Course: Automatic Control System Technology

Lecture 2: Describe Laplace transform fundamentals

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Session objectives

By the end of this lecture, students will be able to :

- Explain why Laplace Transform is needed
- Define Laplace transform: F(s)
- Define the region of convergence (ROC) of Laplace transform
- Describe the position of poles and zeros of F(s) versus ROC in s-plane
- Calculate Laplace transform of test signals: Examples
- Use Laplace transform table
- Describe the Laplace transform properties and theorems

Explain why Laplace Transform is needed

- The lecture 1 dealt with control system description and;
- The control system was described by its input signal, control system components or weighted function and output signal.
- The **weighted function** of a control system is its **impulse response**, which is the response or output of the system to unit-impulse input.

Explain why Laplace Transform is needed

- The lecture 1 concluded that this course of Automatic Control System Technology is going to focus on design and analysis of causal, SISO, deterministic, Linear Time-Invariant (LTI) control systems in time and S domains;
- The fundamental input-output relationship for LTI systems, in time domain, is described in terms of a convolution operation, refer to Figure 1.

Explain why Laplace Transform is needed

- In other words: the output y(t) of a LTI system is defined as the convolution of its weighting function(impulse response) h(t) and the driving signal(input) x(t) as it is shown in Figure 1.

Figure 1: A linear time invariant (LTT) system

Hwei P. Hsu, Ph.D.(1995), Schaum's outlines of Theory and Problems of Signals and Systems, McGraw-Hill, page 57.

Explain why Laplace Transform is needed

- The convolution integral operation is commutative and it can be written as follows;

$$y(t) = h(t) * x(t) = x(t) * h(t)$$

$$= \int_0^t h(\tau)x(t-\tau)d\tau = \int_0^t x(\tau)h(t-\tau)d\tau$$

Explain why Laplace Transform is needed

❖ **Example:**

The impulse response of an RC circuit $\boldsymbol{h(t) = \frac{1}{Rc} e^{-\frac{t}{RC}}, t \geq 0}$,

determine the response **y(t)** of RC circuit to a step signal

defined $\boldsymbol{x(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}}$.

Explain why Laplace Transform is needed

❖ **Solution: Method 1.**

The output a LTI system in time domain is given by**:**

$$\boldsymbol{y(t) = h(t) * x(t) = \int_0^t h(\tau)x(t-\tau)d\tau}$$

$$= \int_0^t \frac{1}{Rc} e^{-\frac{\tau}{RC}}\, d\tau = \frac{1}{Rc}\int_0^t e^{-\frac{\tau}{RC}}\, d\tau = \frac{1}{\cancel{Rc}} * \frac{1}{-\frac{1}{\cancel{Rc}}}\left[e^{-\frac{\tau}{RC}}\right]_0^t$$

$$= -\left[e^{-\frac{\tau}{RC}}\right]_0^t = -(e^{-\frac{t}{RC}}-1) = 1 - e^{-\frac{t}{RC}},\ \mathrm{t} \geq 0$$

Explain why Laplace Transform is needed

❖ **Solution: Method 2.**

The output a LTI system in time domain is given by**:**

$$\boldsymbol{y(t) = x(t) * h(t) = \int_0^t x(\tau)h(t-\tau)d\tau}$$

$$\boldsymbol{= \int_0^t \frac{1}{Rc} e^{-\frac{(t-\tau)}{RC}}\, d\tau = \frac{1}{Rc}\int_0^t e^{-\frac{t}{RC}} e^{\frac{\tau}{RC}} d\tau = \frac{e^{-\frac{t}{RC}}}{Rc}\int_0^t e^{\frac{\tau}{RC}} d\tau}$$

$$= \frac{e^{-\frac{t}{RC}}}{\cancel{Rc}} \frac{1}{\frac{1}{\cancel{RC}}} \left[e^{\frac{\tau}{RC}}\right]_0^t = e^{-\frac{t}{RC}}\left(e^{\frac{t}{RC}} - 1\right) = 1 - e^{-\frac{t}{RC}},\ \mathrm{t} \geq 0$$

Explain why Laplace Transform is needed

- The convolution integral operation helps to determine the response/output of a LTI control system.
- However its computation is difficult and time consuming.
- To overcome this challenge, the **output of a LTI control system** is computed in frequency domain or **s-domain** and later on converted into time domain.

Explain why Laplace Transform is needed

❖ The mathematical tool used convert a LTI control system from time domain to s-domain is the **Laplace Transform**.

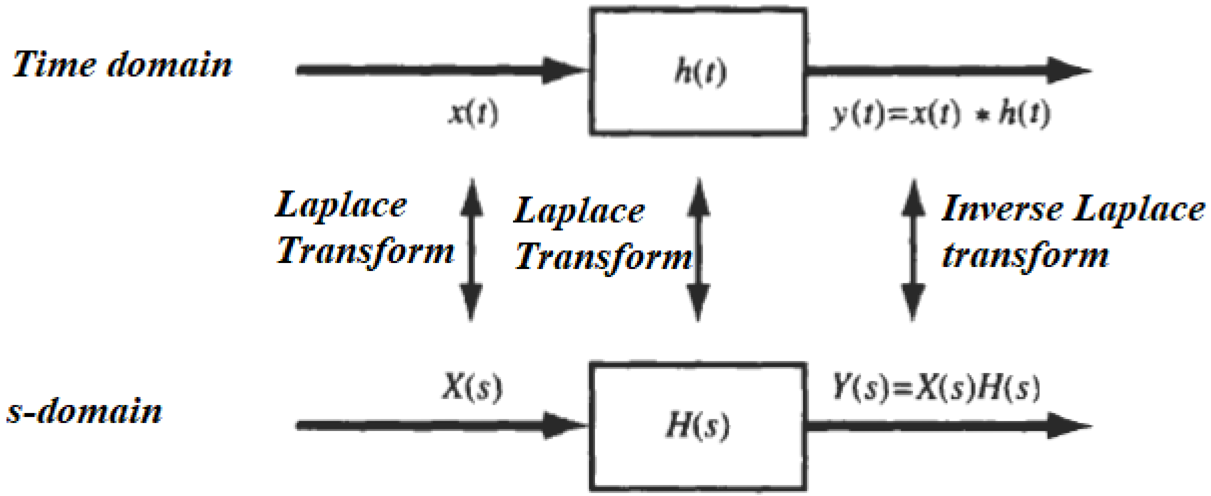

Figure 2: Conversion from time domain to s-domain

Hwei P. Hsu, Ph.D.(1995), Schaum's outlines of Theory and Problems of Signals and Systems, McGraw-Hill, page 121.

Define Laplace transform

- The Laplace Transform of continuous time function f(t), denoted as $F(s) = \mathcal{L}[f(t)]$, is defined as $\boldsymbol{F(s) = \int_{-\infty}^{\infty} f(t) e^{-st} dt}$ **.**
- The variable s is referred to as **the Laplace operator**, which is a complex variable; that is, s=Re(s)+Im(s) = σ + jω, where σ is the real component, $j = \sqrt{-1}$, and ω is the imaginary component.
- This Laplace transform definition is often called the **bilateral (or two-sided) Laplace transform** in contrast to the **unilateral (or one-sided) Laplace transform**.

Define Laplace transform

- A **unilateral (or one-sided) Laplace transform** is defined as follows: $\boldsymbol{F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt}$
- For the unilateral (or one-sided) Laplace transform, the integration is evaluated from t = 0 to + ∞
- In the bilateral (or two-sided) Laplace transform, the integration is evaluated from from t = - ∞ to + ∞
- The **bilateral and unilateral Laplace Transforms** are equivalent for causal function or signal.

Define Laplace transform

- That is when f(t)=0, for $t < 0$
- In this course, we will use the unilateral Laplace transform definition : $\boldsymbol{F}(\boldsymbol{s}) = \int_{\mathbf{0}^-}^{\infty} \boldsymbol{f}(\boldsymbol{t})\boldsymbol{e}^{-\boldsymbol{st}}\boldsymbol{dt}$,but we will omit the word unilateral or one sided as we will be dealing with causal or physically realizable control systems
- That is also because, in time-domain studies, time reference is often chosen at t = 0. And when an input is applied at t = 0, the response of a physical system does not start sooner than t = 0

Define the region of convergence (ROC) of Laplace transform

- We call the **region of convergence** (ROC) of Laplace transform: The condition of existence of the integral $\boldsymbol{F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt}$
- In other words the **region of convergence** (ROC) of Laplace transform is defined as the range of values of the **complex variables s** for which the Laplace transform converges.

Define the region of convergence (ROC) of Laplace transform

Example :

Exponential function: $f(t) = e^{-at}u(t) = \begin{cases} e^{-at} & ,t \geq 0 \\ 0 \ , & t < 0 \end{cases}$, a real

$$F(s) = \int_0^{\infty} e^{-at} e^{-st} dt = -\frac{1}{s+a}\left[e^{-(s+a)t}\right]_0^{\infty} = \frac{1}{s+a}$$

This integral is defined only if Re(s+a) > 0

The ROC: **Re(s)** $>$ **−Re(a)= −a**

The ROC is an infinite half plane that lies strictly to the right of the vertical line through s=-a

Describe the position of poles and zeros of F(s) versus ROC in s-plan

- Very often, Laplace transforms are rational fractions, that is as the ratio of two constant-coefficient polynomials of the form:

$$F(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}, \quad m<n$$

Where a_i and b_j are real

- The Laplace transform zeros are the values of s for which F(s)=0. They are found by setting the numerator polynomial to 0 and solving the resulting equation.

Describe the position of poles and zeros of F(s) versus ROC in s-plane

- The Laplace transform poles are the values of s for which F(s) is infinite. They are found by setting the denominator polynomial to o and solving the resulting equation.
- Therefore, the poles of F(s) lie outside the ROC since F(s) does not converge at the poles.
- The zeros, on the other hand, may lie inside or outside the ROC.
- **Note**: The ROC does not contain any poles.

Calculate Laplace transform for test signals: Examples

- In control engineering, standard test input signals are used, both analytically and during testing, to verify the design. Thus, it is of great importance, for control engineers, to know their Laplace transforms.
- The standard test signals are: an impulse input, a step input, a ramp input, and a parabolic input.

Calculate Laplace transform for test signals: Examples

- Standards test signals may be a sudden shock(impulse input),a sudden change(step input), a linear change with time (ramp input) or faster changes with time(parabola input)
- A sinusoidal input is also a standard test signal and it is used to analyze linear time invariant systems in frequency domain.
- Standard test signals may be used as command (or reference) input or the disturbance inputs

Calculate Laplace transform for test signals: Test signals

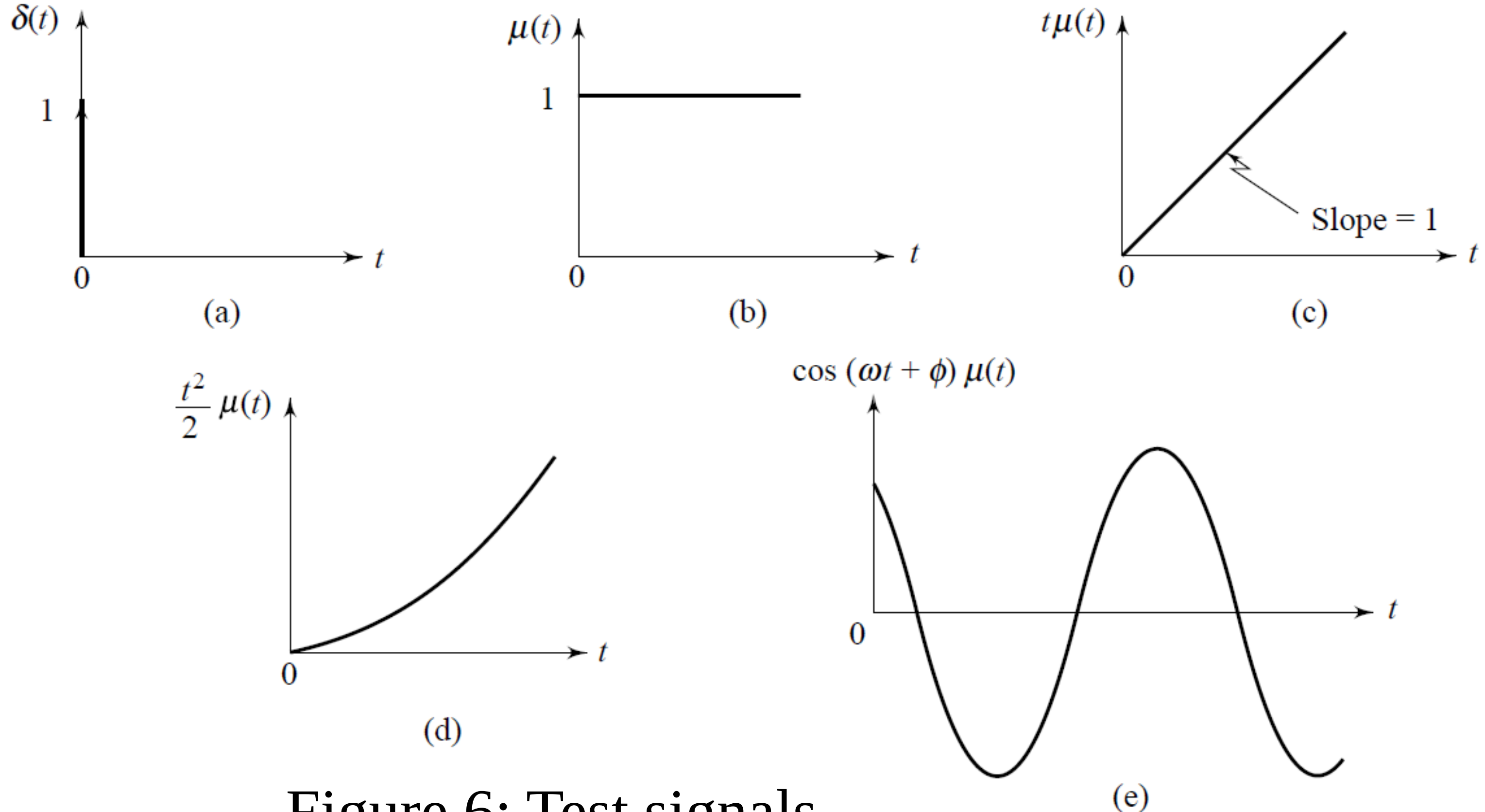

Figure 6: Test signals

Gopal M (2008),Control Systems: Principles and Design, 3rd Edition, Tata McGraw-Hill, page 39

Calculate Laplace transform for test signals: Examples

Example 1:

- ❖ A unit impulse function, $\boldsymbol{\delta(t)}$ is defined as $\delta(t) = \begin{cases} 1 \ \ for\ t = 0 \\ 0 \ \ for\ t \neq 0 \end{cases}$
- ❖ Impulse signal properties:
  - ✓ Sampling property: $\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$
  - ✓ Shifting property: $\int_{-\infty}^{\infty} f(t)\delta(t - t_o)dt = f(t_o)$
- ❖ From the sampling property,

$$\boldsymbol{F(s) = \mathcal{L}\{\delta(t)\} = \int_0^{\infty} \delta(t)\, e^{-st} dt = e^{-s.0} = 1}$$, ROC: **all s**

Calculate Laplace transform for test signals: Examples

Example 2:

A unit step function, $\boldsymbol{u(t)}$ is defined as $f(t) = \begin{cases} 1 & for\ t \geq 0 \\ 0 & for\ t < 0 \end{cases}$

$$\boldsymbol{F(s)} = \int_{\boldsymbol{0}}^{\infty} \boldsymbol{f(t)e^{-st}dt} = \int_0^{\infty} u(t)e^{-st}dt = \int_0^{\infty} e^{-st}dt$$

$$= -\frac{1}{s}e^{-st}\Big|_0^{\infty} = -\frac{1}{s}[0-1] = \frac{1}{s}$$

The ROC: $\mathbf{Re(s) > 0}$

Calculate Laplace transform for test signals: Examples+ROC

Example 3:

A ramp function: $f(t) = tu(t) = \begin{cases} t & ,t \geq 0 \\ 0 & ,t < 0 \end{cases}$

$$F(s) = \int_{\mathbf{0}}^{\infty} \boldsymbol{f(t)e^{-st}dt} = \int_0^{\infty} te^{-st}dt$$

$$= -\frac{1}{s}[te^{-st}]_0^{\infty} + \frac{1}{s}\int_0^{\infty} e^{-st}dt = \frac{1}{s^2}$$

The ROC: $\mathbf{Re(s) > 0}$

Use Laplace transform table

Instead of computing ***Laplace transform for each function***, ***and/or memorizing complicated Laplace transforms***, you can use the ***Laplace transform table*** or ***pairs***

Use Laplace transform table

Table 1.Table of Laplace transforms

$f(t);\ t \geq 0$	$F(s)$	$f(t);\ t \geq 0$	$F(s)$
$\delta(t)$: unit impulse	1	$1-e^{-at}$	$\dfrac{1}{s(s+a)}$
$\mu(t)$: unit step	$\dfrac{1}{s}$	$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
t	$\dfrac{1}{s^2}$	$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
$\dfrac{t^2}{2}$	$\dfrac{1}{s^3}$	$e^{-at} \sin \omega t$	$\dfrac{\omega}{(s+a)^2+\omega^2}$
e^{-at}	$\dfrac{1}{s+a}$	$e^{-at} \cos \omega t$	$\dfrac{s+a}{(s+a)^2+\omega^2}$
te^{-at}	$\dfrac{1}{(s+a)^2}$		

Gopal M (2008),Control Systems: Principles and Design, 3rd Edition, Tata McGraw-Hill, page 43.

Describe the Laplace transform properties and theorems

- Linearity: $\mathcal{L}\{\alpha_1 f_1(t) + \alpha_2 f_2(t)\} = \alpha_1 F_1(s) + \alpha_2 F_2(s)$
- Time delay: $\mathcal{L}\{f(t-T)u(t-T)\} = e^{-Ts}F(s)$
- Differentiation: $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$

  For n^{th} order derivative

  $$\mathcal{L}\{f^n(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

- Integration: $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$

Describe the Laplace transform properties and theorems

- ✓ The Laplace variable s is considered as the **differential operator**: $s \equiv \frac{d}{dt}$.
- ✓ Then, we also have **integral operator**: $\frac{1}{s} \equiv \int_0^t d\tau$.

Describe the Laplace transform properties and theorems

❖ **Real convolution :**

$$\mathcal{L}[f_1(t) * f_2(t)] = \mathcal{L}\left[\int_0^t f_1(\tau) f_2(t-\tau) d\tau\right]$$

$$= \mathcal{L}\left[\int_0^t f_2(\tau) f_1(t-\tau) d\tau\right] = F_1(s) F_2(s)$$

Where $*$ denotes convolution in time domain

❖ **Complex convolution :** $\mathcal{L}[f_1(t) f_2(t)] = F_1(s) * F_2(s)$

In this case * denotes complex convolution

Farid Golnaraghi & Benjamin C. Kuo (2019), Automatic Control Systems,10th Edition, McGraw-Hill Education, page 180.

Describe the Laplace transform properties and theorems

❖ **Final value theorem:** $\lim_{t\longrightarrow\infty} f(t) = \lim_{s\longrightarrow 0} sF(s)$,

if all poles of sF(s) are in the left half plane.

This theorem is very important since it helps to determine the final value of time function (i.e. steady state value) by determining its Laplace transform as $s \longrightarrow 0$.

❖ **Initial value theorem:** $\lim_{t\longrightarrow 0} f(t) = \lim_{s\longrightarrow\infty} sF(s)$ ***if the limit exist***.

Also, this theorem helps to determine the initial value of f(t) from F(s) when $s \longrightarrow \infty$

Describe the Laplace transform properties and theorems

- ❖ Examples of determining the initial and final values of a time varying function when you know its Laplace transform:

  - ✓ **Example 1:** Consider the function: $\boldsymbol{F(s) = \frac{5}{s(s^2+s+2)}}$
  - ✓ Because sF(s) does not have poles on imaginary axis and in the right half s-plane; which means that the system is stable; then the final value theorem may be applied.

$$\boldsymbol{f(\infty) = \lim_{t\longrightarrow\infty} f(t) = \lim_{s\longrightarrow 0} sF(s) = \lim_{s\longrightarrow 0} \frac{5}{(s^2+s+2)} = \frac{5}{2}}$$

Describe the Laplace transform properties and theorems

- ✓ **Example 2:** Consider the function: $\boldsymbol{F(s) = \frac{\omega}{s^2+\omega^2}}$
- ✓ Which is the Laplace transform of $f(t) = \sin(\omega t)$. Because the function **sF(s)** has two poles on the **imaginary axis of the s-plane**, the final-value theorem **cannot** be applied in this case. In other words, although the final-value theorem would yield a value of zero as the final value of 0, the result is erroneous.

References

1. Farid Golnaraghi & Benjamin C. Kuo (2019), Automatic Control Systems,10th Edition, McGraw-Hill Education.
2. Gopal M (2008),Control Systems: Principles and Design, 3rd Edition, Tata McGraw-Hill.
3. Hwei P. Hsu, Ph.D.(1995), Schaum's outlines of Theory and Problems of Signals and Systems, McGraw-Hill.

THANK YOU