Course: Advanced Algorithm and Problem Solving

WEEK 9 Minimum Spanning Tree Algorithms

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WEEK 9 : Minimum Spanning Tree Algorithms

Content

- Introduction to Minimum Spanning Tree (MST)
- Types of Minimum Spanning Tree
    - Boruvka's algorithm
    - Kruskal’s Algorithm
    - Prim’s Algorithm
- Differences Between Kruskal's Algorithm and Prim's Algorithm
- Applications of minimum spanning Tree

Lecture Learning Outcome

- Understand spanning tree and minimum weight spanning tree
- Understand general properties of minimum spanning tree
- Understand Boruvka's algorithm and how it works
- kruskal’s algorithm and how it works
- Understand prim’s algorithm and how it works
- Differentiate kruskal's algorithm and prim's algorithm
- Understand applications of minimum spanning Tree

Introduction to Minimum Spanning Tree (MST)

What is a Spanning Tree?

- In an undirected and connected graph G=(V,E), a spanning tree is a subgraph that is a tree which includes all of the vertices of G, with minimum possible number of edges.
- A graph may have several spanning trees.
- The cost of the spanning tree is the sum of the weights of all the edges in the tree.
- To understand what a spanning tree is, one must be familiar with undirected graphs.
- An undirected graph is a type of graph where edges are not directed in any specific direction.

[1]. Introduction to Minimum Spanning Tree (MST),Tutorial Horizon, 2022. https://tutorialhorizon.com/algorithms/introduction-to-minimum-spanning-tree-mst/

Introduction to Minimum Spanning Tree (MST) ...cont'd

- A spanning tree consists of (n-1) edges, where 'n' denotes the number of vertices or nodes in the graph.
- Formula for calculating the number of Spanning Tree of a complete graph:

$$n^{(n-2)}$$, where 'n' is the number of vertices in the graph

Example:-

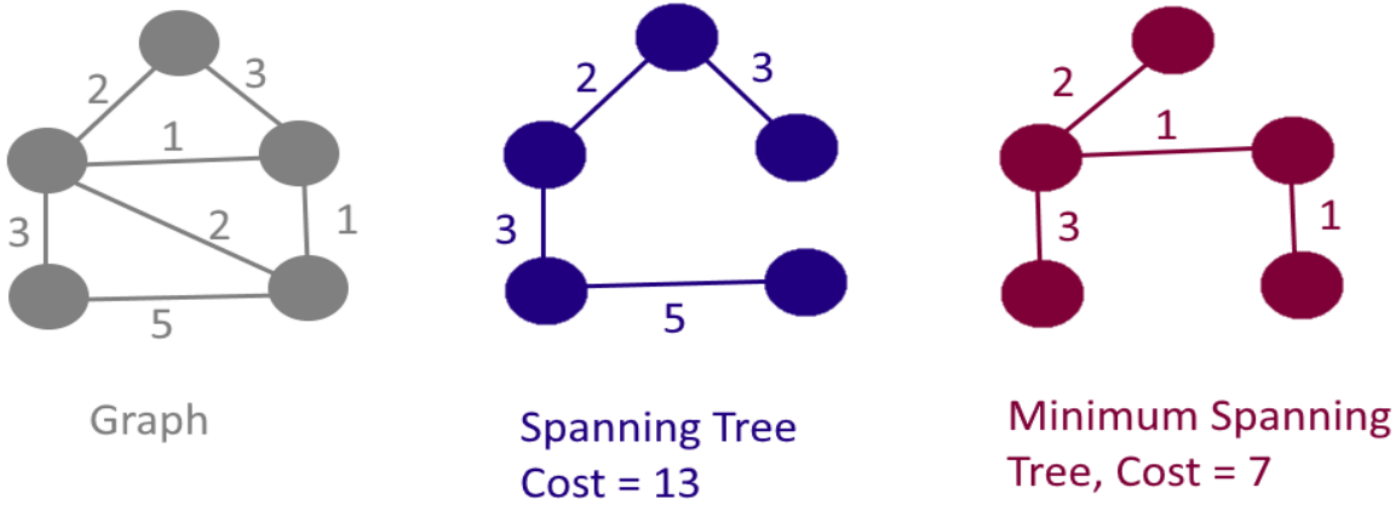
- If a complete graph has 4 vertices (n = 4) then the maximum number of possible spanning trees will be 4^(4-2) = 16.

[1]. Introduction to Minimum Spanning Tree (MST),Tutorial Horizon, 2022. https://tutorialhorizon.com/algorithms/introduction-to-minimum-spanning-tree-mst/

Introduction to Minimum Spanning Tree (MST) ...cont'd

- A **minimum spanning tree (MST)** or **minimum weight spanning tree** is a subset of the edges of a connected, edge-weighted (un)directed graph that connects all the vertices together, without any cycles and with the minimum possible total edge weight.
- That is, it is a spanning tree whose sum of edge weights is as small as possible. Number of edges in MST: **V-1 (V – no of vertices in Graph).**

Example:-



[1]. Introduction to Minimum Spanning Tree (MST),Tutorial Horizon, 2022.
https://tutorialhorizon.com/algorithms/introduction-to-minimum-spanning-tree-mst/

Introduction to Minimum Spanning Tree (MST) ...cont'd

General properties of a minimum spanning tree:

1. Removing Edge Breaks Connectivity

- Removing even one edge from a spanning tree will no longer connect all the vertices. The graph will become disconnected, and it will no longer be a tree.

2. Adding an Edge Forms a Cycle

- A tree is, by definition, acyclic. So, if you try to add any extra edge to a spanning tree, it creates a loop. That loop violates the tree structure.

3. Unique Tree for Distinct Weights

- If every edge in the graph has a unique weight, then only one unique minimum spanning tree is possible. However, if weights are repeated, multiple valid MSTs with the same total weight could exist.

[1]. Introduction to Minimum Spanning Tree (MST),Tutorial Horizon, 2022.
https://tutorialhorizon.com/algorithms/introduction-to-minimum-spanning-tree-mst/

Introduction to Minimum Spanning Tree (MST) ...cont'd

General properties of a minimum spanning tree:

4. Spanning Trees in a Complete Graph

- A complete undirected graph with **n vertices** can have exactly **n^(n−2)** spanning trees. This result comes from Cayley's Theorem and shows how more possible trees increase rapidly with more nodes.

5. Every Connected Graph Has a Spanning Tree

- As long as a graph is connected and undirected, it will have at least one spanning tree. That's because there's always a way to connect all vertices without cycles.

Introduction to Minimum Spanning Tree (MST) ...cont'd

General properties of a minimum spanning tree: ....

6. Disconnected Graphs Have None

- If the graph is not connected, creating a spanning tree is impossible because we can't reach all the nodes.

7. Edge Reduction in Complete Graphs

- In a complete graph, we can remove up to $(e - n + 1)$ edges, where e is the number of edges and n is the number of nodes. It will still be able to form a spanning tree. Beyond that, the graph becomes disconnected.

[1]. Introduction to Minimum Spanning Tree (MST),Tutorial Horizon, 2022.
https://tutorialhorizon.com/algorithms/introduction-to-minimum-spanning-tree-mst/

Introduction to Minimum Spanning Tree (MST) ...cont'd

Applications of Minimum spanning trees :

- Minimum spanning trees have direct applications in the design of networks, including computer networks, telecommunications networks, transportation networks, water supply networks, and electrical grids.

Also used in:-

- Approximating travelling salesman problem
- Approximating multi-terminal minimum cut problem
- Approximating minimum-cost weighted perfect Cluster Analysis
- Image segmentation
- Circuit design

Algorithms for Minimum Spanning Tree

a) Boruvka's algorithm

- This is one of the oldest algorithms.
- It is a graph traversal algorithm used to find the minimum spanning tree of a connected, undirected graph.
- The algorithm works by iteratively building the minimum spanning tree, starting with each vertex in the graph as its own tree.
- In each iteration, the algorithm finds the cheapest edge that connects a tree to another tree, and adds that edge to the minimum spanning tree.

[2]. Boruvka’s algorithm | Greedy Algo-9, GeeksforGeeks, 2023.
https://www.geeksforgeeks.org/boruvkas-algorithm-greedy-algo-9/

a) Boruvka's algorithm

Below is a complete algorithm.

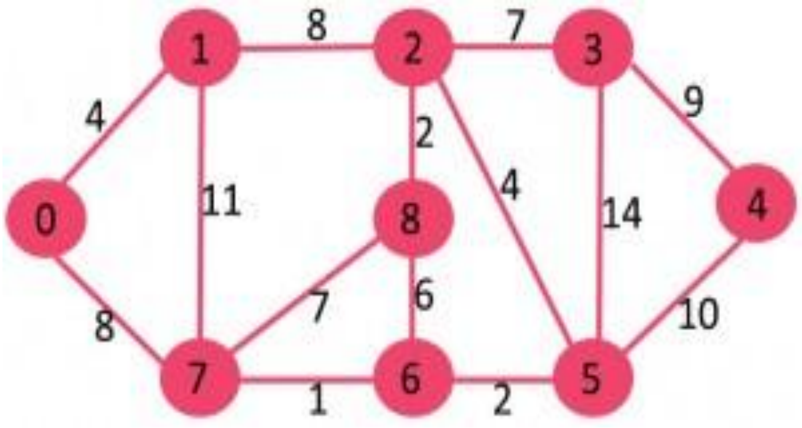
- The algorithm can be implemented using a data structure such as a priority queue to efficiently find the cheapest edge between trees.
- It is simple and easy-to-implement algorithm for finding minimum spanning trees, but it may not be as efficient as other algorithms for large graphs with many edges.

```
1) Input is a connected, weighted and un-directed
graph.
2) Initialize all vertices as individual components
(or sets).
3) Initialize MST as empty.
4) While there are more than one components, do
following
   for each component.
      a)  Find the closest weight edge that connects
this
          component to any other component.
      b)  Add this closest edge to MST if not already
added.
5) Return MST.
```

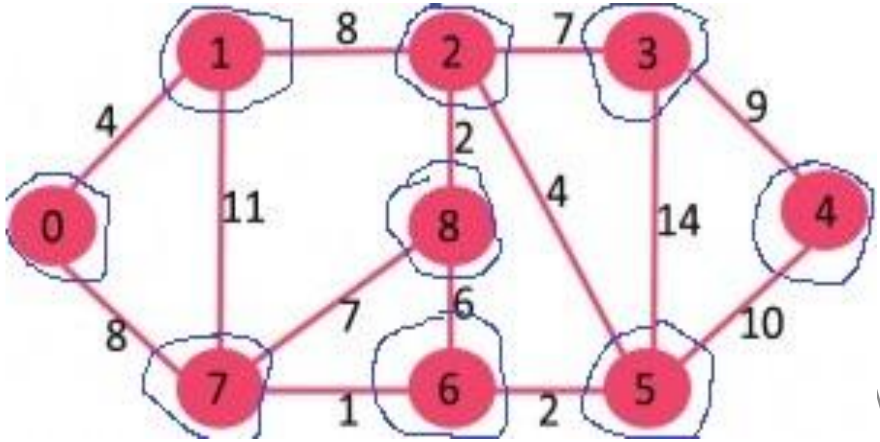
[2]. Boruvka's algorithm | Greedy Algo-9, GeeksforGeeks, 2023.
https://www.geeksforgeeks.org/boruvkas-algorithm-greedy-algo-9/

Algorithms for Minimum Spanning Tree ..cont'd

- Let us consider this graph to understand the algorithm.



Step1. Initially, MST is empty. Every vertex is single component as highlighted in blue color in the below diagram.

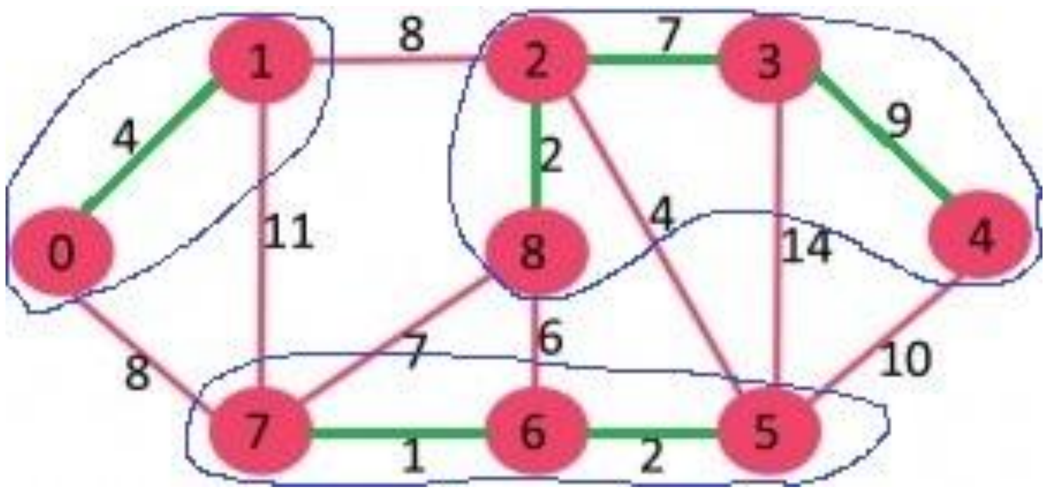


Step 2. For every component, find the cheapest edge that connects it to some other component.

Component	Cheapest Edge that connects it to some other component
{0}	0-1
{1}	0-1
{2}	2-8
{3}	2-3
{4}	3-4
{5}	5-6
{6}	6-7
{7}	6-7
{8}	2-8

Algorithms for Minimum Spanning Tree ..cont'd

- The cheapest edges are highlighted with green color. Now MST becomes {0-1, 2-8, 2-3, 3-4, 5-6, 6-7}.



- We again repeat the step, i.e., for every component, find the cheapest edge that connects it to some other component.

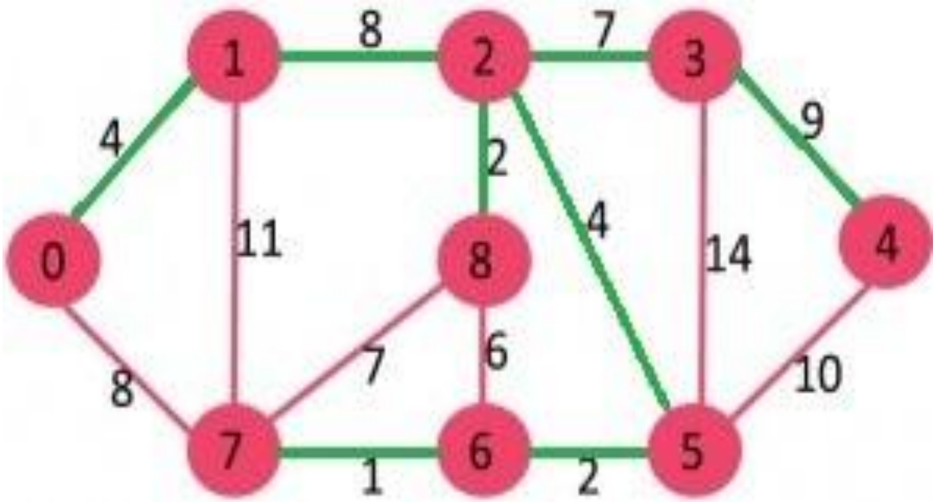
Component	Cheapest Edge that connects it to some other component
{0,1}	1-2 (or 0-7)
{2,3,4,8}	2-5
{5,6,7}	2-5

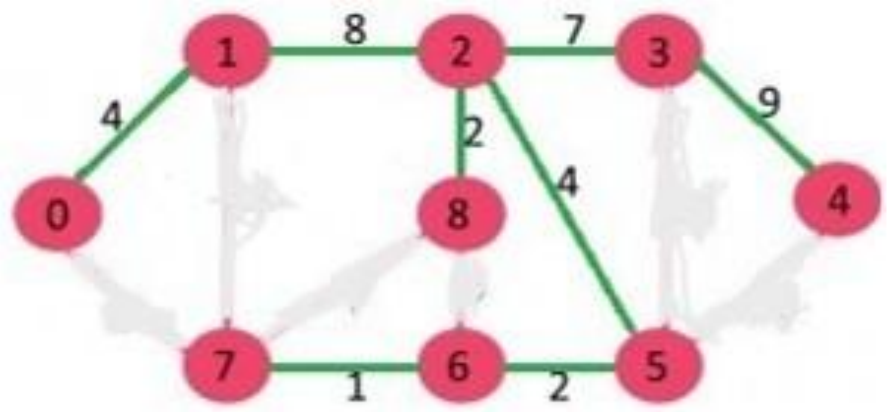
- After above step, components are {{0,1}, {2,3,4,8}, {5,6,7}}. The components are encircled with blue color.

[2]. Boruvka's algorithm | Greedy Algo-9, GeeksforGeeks, 2023. https://www.geeksforgeeks.org/boruvkas-algorithm-greedy-algo-9/

Algorithms for Minimum Spanning Tree ..cont'd

- The cheapest edges are highlighted with green color.
- Now MST becomes {0-1, 2-8, 2-3, 3-4, 5-6, 6-7, 1-2, 2-5}





Finally, these are the only one component {0, 1, 2, 3, 4, 5, 6, 7, 8} which has all edges.

Minimum Spanning Tree (MST)
Total cost : 4+8+7+2+4+9+2+1= 37

[2]. Boruvka's algorithm | Greedy Algo-9, GeeksforGeeks, 2023.
https://www.geeksforgeeks.org/boruvkas-algorithm-greedy-algo-9/

Algorithms for Minimum Spanning Tree ..cont'd

b) Kruskal Algorithm

- Kruskal's algorithm for finding the Minimum Spanning Tree(MST), which finds an edge of the least possible weight that connects any two trees in the forest
- It is a greedy algorithm.
- It finds a subset of the edges that forms a tree that includes every vertex, where the total weight of all the edges in the tree is minimized.
- If the graph is not connected, then it finds a minimum spanning forest (a minimum spanning tree for each connected component).
- Number of edges in **MST: V-1 (V – no of vertices in Graph).**

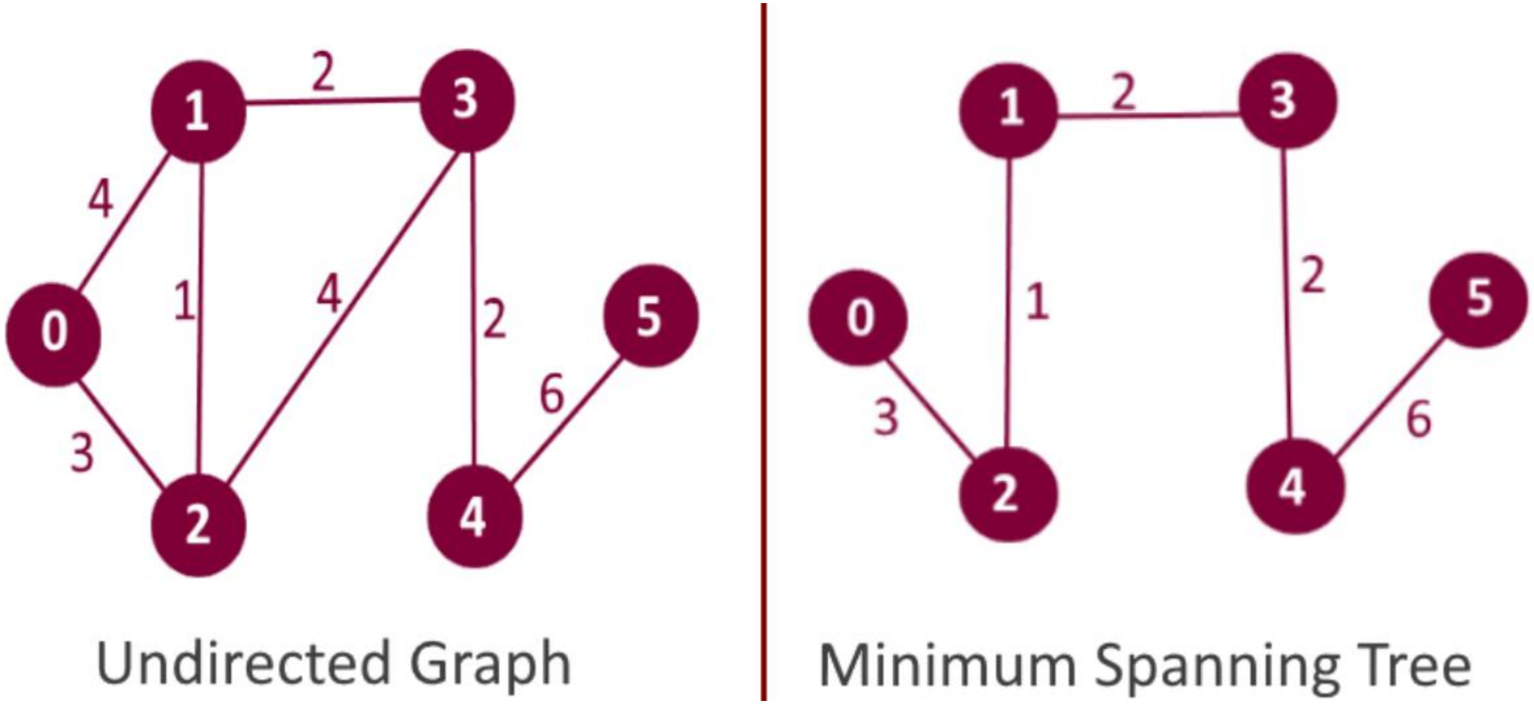
Algorithms for Minimum Spanning Tree ..cont'd

- **Kruskal's algorithm** starts by sorting all the edges of the graph in ascending order, based on their weights.
- Then, it iterates through these edges in ascending order and adds them to the MST if they don't form a cycle.
- It uses the disjoint-set data structure to check whether the addition of an edge is forming a cycle in the MST or not.
- These steps are repeated until all vertices are included in the MST (minimum spanning tree), or we can say until (n-1) edges are added, where 'n' is the the number of vertices.

[3]. Minimum Spanning Tree (MST) using Kruskal's and Prim's Algorithm, Sahil Mattoo, Intellipaat, 2025. https://intellipaat.com/blog/minimum-spanning-tree-algorithm/

Algorithms for Minimum Spanning Tree ..cont'd

Example :- Kruskal Algorithm



[4]. Prim's Algorithm - Minimum Spanning Tree (MST), 2022.
https://tutorialhorizon.com/algorithms/prims-algorithm-minimum-spanning-tree-mst/

Algorithms for Minimum Spanning Tree ..cont'd

Steps

1. Sort the edges in ascending order of weights.

2. Pick the edge with the least weight. Check if including this edge in spanning tree will form a cycle is Yes then ignore it if No then add it to spanning tree.

3. Repeat the step 2 untill spanning tree has V-1 edges (V – vertices in Graph).

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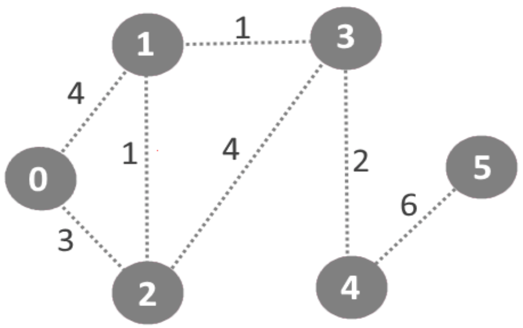
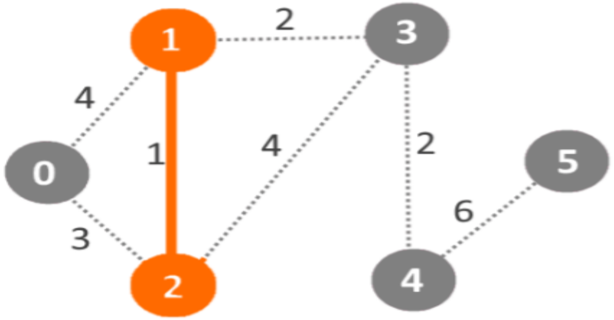
Edges	1-2	1-3	3-4	0-2	0-1	2-3	4-5
Weights	1	2	2	3	4	4	6



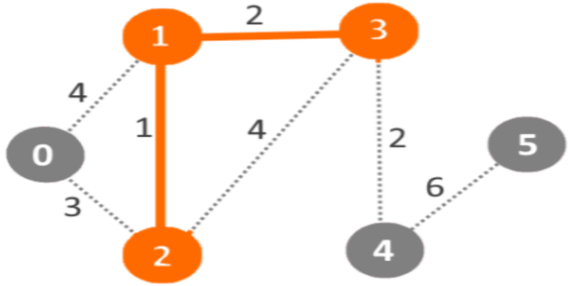
Edges	1-2	1-3	3-4	0-2	0-1	2-3	4-5
Weights	1	2	2	3	4	4	6



Edge 0-1:
No cycle formed by including edge 0-1 so include it

MST Edges : 1

Edges	1-2	1-3	3-4	0-2	0-1	2-3	4-5
Weights	1	2	2	3	4	4	6



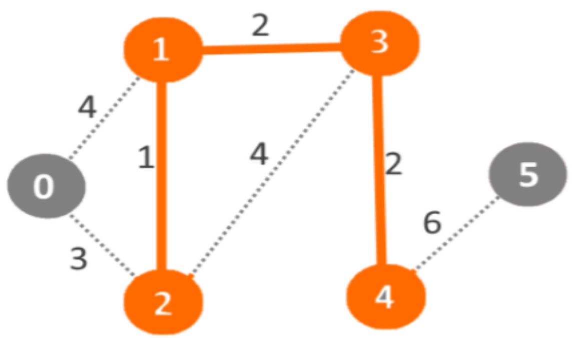
Edge 1-3:
No cycle formed by including edge 1-3 so include it.

MST Edges : 2

[5]. Kruskal's Algorithm – Minimum Spanning Tree (MST) - Complete Implementation, 2022. https://tutorialhorizon.com/algorithms/kruskals-algorithm-minimum-spanning-tree-mst-complete-implementation/

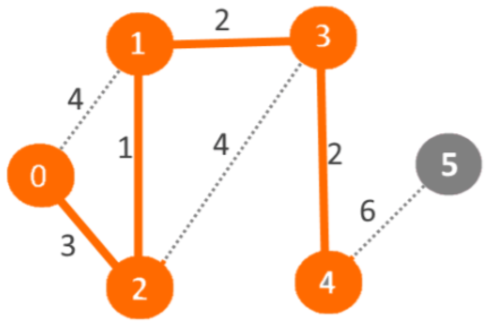
Algorithms for Minimum Spanning Tree ..cont'd

Edges	1-2	1-3	3-4	0-2	0-1	2-3	4-5
Weights	1	2	2	3	4	4	6



Edge 3-4:
No cycle formed by including edge 3-4 so include it.

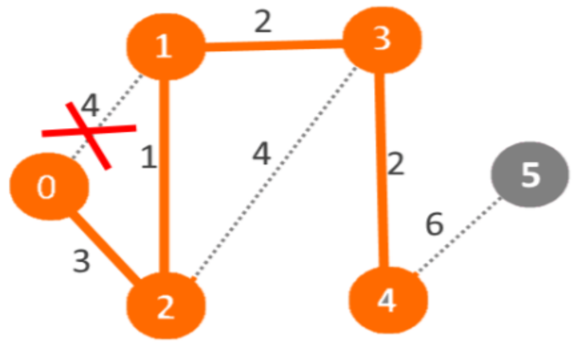
Edges	1-2	1-3	3-4	0-2	0-1	2-3	4-5
Weights	1	2	2	3	4	4	6



Edge 0-2:
No cycle formed by including edge 0-2 so include it.

MST Edges : 4

Edges	1-2	1-3	3-4	0-2	0-1	2-3	4-5
Weights	1	2	2	3	4	4	6



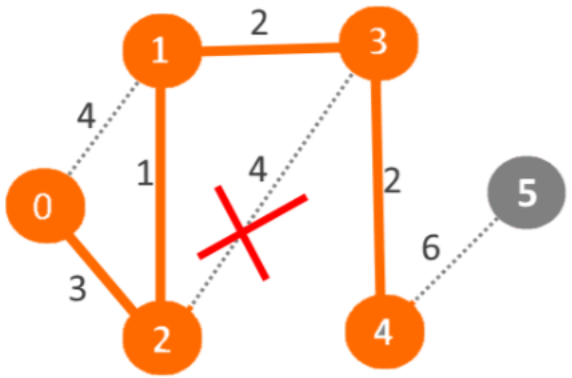
Edge 0-1:
If edge 0-1 is included then CYCLE will be formed so ignore this edge

MST Edges : 4

[5]. Kruskal's Algorithm – Minimum Spanning Tree (MST) - Complete Implementation, 2022. https://tutorialhorizon.com/algorithms/kruskals-algorithm-minimum-spanning-tree-mst-complete-implementation/

Algorithms for Minimum Spanning Tree ..cont'd

Edges	1-2	1-3	3-4	0-2	0-1	2-3	4-5
Weights	1	2	2	3	4	4	6

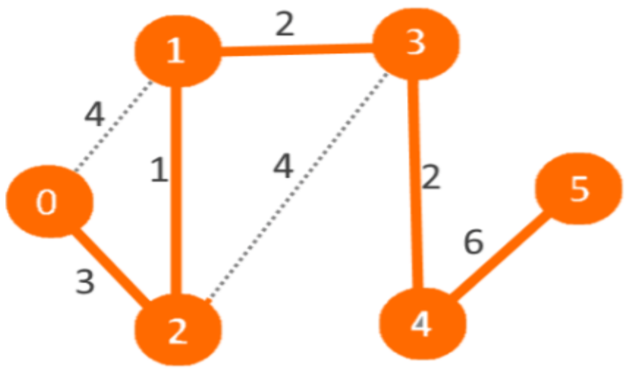


Edge 2-3:

If edge 2-3 is included then CYCLE will be formed so ignore this edge.

MST Edges : 4

Edges	1-2	1-3	3-4	0-2	0-1	2-3	4-5
Weights	1	2	2	3	4	4	6

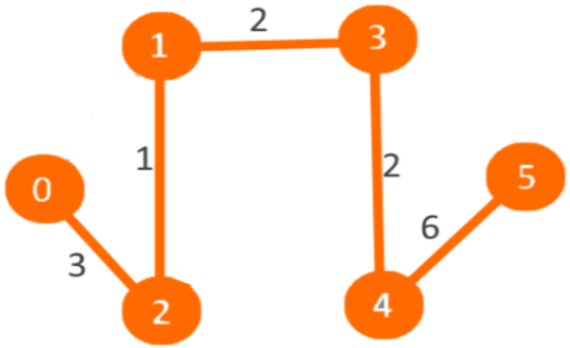


Edge 4-5:

No cycle formed by including edge 4-5 so include it

MST Edges : 5 (equal to no of vertices-1, Stop iteration)

Edges	1-2	1-3	3-4	0-2	0-1	2-3	4-5
Weights	1	2	2	3	4	4	6



Minimum Spanning Tree (MST)

Total Cost = 3+1+2+2+6 = 13

MST Edges : 5

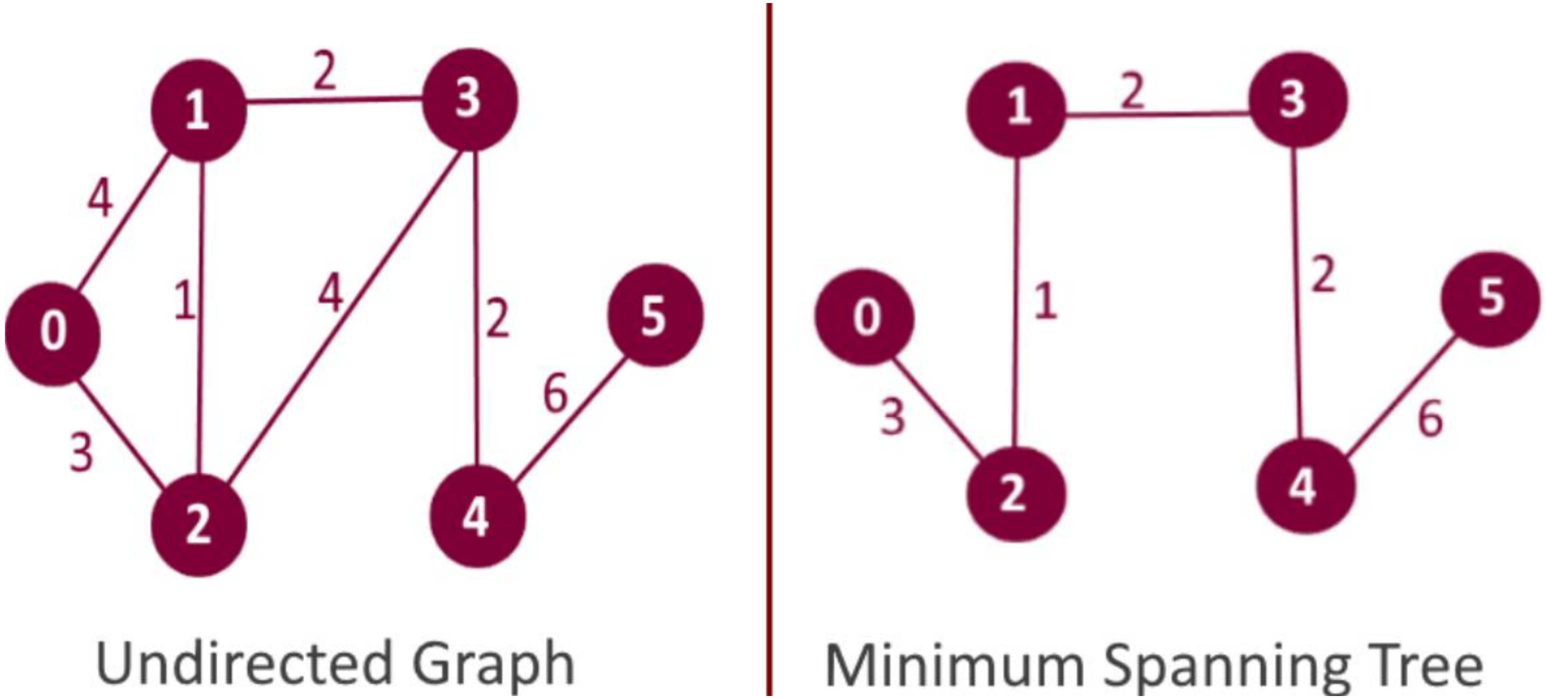
[5]. Kruskal's Algorithm – Minimum Spanning Tree (MST) - Complete Implementation, 2022. https://tutorialhorizon.com/algorithms/kruskals-algorithm-minimum-spanning-tree-mst-complete-implementation/

Algorithms for Minimum Spanning Tree ..cont’d

Prim’s algorithm

- Prim's algorithm is a greedy algorithm.
- It finds a minimum spanning tree for a weighted undirected graph.
- This means it finds a subset of the edges that forms a tree that includes every vertex, where the total weight of all the edges in the tree is minimized.
- The algorithm operates by building this tree one vertex at a time, from an arbitrary starting vertex, at each step adding the cheapest possible connection from the tree to another vertex.

Example:



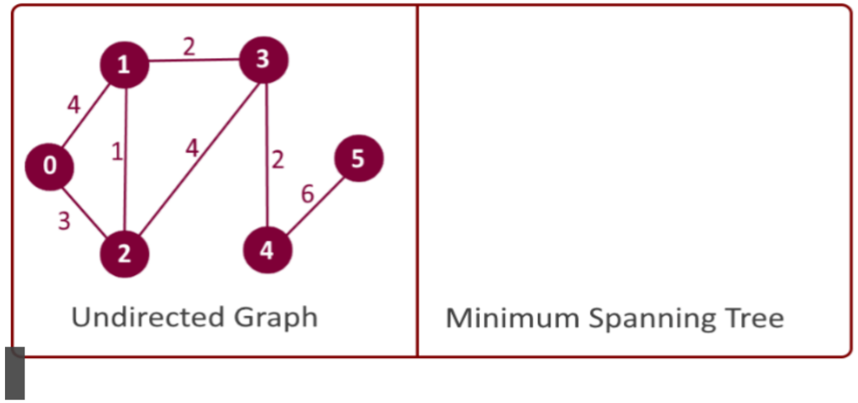
[4]. Prim's Algorithm - Minimum Spanning Tree (MST), 2022. https://tutorialhorizon.com/algorithms/prims-algorithm-minimum-spanning-tree-mst/

Algorithms for Minimum Spanning Tree ..cont'd

How to implement Prim's algorithm?

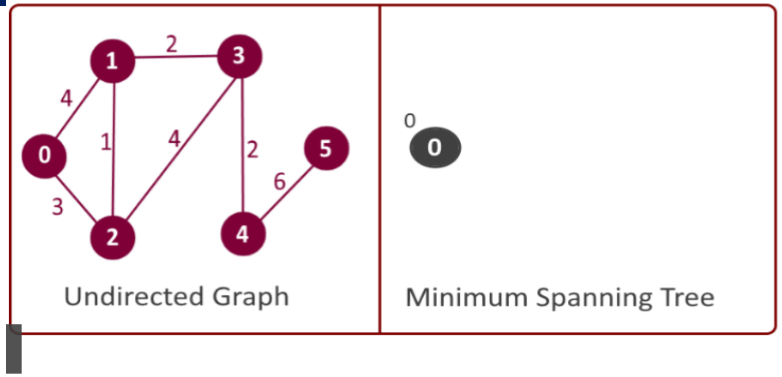
- Start with the empty spanning tree.
- Maintain a set mst[] to keep track to vertices included in minimum spanning tree.
- Assign a key value to all the vertices, (say key []) and initialize all the keys with +∞ (Infinity) except the first vertex. (We will start with this vertex, for which key will be 0).
- Key value will be used in making decision that which next vertex and edge will be included in the **mst[].** We will **pick the vertex which is not included in mst[] and has the minimum key**. So at the beginning the first vertex will be picked first.
- Repeat the following steps until all vertices are processed

Algorithms for Minimum Spanning Tree ..cont'd



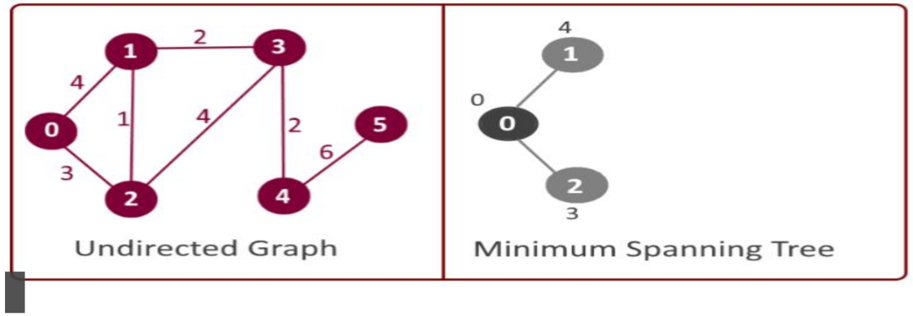
Vertices	0	1	2	3	4	5
Key	0	+∞	+∞	+∞	+∞	+∞

Initialize all the keys with +∞ except the first vertex.
(We will start with this vertex, for which key will be 0).



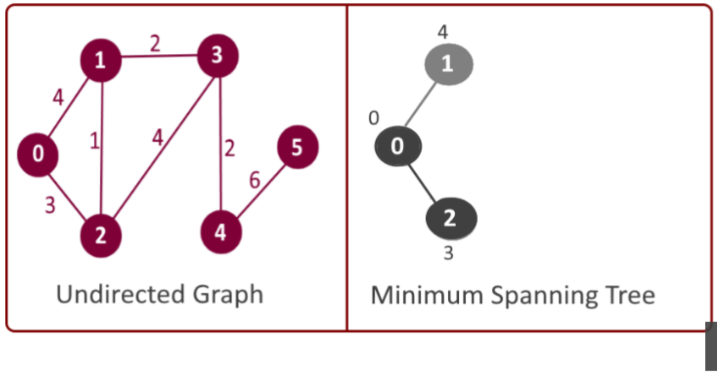
Vertices	0	1	2	3	4	5
Key	0	+∞	+∞	+∞	+∞	+∞

Pick Minimum Key:
Pick the vertex not in MST and has minimum Key: vertex 0
Include vertex 0 in MST



Vertices	0	1	2	3	4	5
Key	0	4	3	+∞	+∞	+∞

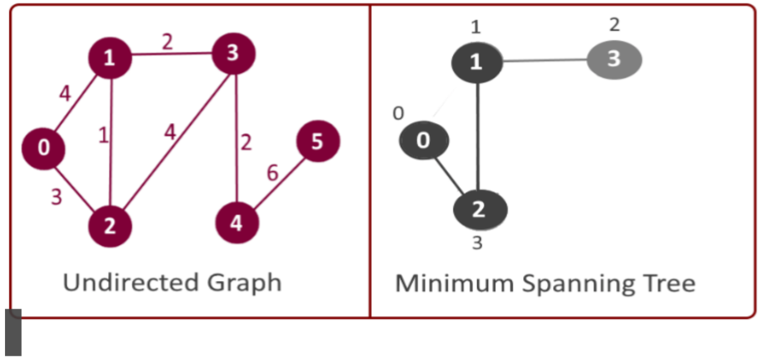
Update keys:
Loop over all the adjacent vertices of vertex 0
For adjacent vertex 1, vertex 1 is not in MST and edge 0-1 weight=4, key[1]=+∞ weight<key[1] update the key[1]= 4
For adjacent vertex 2, vertex 2 is not in MST and edge 0-2 weight=3, key[2]=+∞ weight<key[2] update the key[2]= 3



Vertices	0	1	2	3	4	5
Key	0	4	3	+∞	+∞	+∞

Pick Minimum Key:
Pick the vertex not in MST and has minimum Key: vertex 2
Include vertex 2 in MST

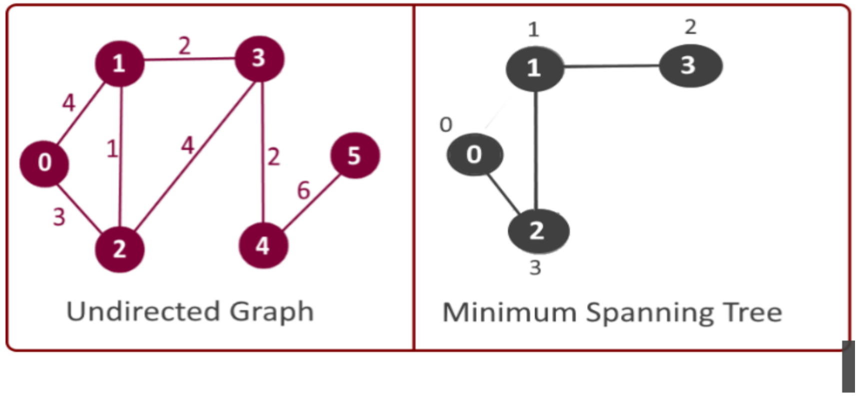
Algorithms for Minimum Spanning Tree ..cont'd



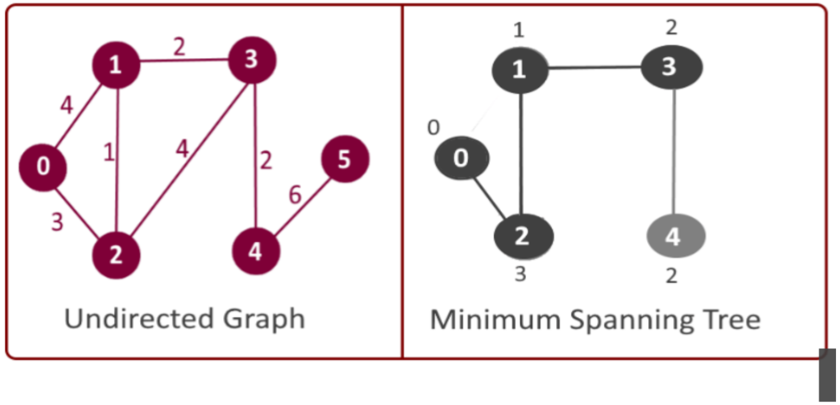
Vertices	0	1	2	3	4	5
Key	0	1	3	2	+∞	+∞

Update keys:

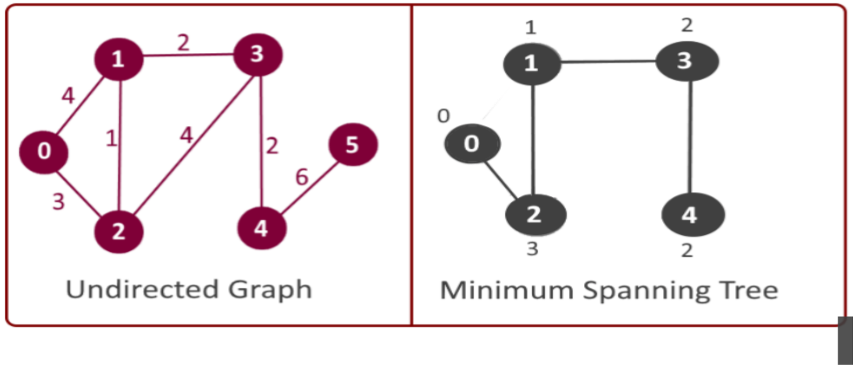
Loop over all the adjacent vertices of vertex 1
- For adjacent vertex 2, - already in MST
- For adjacent vertex 3, vertex 3 is not in MST and edge 1-3 weight=2, key[3]=4
  - weight<key[3] update the key[3]= 2
- For adjacent vertex 0, - already in MST



Vertices	0	1	2	3	4	5
Key	0	1	3	2	+∞	+∞



Vertices	0	1	2	3	4	5
Key	0	1	3	2	2	+∞

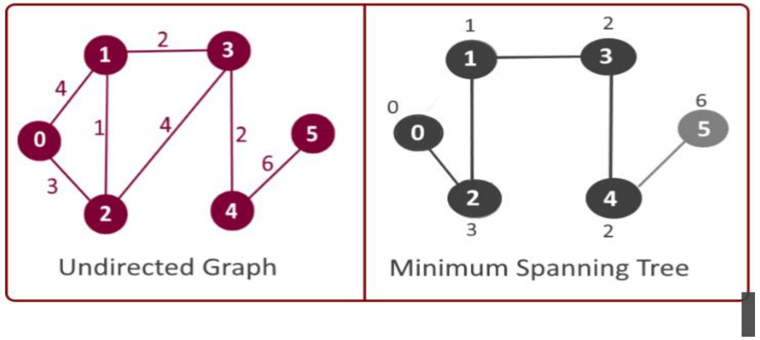


Vertices	0	1	2	3	4	5
Key	0	1	3	2	2	+∞

Pick Minimum Key:

Pick the vertex not in MST and has minimum Key: vertex 4

Include vertex 4 in MST

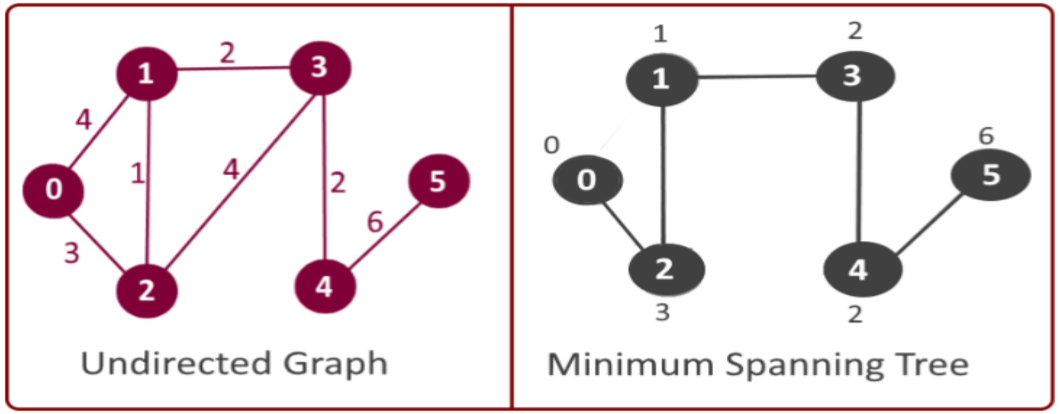


Vertices	0	1	2	3	4	5
Key	0	1	3	2	2	6

Update keys:

Loop over all the adjacent vertices of vertex 4
- For adjacent vertex 5, vertex 5 is not in MST and edge 4-5 weight=6, key[5]=+∞
  - weight<key[5] update the key[5]= 6
- For adjacent vertex 3, - already in MST

Algorithms for Minimum Spanning Tree ..cont'd



Vertices	0	1	2	3	4	5
Key	0	1	3	2	2	6

Minimum Spanning Tree is Found
Total Weight : 14

[4]. Prim's Algorithm - Minimum Spanning Tree (MST), 2022.
https://tutorialhorizon.com/algorithms/prims-algorithm-minimum-spanning-tree-mst/

Algorithms for Minimum Spanning Tree ..cont'd

Application of Prim's Algorithm

- Cluster analysis
- Image segmentation
- Handwriting recognition
- Medical Image Processing
- Identify patterns in gene expression
- Approximation to NP hard problems, i.e., No polynomial-time solution for exact answers.
- Document categorization for web search Network design for telephone, electrical, hydraulic, TV cable, computer, road etc.
- There are many indirect applications also.

[5]. Minimum Spanning Tree-MST Kruskal and Prims algorithm Explanation, Vivek Kumar, Codingeek, 2017https://www.codingeek.com/data-structure/application-of-graph-minimum-spanning-tree/#1_kruskals_algorithm

Differences Between Kruskal's Algorithm and Prim's Algorithm

Feature	Kruskal's Algorithm	Prim's Algorithm
Approach	Edge-based	Vertex-based
Starting Point	Starts with the smallest edge	Starts from a chosen vertex
Edge Selection	Picks edges in increasing weight	Picks minimum weight edge from visited
Cycle Detection	Uses Union-Find	Handled by mstSet[] logic
Suitable For	Sparse graphs	Dense graphs
Data Structure Used	Disjoint Set (Union-Find)	Arrays or Priority Queue
Graph Type Required	Works on disconnected graphs, too	Requires a connected graph
Time Complexity	O(E log E)	$O(V^2)$ with matrix, O(E log V) with heap

[6]. Minimum Spanning Tree: Algorithms Explained with Examples, CCBP, 2025.
https://www.ccbp.in/blog/articles/minimum-spanning-tree

Applications of Minimum Spanning Tree

The Minimum Spanning Tree (MST) algorithm has numerous practical real-world applications in various fields. Here are some of the most popular cases of application in various business sectors:

- **Telecommunication Networks:** MSTs are used in designing efficient communication networks, like telephone or internet networks, to establish reliable connections among different locations that minimize the overall cost or distance of laying cables.

- **Transportation and Logistics:** In logistics and transportation, MST algorithms assist in optimizing routes for delivery trucks, minimizing travel costs, and streamlining supply chain management by establishing the most efficient connections between different locations.

[7]. Minimum Spanning Tree (MST) using Kruskal's and Prim's Algorithm, Sahil Mattoo ,Intellipaat, 2025. https://intellipaat.com/blog/minimum-spanning-tree-algorithm/

Applications of Minimum Spanning Tree …cont'd

- **Circuit Design**: Electronic circuit design benefits from MSTs by optimizing the layout of components on a circuit board, reducing wire length, and minimizing manufacturing costs while maintaining connectivity.
- **Computer Networking:** In computer networks, MSTs help in designing efficient network protocols, optimizing routing, and ensuring reliable data transmission among devices in a network.
- **Wireless Sensor Networks:** MSTs are utilized in wireless sensor networks for optimal data collection, routing, and energy conservation by minimizing the total distance data needs to travel.
- **Image Segmentation**: In image processing, MSTs assist in image segmentation, where pixels with similar properties are grouped efficiently to help in object recognition and analysis.

[7]. Minimum Spanning Tree (MST) using Kruskal's and Prim's Algorithm, Sahil Mattoo ,Intellipaat, 2025. https://intellipaat.com/blog/minimum-spanning-tree-algorithm/

Applications of Minimum Spanning Tree ...cont'd

- **Oil and Gas Pipelines**: MST algorithms are applied in laying pipelines for oil and gas transportation, optimizing the network to reduce construction costs and improve efficiency.
- **Spanning Tree Protocol (STP) in Computer Networks:** The Spanning Tree Protocol is based on MST algorithms and ensures a loop-free topology in Ethernet networks by creating a spanning tree that prevents network loops and ensures redundancy.

[7]. Minimum Spanning Tree (MST) using Kruskal's and Prim's Algorithm, Sahil Mattoo ,Intellipaat, 2025. https://intellipaat.com/blog/minimum-spanning-tree-algorithm/

Summary

- The Minimum Spanning Tree (MST) algorithm has a wide range of practical applications in various industries. It optimizes network connectivity in telecommunications and streamlines processes in logistics and circuit design.
- MST algorithms efficiently minimize costs while establishing optimal connections between points. Understanding their application is significant for engineers and computer scientists aiming for optimal network design and performance.
- Boruvka's algorithm is one of the oldest algorithms. It is a graph traversal algorithm used to find the minimum spanning tree of a connected, undirected graph.
- Kruskal's algorithm for finding the Minimum Spanning Tree(MST), which finds an edge of the least possible weight that connects any two trees in the forest.It is a greedy algorithm.
- Prim’s Algorithm operates by building this tree one vertex at a time, from an arbitrary starting vertex, at each step adding the cheapest possible connection from the tree to another vertex

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7. Minimum Spanning Tree (MST) using Kruskal’s and Prim's Algorithm, Sahil Mattoo ,Intellipaat, 2025. https://intellipaat.com/blog/minimum-spanning-tree-algorithm

Thank You!

For your attention