Course: Mathematical statistics

Week 7: p-value in hypothesis test

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April 14, 2025

Outline

1. P-value hypothesis test
2. test for μ when σ is unknown
3. Test of population variance σ^2
4. Test of population proportion p

Intended learning outcomes

- Differentiate between the p-value approach and the critical value approach to hypothesis testing.
- Formulate null and alternative hypotheses for testing the population mean when the population standard deviation is unknown.
- Use the chi-square distribution to find critical values and p-values.
- Calculate the test statistic using the sample proportion and hypothesized population proportion.

P-value hypothesis test

- It gives the decision maker no ideas about whether the computed value of the test statistic was just bearing in the rejection region or referred in the rejection region
- The statement of results in this approach imposes a predetermined level of significance on other users of the information
- **p-value** is the smallest level of significance that will lead to rejection of the null hypothesis for a given data.

For a normal distribution test with known SD if z is a computed test value, the p-value is defined as

$$p = \begin{cases} 2(1-\phi|z|) & for, H_0 : \mu = \mu_0, H_1 : \mu \neq \mu_0 \\ 1-\phi(z) & for, H_0 : \mu = \mu_0, H_1 : \mu > \mu_0 \\ \phi(z) & for, H_0 : \mu = \mu_0, H_1 : \mu < \mu_0 \end{cases}$$

Normal distribution with known σ

- From the problem context identify the parameter of interest
- State the null hypothesis e.g. $H_0 : \mu = \mu_0$, $H_0 : \mu_1 - \mu_0 = 0$ or $H_0 : p = p_0, H_0 : p - p_0 = 0$
- specify and approximate alternative hypothesis $H_1 : \mu \neq \mu_0, H_1 : \mu_1 - \mu_0 \neq 0, H_1 : p \neq p_0, H_1 : p - p_0 \neq 0, H_1 : \mu < \mu_0$
- Compute $p - values$
- Take a decision

Example

A distributor of water equipment is considering buying new synthetic water pipes from a manufacturer who claims that the pipes have an average length of $3.8m$.

A random sample of 35 pipes are measured and were found to have a mean length of $3.5m$ with standard deviation of $0.5m$. is there significant evidence at 0.01 level of significance to conclude that the mean length of the pipes is different from $3.8m$.

solution

parameter of interest is the mean μ

$$H_0 : \mu = 3.8$$

$$H_1 : \mu \neq 3.8, \alpha = 0.01, n = 35, \bar{x} = 3.5, s = 0.5$$

since $n \geq 30$ by CLT the sampling distribution of $\bar{x} \approx \mathit{Normal}$

$$z = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}}$$

this is a two tailed test, critical region $|z| > z_{\frac{\alpha}{2}}$

$$|z| > z_{0.005} = 2.576$$

$$z = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} = \frac{3.5 - 3.8}{\frac{0.5}{\sqrt{35}}} = -3.54$$

$$p = 2(1 - \phi| - 3.54|), | - 3.54|$$

$$= 0.5 + \phi(3.54)$$

$$P = 2(1 - 0.9998) = 0.0004$$

since $0.0004 < 0.01$
we fail to reject the H_0

test for μ when σ is unknown

population is normal
n is small- $z = \frac{\bar{x}-\mu}{\frac{s}{\sqrt{n}}}$
steps

- From the problem context identify the parameter of interest
- State the null hypothesis e.g. $H_0 : \mu = \mu_0$, $H_0 : \mu_1 - \mu_0 = 0$ or $H_0 : p = p_0, H_0 : p - p_0 = 0$
- specify and approximate alternative hypothesis $H_1 : \mu \neq \mu_0, H_1 : \mu_1 - \mu_0 \neq 0, H_1 : p \neq p_0, H_1 : p - p_0 \neq 0, H_1 : \mu < \mu_0$

- select and appropriate level of significance α e.g. $1\%.5\%, 10\%$
- Determine an appropriate test statistic e.g. $\bar{x}, \bar{p}, \bar{x_1} - \bar{x_2}$
- state the critical region for the test statistic
- critical values $H_0 : \mu = \mu_0, H_1 : \mu \neq \mu_0$

$$z > t_{\frac{\alpha}{2}, n-1} or z < -t_{\frac{\alpha}{2}, n-1}, |z| > t_{\frac{\alpha}{2}, n-1}$$

$$H_0 : \mu = \mu_0, H_1 : \mu > \mu_0$$

$$H_0 : \mu = \mu_0, H_1 : \mu < \mu_0$$

$$t < -t_{\alpha,n-1}$$

Example

The specification of a certain type of ribbon calls for a mean breaking strength of $185kg$. if 5 pieces are randomly selected ribbon from different rolls have breaking strength of $171.6, 198.8, 178.3, 184.9, 189.1kg$ is there sufficient evidence at 0.05 level of significance to believe that the mean breaking strength is less than $185kg$. state any assumptions made.

solution

parameter is μ

$$H_0 : \mu = 185, H_1 : \mu < 185, \alpha = 0.05$$

Determination of test statistics

$$n = 5, \sigma = unknown$$

Distribution of strength of ribbon :silent

Assumption: strength of the ribbon is normally distributed

Test statistic is $t = \frac{\bar{x}-\mu}{\frac{s}{\sqrt{n}}}$

$$\bar{x} = \frac{\sum_{i=1}^{\infty}}{n=5} = 183.1$$

$$S = \sqrt{\frac{\sum(x_i - \bar{x})^2}{n-1}} = 8.2$$

$$t = \frac{183.1 - 185}{\frac{8.2}{\sqrt{5}}} = -0.49$$

critical region $t < -t_{0.05,4} = -2.132$
Decision: since $-0.49 > -2.132$
we fail to reject the null hypothesis and we have reasons that the mean breaking strength is 185kg

Example

a random sample of 8 cigarette of a certain run has an average nicotine content of $4.2mg$ and S.D is $1.4mg$. Test the hypothesis that the average nicotine content is $3.5mg$ assuming the distribution of nicotine content is normal. use the 0.01 level of significance

solution

parameter is μ

$$H_0 : \mu = 3.5, H_1 : \mu \neq 3.5, \alpha = 0.01$$

Determination of test statistics

$$n = 8, s = 3.5, \bar{x} = 4.2$$

Distribution is normal
critical region

$$|t| > t_{\frac{\alpha}{2}, n-1}, t_{0.005,7} = 3.499$$

$$t = \frac{4.2 - 3.5}{\frac{1.4}{\sqrt{8}}} = 1.414$$

falls in acceptance region since $t = 1.414 < 3.499$
we fail to reject the null hypothesis. The average nicotine content is 3.5kg

Example

A university secretary in a certain university claims that the pay per hour for part time lecturers in a private universities is 30000 shillings per hour. a random sample of 8 private universities are selected and their teaching cost per hour are 30000, 25000, 35000, 50000, 45,000,35000, 20000,40000 shillings per hour. is there enough evidence to support this claim at 0.01 level of significance. state any underlined assumptions

solution

parameter of interest is μ

$$H_0 : \mu = 30000, H_1 : \mu \neq 30000, \alpha = 0.01$$

Determination of test statistic

$$n = 8, s = 10000, \bar{x} = 35,000$$

critical region

$$|t| > t_{\frac{\alpha}{2},n-1}, |t| > t_{0.005,7} = 3.499$$

$$t = \frac{35000 - 30000}{\frac{10000}{\sqrt{8}}}, t = 1.41$$

we fail to reject the null hypothesis. there is enough evidence to support this claim

Test of population variance σ^2

To test hypothesis that variance $\sigma^2 = \sigma_0^2$ where σ_0^2 is a specified value.
For a random sample $x_1, x_2, ..., x_n$ from the population to test

$$H_0 : \sigma^2 = \sigma_0^2$$

Test statistic is $\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$ with $n-1$ degree of freedom critical region

$$\chi^2 < \chi^2_{1-\alpha/2,n-1}$$

or

$$\chi^2 > \chi^2_{\alpha/2,n-1}$$

for $H_1 : \sigma^2 > \sigma_0^2$.

$$\chi^2 > \chi^2_{\alpha,n-1}$$

$$for H_1 : \sigma^2 < \sigma_0^2$$

$$\chi^2 < \chi^2_{1-\alpha,n-1}$$

Example

A random sample of 20 500 millilitre bottles of mineral water is found to have an average of 498.2 ml of water with standard deviation of 2.5ml. if the variance of the fill volume exceeds 5ml an acceptable proportion of the bottles will be over filled or under filled. is there sufficient evidence in this data to suggest that the bottle has the problem with under filled or overfilled bottles. use $\alpha = 0.05$ and assume that the fill volume is normally distributed.

solution

parameter of interest is σ^2
$H_0 : \sigma^2 = 5$
$H_1 : \sigma^2 > 5$
$\sigma = 0.05$

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$$

$$\frac{19(6.25)}{25} = 23.75$$

critical region $\chi > \chi_{0.005,19} = 30.14$
we fail to reject the null hypothesis since the computed value is less than the critical value.
The variance of fill volume does not exceed 5ml.

Test of population proportion p

if there x objects out of n with a specified characteristics then the sample proportion $\hat{p} = \frac{x}{n}$ is unbiased estimator of p.
sampling distribution of $\hat{p}$

$$\mu_{\hat{p}} = p$$

$$\sigma^2_{\hat{p}} = \frac{p(1-p)}{n}$$

$$z = \frac{\hat{p} - p}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}}$$

Is standard normal random variable but since p is a parameter to be estimated we use $\hat{p}$

$$z = \frac{\hat{p} - p}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}}$$

$$n\hat{p} \geq 5, n\hat{p}\hat{q} \geq 5$$

To test the null hypothesis

$$H_0 : p = p_0, H_1 : p \neq p_0$$

level of significance α the test statistic

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

critical region $z < -z_{\alpha/2}, z > z_{\alpha/2}$ or $|z| > z_{\alpha/2}$
for alternate hypothesis $H_1 : p > p_0$
critical region is $z > z_{\alpha}$, $H_1 : p < P_0$ critical region is $z < -z_{\alpha}$

Example

An educator estimates that the drop-out rate at p/s due to early pregnancy is 10%. last year 18 girls of a sample of 200 were found to be pregnant. assume that the pregnancy rate is normally distributed. is there evidence at 0.05 level of significance to reject the educators claim.

solution

parameter of interest proportion p

$H_0 : p = 0.1$

$H_1 : p \neq 0.1$

$\sigma = 0.05$

critical region $|z| > z_{\alpha/2} = z_{0.025} = 1.96$

$$\hat{z} = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

$$= \frac{18}{200} = 0.09$$

$$p_0 = 0.1, n = 200$$

$$z = \frac{0.09 - 0.1}{\sqrt{\frac{0.1(1-0.1)}{200}}}$$
$$= -0.47$$

$|z| = 0.47 < 1.96$
we fail to reject the null hypothesis since the computed value is less than the critical value.
We do not have evidence to reject the claim that the pregnancy rate is 0.1

Example

The regional traffic police officer claim that more than 25% of the drivers in the region are using forged driving permits. a sample of 200 drivers in the region showed that 63 had forged permits. at 0.01 level of significance is there enough evidence to support he officers claim. state any assumptions made.

parameter of interest is proportion p

$$H_0 : p = 0.25$$
$$H_1 : p > 0.25$$

- The sampling distribution of $\hat{p}$ is approximately normal since:

$$np_0 = 200 \times 0.25 = 50 \geq 5, \quad n(1 - p_0) = 200 \times 0.75 = 150 \geq 5$$

We use the z-test for population proportion:

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0.315 - 0.25}{\sqrt{\frac{0.25\times0.75}{200}}} = \frac{0.065}{\sqrt{0.0009375}} = \frac{0.065}{0.0306} \approx 2.125$$

- Critical value for $\alpha = 0.01$ (right-tailed): $z_{0.01} = 2.33$
- Since $z = 2.125 < 2.33$, we fail to reject the null hypothesis.

References

- Hogg,R;Mckean,J;Craig,A(2012).Introduction to mathematical statistics, 7th edition, pearson Prentice Hall, 2012.
- Hastings K.J,(1997) Probability and statistics, Addison Wesley reading,massachusetts.

Thank You!

Next Lecture: Inference for two samples