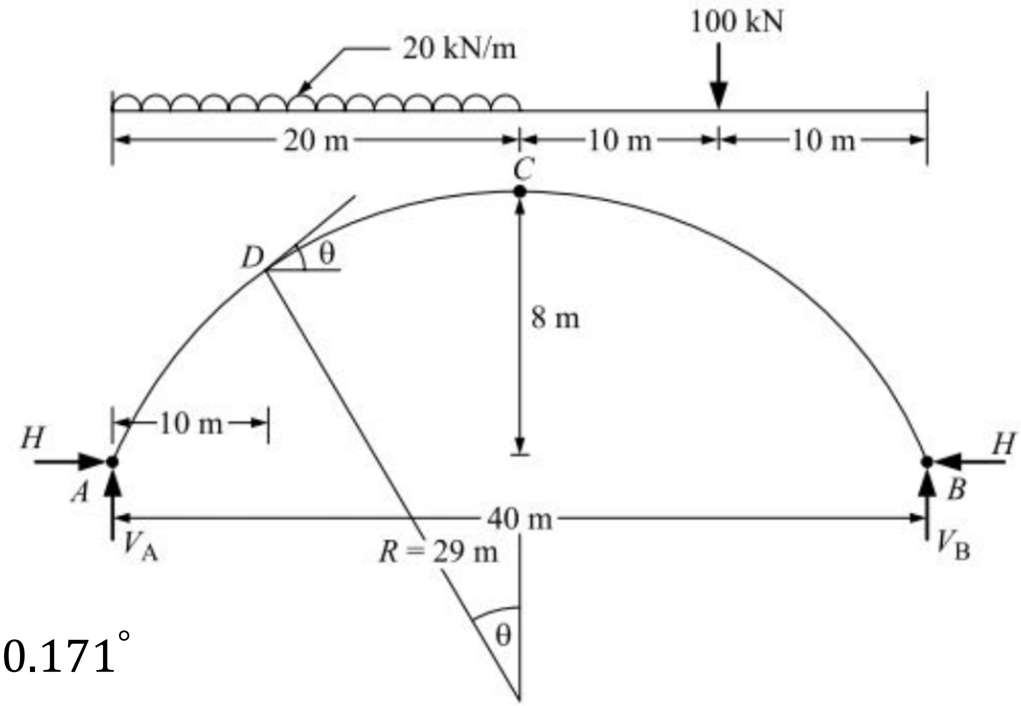
$$V = V_A - 20 * 10$$
$$= 325 - 200 = 125\ kN$$

Normal thrust at D,

$$N = V sin\theta + H cos\theta$$
$$= 125 \sin 20.171^{\circ} + 312.5 \cos 20.171^{\circ}$$
$$= 336.437\ kN$$

Radial shear at D,

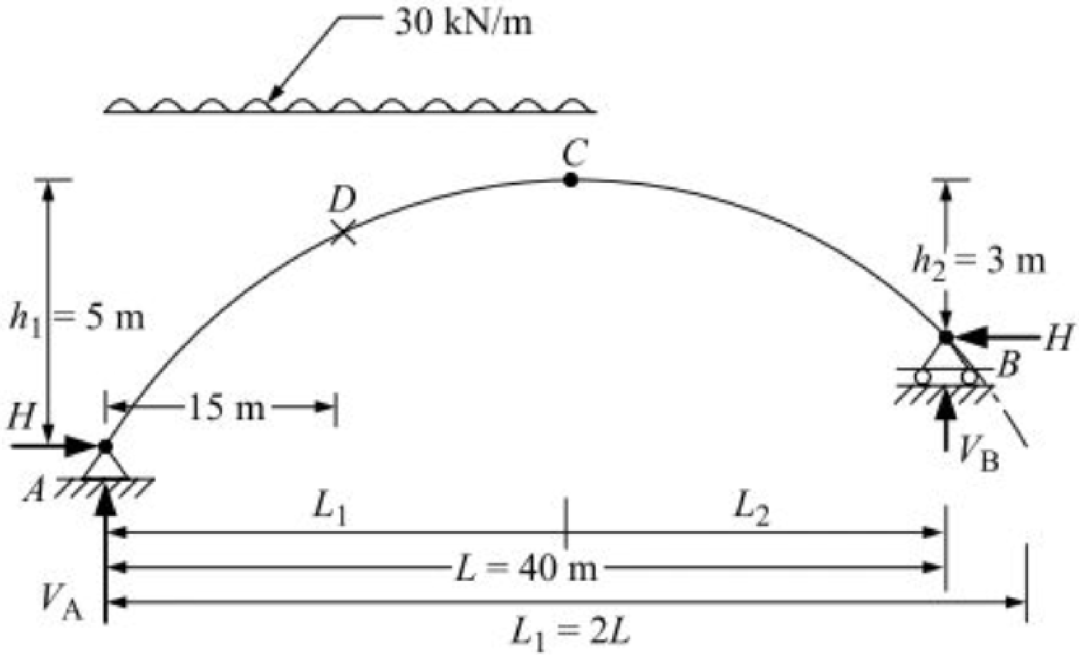
$$Q = V cos\theta - H sin\theta$$
$$= 125\ cos\ 20.171^{\circ} - 312.5\ sin\ 20.171^{\circ}$$
$$= 9.575\ kN$$



Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

6.3 Determination of support reactions, shear forces, normal forces, and bending moments

Numerical #2. A 3-hinged parabolic arch having supports at different levels as shown in the figure carries a UDL of 30 kN/m over the left portion of the crown. Determine the horizontal thrust developed, bending moment, normal thrust, and radial shear developed at a section 15 m from the left support.



Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

6.3 Determination of support reactions, shear forces, normal forces, and bending moments

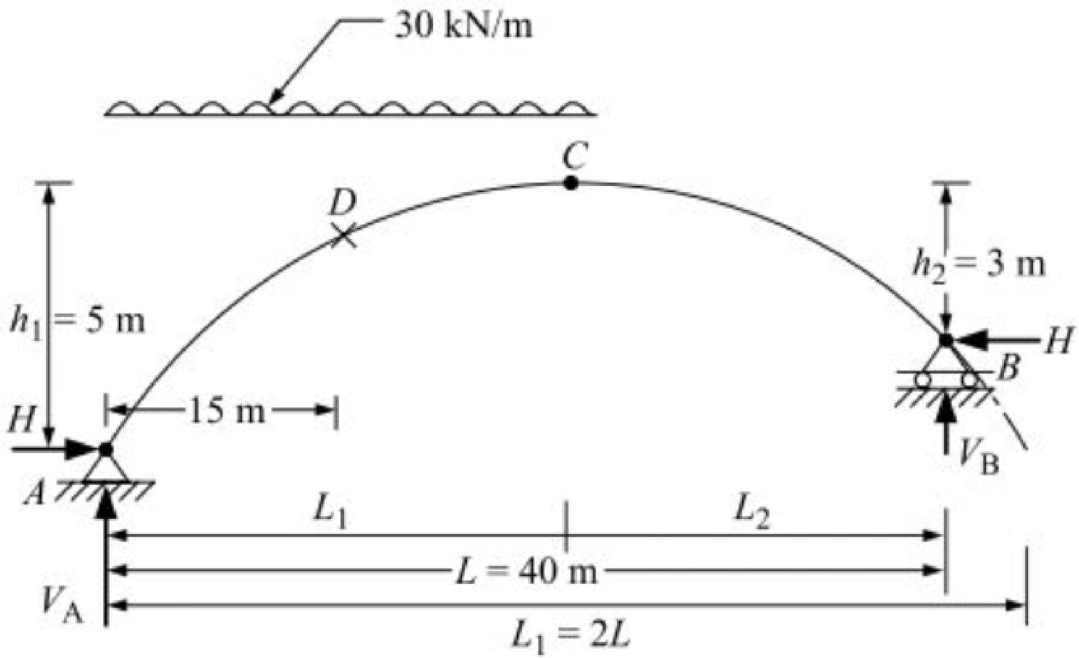
Numerical #2.

Solution:

Taking C as the origin, the equation of the parabola is,

$$\frac{x^2}{y} = a = constant$$

Let the horizontal distance between A and C be L_1 and that between C and D be L_2. Then,
We have,



Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

$$\Rightarrow L_1 = \frac{L\sqrt{h_1}}{\sqrt{h_1}+\sqrt{h_2}} = \frac{40\sqrt{5}}{\sqrt{5}+\sqrt{3}} = 22.54\ m$$

$$\Rightarrow L_2 = L - L_1 = 40 - 22.54 = 17.46\ m$$

6.3 Determination of support reactions, shear forces, normal forces, and bending moments

Numerical #2.

Solution:

Since C is hinged,
$\Sigma M_{C(right)} = 0$

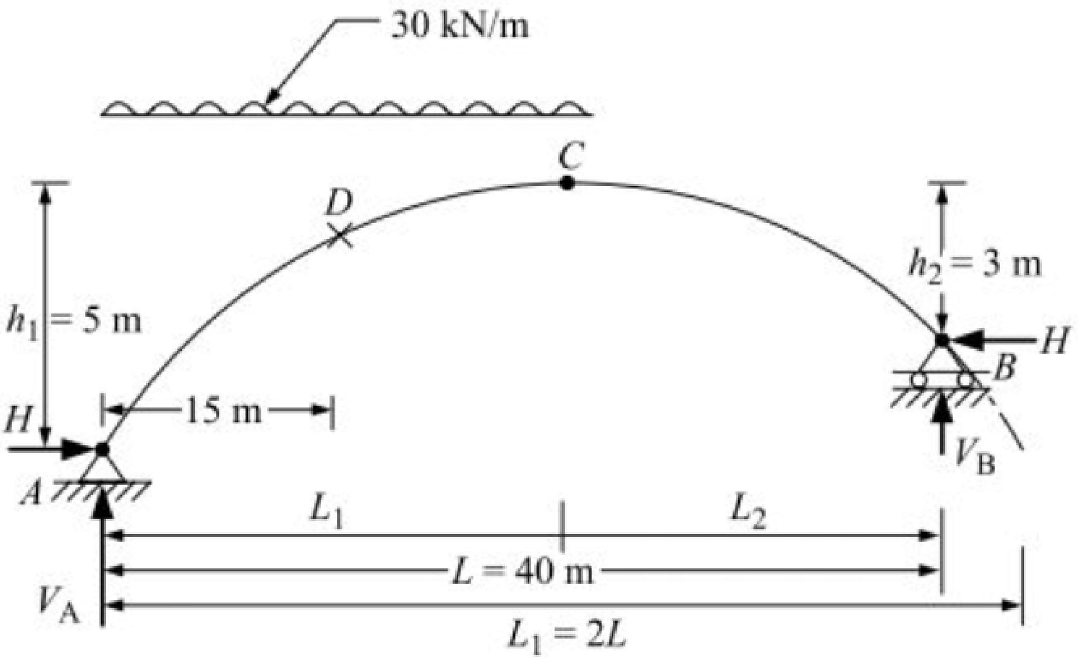
Or, $V_B * 17.46 - H * 3 = 0$

$\Rightarrow H = 5.82\, V_B \quad --(1)$

We have,
$\Sigma M_A = 0$

Or, $V_B * 40 + H * (5-3) - 30 * 22.54 * \frac{22.54}{2} = 0$

$\Rightarrow 40V_B + 2H = 7620.774 --(2)$



Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

6.3 Determination of support reactions, shear forces, normal forces, and bending moments

Numerical #2.

Solution:

From equations (1) and (2), we get

$\Rightarrow 40V_B + 2 * 5.82\ H = 7620.774$

$\Rightarrow V_B = 147.58\ k\mathrm{N}$

Then,

$\Rightarrow \mathrm{H} = 5.82 * 147.58$

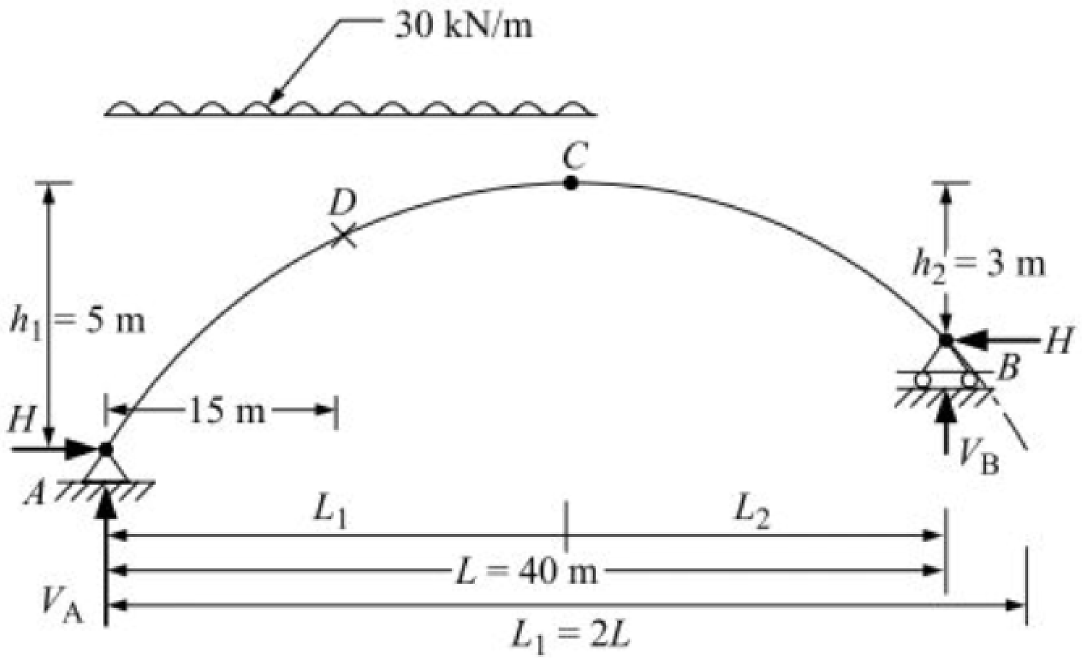
$\Rightarrow H = 858.92\ k\mathrm{N}$

And,

$\Rightarrow V_A = 30 * L_1 - V_B$

$\Rightarrow V_A = 30 * 22.54 - 147.58$

$\Rightarrow V_A = 528.62\ kN$



Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

6.3 Determination of support reactions, shear forces, normal forces, and bending moments

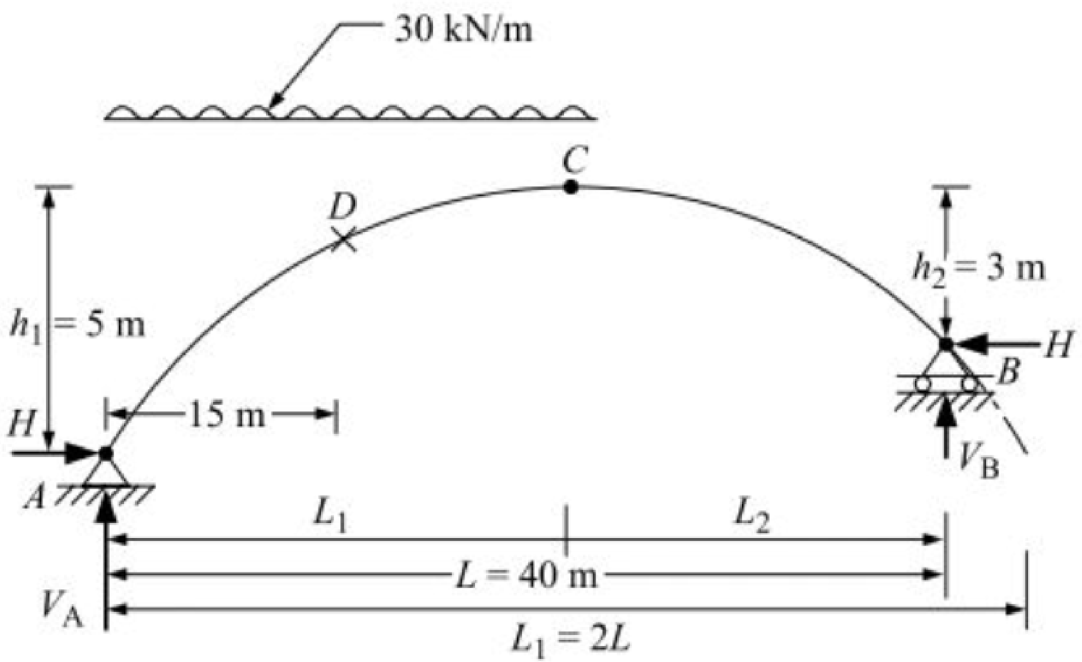
Numerical #2.

Solution:

Now, the portion left of C may be treated as a parabola of span
L' = $2L_1$ =2*22.54 = 45.08 m.
Then,
the equation of the parabola is:

$$y = \frac{4h_1x}{L'^2}(L' - x)$$

$$OR, y = \frac{4*5*x}{45.08^2}(45.08 - x)$$



Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

6.3 Determination of support reactions, shear forces, normal forces, and bending moments

Numerical #2.

Solution:

$At\ x = 15m,$

$$y = \frac{4*5*15}{45.08^2}(45.08 - 15)$$

$$Or, y = 4.44\ m$$

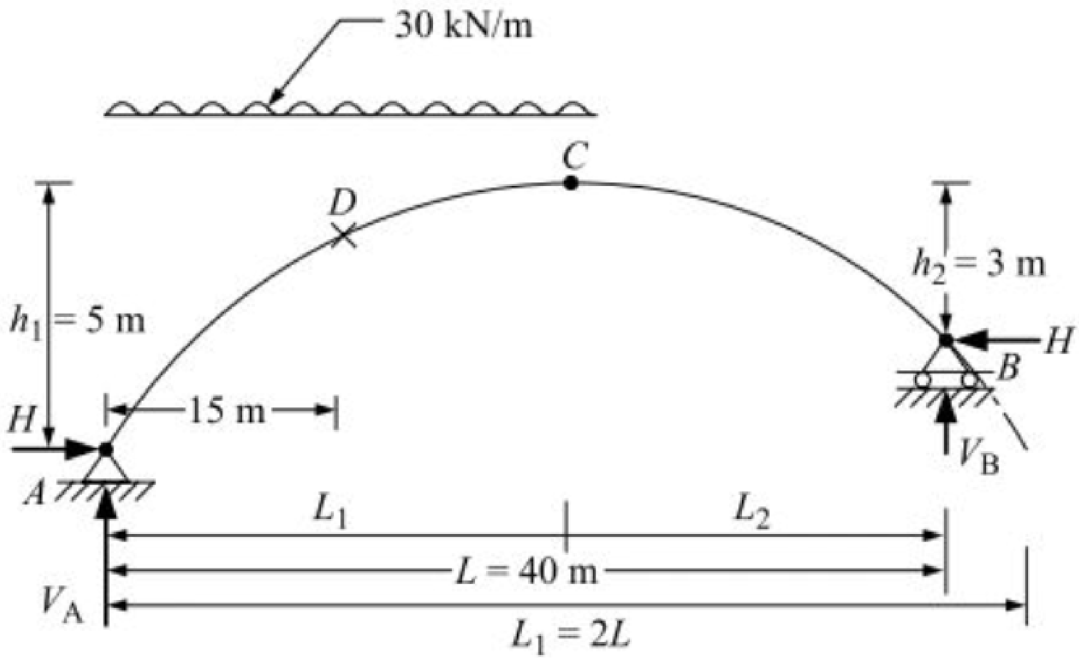
Then,

M at this section

$$= V_A * 15 - H * 4.44 - 30 * 15 * \frac{15}{2}$$

$$= 528.62 * 15 - 858.92 * 4.44 - 30 * 15 * \frac{15}{2}$$

= 740.7 kNm



Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

6.3 Determination of support reactions, shear forces, normal forces, and bending moments

Numerical #2.

Solution:

We have,

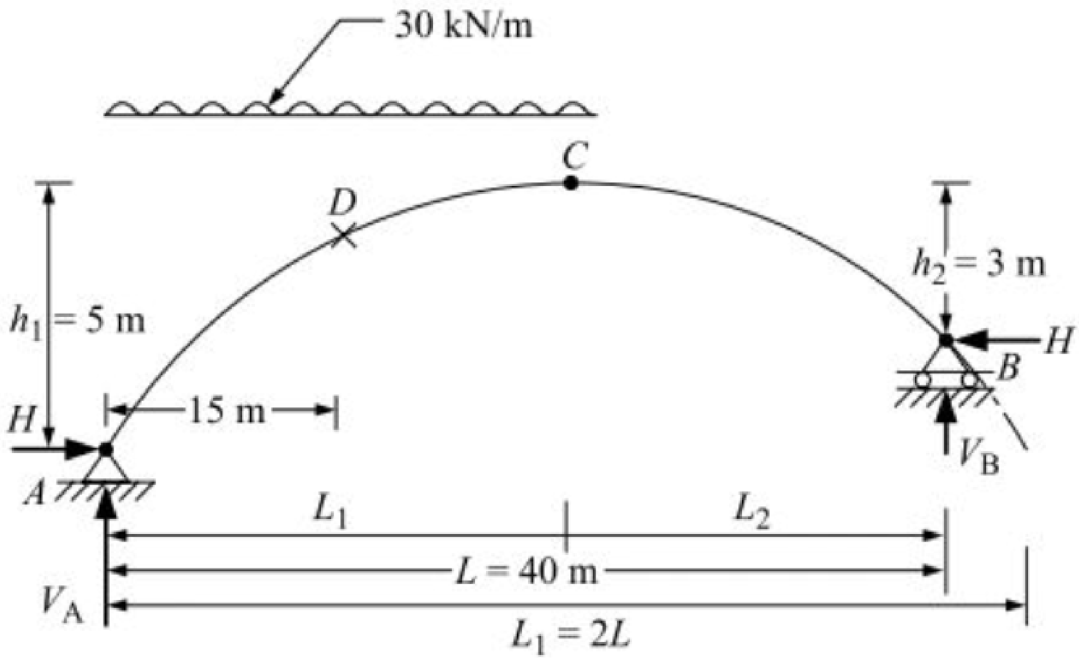
$$y = \frac{4h_1 x}{L'^2}(L' - x)$$

$$\frac{dy}{dx} = tan\theta = \frac{4h_1}{L'^2}(L' - 2x)$$

$At\ x = 15m,$

$$tan\theta = \frac{4 * 5}{45.08^2}(45.08 - 2 * 15)$$

$$\Rightarrow \theta = 8.44^\circ$$



Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

6.3 Determination of support reactions, shear forces, normal forces, and bending moments

Numerical #2.

Solution:

Vertical shear at D,

$V = V_A - 30 * 15$

$= 528.62 - 450 = 78.62\ kN$

Normal thrust at D,

$N = V sin\theta + H cos\theta$

$= 78.62\ \sin 8.44^{\circ} + 858.92 \cos 8.44^{\circ}$

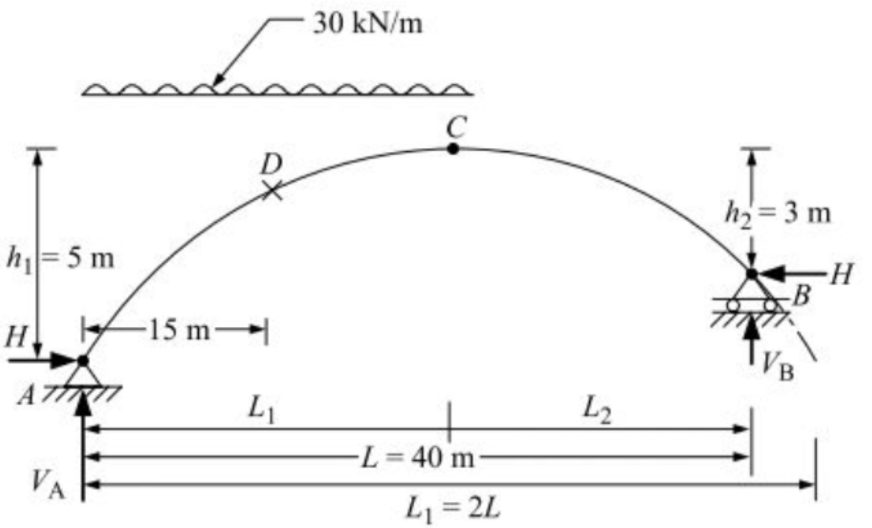
$= 861.16\ kN$

Radial shear at D,

$Q = V cos\theta - H sin\theta$

$= 78.62\ cos\ 8.44^{\circ} - 858.92\ sin\ 8.44^{\circ}$

$= -48..3\ kN$



Source: Bhavikatti, S. S. (2011). *Structural Analysis –I* (4th ed.).New Delhi: Vikas Publishing House.

-------------End of Lecture #10------------------

-------------End of Part I of II for Chapter 6--------------

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[4] Kindij, A., Ivankovic, A. M., & Vasilj, M. (2013). Assessment of masonry arch bridge with concrete deck. In *7th International Conference on Arch Bridges, ARCH* (Vol. 13).

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