Course: Mathematical statistics

Week 6: Tests of hypothesis i.e.. null and alternate hypothesis

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Outline

Intended learning outcomes

- Explain the concept of statistical hypothesis testing.
- Differentiate between a null hypothesis H_0 and an alternative hypothesis H_1.
- State appropriate null and alternative hypotheses for various research scenarios.

Tests of hypothesis

Is an assertion, conjuncture or statement about parameters of one or more population.

categories of statistical hypothesis

- Simple statistical hypothesis (SSH)
- Composite statistical hypothesis (CSH)

It is SSH if it specifies the distribution of the parameter to be tested otherwise it is CSH

Null and alternate hypothesis

Null hypothesis is a statistical hypothesis that states there is no difference between a population parameter and a specified value or there is no difference between two population parameter.

If θ is a population parameter. $H_0 : \theta = \hat{\theta}$ where θ is the population and $\hat{\theta}$ a specified value.

$H_0 : \theta_1 = \theta_2$ e.g $H_0 : \mu = 8, H_0 : \mu_1 = \mu_2,\ H_0 : \mu_1 - \mu_2 = 0$

Alternating hypothesis is a statistical hypothesis that tests the existence of difference of a population and a specified parameter e.g. $H_1 : \theta \neq \theta_1$

Example

A medical research is interested in finding whether a new medication will have effect on the pulse rate. Does medication result in an increase, decrease in pulse rate or remain lately unchanged. The researchers knowns that the pulse rate for the population study is 80. state the Null and alternate hypothesis.

solution

means pulse rate of population is μ

$$\mu = 80, H_0 : \mu = 80, H_1 : \mu \neq 80, \mu > 80, or \mu < 80$$

Two sided null hypothesis

case II: If the researcher is interested whether the medication will lead to decrease in the pulse rate

$$H_0 : \mu = 80, H_1 : \mu < 80$$

one sided null hypothesis.

Example

state the null and alternate hypothesis in each of the following situation

(i) A medical researcher thinks that if expectant mothers who use vitamin pills, the birth weight of their babies increase the average birth weight is 3.5kg.

solution

$$H_0 : \mu = 3.5,$$

$$H_1 : \mu > 3.5$$

(ii) An engineer hypothesizes that the mean number of defects in manufacture of a certain product can be decreased by using robots instead of human beings. the mean number of defective items per 1500 is 24.

solution

$$H_0 : \mu = 24$$
$$H_1 : \mu < 24$$

(iii) a lecturer thinks that a new method of instruction will change the results of achievement test. The lecturer is not certain whether the grades will be higher or lower. in the past the mean score after using the traditional method was found o be 75.

solution

$$H_0 : \mu = 75$$

$$H_1 : \mu \neq 75$$

Example

For each of the following situations state the null and alternate hypothesis.

(i) The average age of lecturers at iuiu is 45 years

(ii) The average electricity bills of residents of an LC cell in mbale is atmost 50000 shillings

(iii) The average mirage for a new brand of tyres is greater than 12760 km

solution

(i)

$$H_0 : \mu = 45$$

$$H_1 : \mu \neq 45$$

(ii)

$$H_0 : \mu = 50000$$

$$H_1 : \mu \leq 50000$$

(iii)

$$H_0 : \mu = 12760$$

$$H_1 : \mu > 12760$$

Test Of statistical hypothesis

- Having selected H_0 and H_1 we need to select an appropriate statistic test.
- Test of statistical hypothesis is a rule which when the experimental sample values have been obtained leads to a decision of accepting or rejecting the null hypothesis under consideration
- Numerical value obtain from a statistical test is called a test value.
- For a critical region of a test statistic, if the computed test value falls in the region that leads to rejection of the null hypothesis, then that region is called a critical region for a test statistic.

The computed test values falls in the region that leads to acceptance then it is called the acceptance region

suppose our

$$H_0 : \mu = 40$$

$$H_1 : \mu \neq 40$$

suppose the rule is that we accept H_0 when $38.5 \leq \bar{x} \leq 41.8$ and reject H_0 otherwise.

critical region: $\bar{x} > 41.8$ or $\bar{x} < 38.5$

- Accept H_0 when it is true, correct decision
- Accept H_0 when it is false, wrong decision
- reject H_0 when it is true , wrong decision
- Reject H_0 when it is false, correct decision

Type I error - rejectH_0 when it is true $\alpha = P(typeIerror)$

Type II error- Accept H_0 when it is false

Decision	H_0 is true	H_0 is false
Accept	correct decision	wrong decision (type II error)
Reject	wrong decision (Type I error)	correct decision

Level of significance (α-error) is defined as the probability of committing type I error. $\alpha =$P(type I error), $\alpha =$P(Rejecting $H_0$when true).

β is the probability of type II error. $\beta =$ P(type Ii error)=P(Accept H_0 when false)

Example

The proportion of adults living in a town who are graduates is estimated to be $p = 0.3$, to test the null hypothesis, a random sample of 15 adults is selected if the number of graduates is any where from 2-7 will accept the null hypothesis. evaluate

(i) $\alpha-$ type I error assuming $P = 0.3$

(ii) $\beta-$ type II error for the $H_1 : P \neq 0.2$ and $H_1 : p \neq 0.4$

solution

$$H_0 : P = 0.3, H_1 : p \neq 0.3$$

$$\alpha = P(typeIerror) = p(RejectingH_0 whenitisTrue)$$

Acceptance: $2, 3, 4, 5, 6, 7$
Reject: $0, 1, 8, 9, 10, 11, 12, 13, 14, 15,$

$$P(x = 0, p = 0.3) + P(x = 1, p = 0.3) + P(x > 7, p = 0.3)$$

$$P(x = 0, p = 0.3) + P(x = 1, p = 0.3) + \Sigma_{x=8}^{15} P(x, p = 0.3)$$

$$= 0.0853$$

(ii)

$$\beta = P(typeIIerror) = p(AcceptingH_0 whenitisfalse)$$
$$= P(x = 2, 3, 4, 5, 6, 7, p = 0.2) = \Sigma_{x=2}^{7} C_x^{15} (0.2)^x (0.8)^{15-x}$$
$$= 0.8366$$

(iii)

$$\beta = P(typeIIerror) = p(AcceptingH_0 whenitisfalse)$$
$$= P(x = 2, 3, 4, 5, 6, 7, p = 0.4) = \Sigma_{x=2}^{7} C_x^{15} (0.4)^x (0.6)^{15-x}$$
$$= 0.7817$$

Example

it is known that the probability of a person recovering from a certain disease when treated with a given drug is 0.2. a random sample of 50 persons suffering from the disease are treated with a drug. if 14 people or more recover after getting treatment we reject $H_0 : p = 0.2$ and accept $H_1 : p = 0.4$. determine the probability of committing

(i) Type I error

(ii) Type II error

$\alpha =$ P(reject H_0 when true)

$$= p(x \geq 14, p = 0.2)$$

since n is large we use normal approximation

$$\mu = np = 50X0.2 = 10$$

$$\sigma^2 = npq = 50X0.2X0.8$$

$$\sigma = \sqrt{8}$$

$$= P(x \geq 14, p = 0.2) = P(z \geq \frac{14 - 0.5 - 10}{\sqrt{8}})$$

$$= P(z \geq 1.24) = 0.5 - 0.3925 = 0.1075$$

we know that

$$P(x < a) = a - 0.5, P(x > a) = a + 0.5, P(x \leq a)$$

$$= a + 0.5, P(x \geq a) = a - 0.5$$

(ii)

$$\beta = P(TypeIIerror)$$

=P(accept H_0 when false)

$$= P(x \leq 14, p = 0.4)$$

$$np = (50X0.4) = 20$$

$$\sigma^2 = 50X0.4X0.6 = 12, \sigma = \sqrt{12}$$

$$\beta = P(Z < \frac{13.5 - 20}{\sqrt{12}}) = P(z < -1.88) = 0.5 - 0.4699 = 0.0301$$

Note about type I and type II error

- The size of the critical region and consequently the P(type I error) can always be reduced by appropriate selection of critical values
- Type I and II errors are related. a decrease in the probability of one type of error always results in an increase in the probability of the other type provided that the sample size remains fixed

- An increase in a sample size will generally reduce both type I and II error provided that the critical value are held constant
- when the H_0 is false, the P(type II error) increase as he true value of the parameter approaches hypothesised value in the null hypothesis.

  The value of β decrease as the difference between the true mean and hypothesised value increases.

  If $H_1 : \mu < \mu_0$ then the critical region should lie in the lower tail of the distribution of the test statistics

General procedure for testing hypothesis

1. From the problem context identify the parameter of interest
2. State the null hypothesis e.g. $H_0 : \mu = \mu_0$, $H_0 : \mu_1 - \mu_0 = 0$ or $H_0 : p = p_0, H_0 : p - p_0 = 0$
3. specify and approximate alternative hypothesis $H_1 : \mu \neq \mu_0, H_1 : \mu_1 - \mu_0 \neq 0, H_1 : p \neq p_0, H_1 : p - p_0 \neq 0, H_1 : \mu < \mu_0$
4. select and appropriate level of significance α e.g. $1\%.5\%, 10\%$

1. Determine an appropriate test statistic e.g. $\bar{x}, \bar{p}, \bar{x}_1 - \bar{x}_2$
2. state the critical region for the test statistic
3. compute any necessary sample quantity, substitute this into the equation of the test statistic and compute the test value
4. Take a decision whether or not the null hypothesis to be rejected and you have to report it in the context of the problem

Case I

when μ is known and a population is normal or $n \geq 30$

$$H_0 : \mu = \mu_0, H_1 : \mu \neq \mu_0, H_1 : \mu > \mu_0, H_1 : \mu < \mu_0$$

sampling distribution of sample mean $\bar{x}$

$$\mu_{\bar{x}} = \mu, \sigma_{\bar{x}}^2 = \frac{\sigma^2}{n}, z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = N(0, 1)$$

Test statistic: Z value
critical region $z > z_{\frac{\alpha}{2}} or z < -z_{\frac{\alpha}{2}}, |z| > z_{\frac{\alpha}{2}}$

$$H_0 : \mu = \mu_0, H_1 : \mu < \mu_0$$

critical region is $Z < -z_{\alpha}$

$$H_0 : \mu = \mu_0, H_1 : \mu > \mu_0$$

Example

A distributor of water equipment is considering buying new synthetic water pipes from a manufacturer who claims that the pipes have an average length of $3.8m$.

A random sample of 35 pipes are measured and were found to have a mean length of $3.5m$ with standard deviation of $0.5m$. is there significant evidence at 0.01 level of significance to conclude that the mean length of the pipes is different from $3.8m$.

solution

parameter of interest is the mean μ

$H_0 : \mu = 3.8$

$$H_1 : \mu \neq 3.8, \alpha = 0.01, n = 35, \bar{x} = 3.5, s = 0.5$$

since $n \geq 30$ by CLT the sampling distribution of $\bar{x} \approx \textit{Normal}$

$$z = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}}$$

This is a two tailed test, critical region $|z| > z_{\frac{\alpha}{2}}$

$$|z| > z_{0.005} = 2.576$$

$$z = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} = \frac{3.5 - 3.8}{\frac{0.5}{\sqrt{35}}} = -3.54$$

$$|-3.54| > 2.576$$

Null hypothesis is rejected.

based on sample values, there is no sufficient evidence to conclude at 0.01 level of significance that the mean length of the pipes manufactured by the company is $3.8m$.

The mean length is less than $3.8m$ since if falls in the left tails.

Example

suppose 100 tyres made by a certain manufacturer lasts an average of $21819km$ with a standard deviation $1295km$. Is there sufficient evidence at 0.05 level of significance to conclude that the mean distance is less than $22000km$.

solution

parameter of interest is μ

$H_0 : \mu = 22000, H_1 : \mu < 22000$

$$\alpha = 0.05, n = 100, \bar{x} = 21819, s = 1295$$

since $n \geq 30$ by CLT $z = \frac{\bar{x}-\mu}{\frac{s}{\sqrt{n}}}$
this is a one tailed test. critical region $z < z_{\alpha}, z < z_{0.05} = -1.645$

$$z = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} = \frac{21819 - 22000}{\frac{1295}{\sqrt{100}}}$$

$$z = -1.4$$

Fails in acceptance region. we failed to reject the null hypothesis "we accept" therefore failing to reject means you have no sufficient evidence to reject a null hypothesis.

Example

A random sample of 100 death recorded in a certain country during the past years show an average life span of 71.8 years with SD of 8.9 years. does this seem to suggest that the average lifespan of citizens in that country is greater than 70 years , $\alpha = 0.05$.

solution

parameter of interest is μ

$H_0 : \mu = 70, H_1 : \mu > 70$

$$\alpha = 0.05, n = 100, \bar{x} = 71.8, s = 8.9$$

since $n \geq 30$ by CLT $z = \frac{\bar{x}-\mu}{\frac{s}{\sqrt{n}}}$

this is a one tailed test. critical region $z < z_{\alpha}, z < z_{0.05} = 1.645$

$$z = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} = \frac{71.8 - 70}{\frac{8.9}{\sqrt{100}}}$$

$$z = 2.02$$

$$2.02 > 1.645$$

therefore there is significance evidence that suggest that the average lifespan of a citizen in that country is greater than 70 years.

References

- Hogg,R;Mckean,J;Craig,A(2012).Introduction to mathematical statistics, 7th edition, pearson Prentice Hall, 2012.
- Hastings K.J,(1997) Probability and statistics, Addison Wesley reading,massachusetts.

Thank You!

Next Lecture: p-value in hypothesis test