Course: Automatic Control System Technology

Lecture 5: Model electrical systems

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Model electrical systems

Session objectives:

By the end of this session, students will be able to :

- ❖ Define modeling a system
- ❖ Model passive electrical elements
- ❖ Model Electrical networks

Define modeling a system

- Modeling a system means to find a mathematical or graphical representation of a system.
- A system may have a mathematical or graphical representation/model.

Define modeling a system

- A mathematical model of a dynamic system is a set of equations that represents the dynamics of the system accurately.
- A graphical model of a system is a visual or pictorial representation of functions performed by each system component

Define modeling a system

- A control system can mathematically be represented by a ***differential equation***, ***transfer function*** or ***state space equation.*** Farid Golnaraghi & Benjamin C. Kuo (2010), Automatic Control Systems,9th Edition, John Wiley&Sons, page 147.
- A control system can graphically be represented **by a block diagram or a signal flow graph.**

Define modeling a system

- ❖ System models are useful for analysis and design of control systems.
- ✓ **The analysis of control system** means ***finding the output*** when ***we know the input*** and ***mathematical model***.
- ✓ **Design of control system** means ***finding the mathematical model*** when we know ***the input and the output.***

Define modeling a system

- ❖ In this lecture, we will focus on **determining differential equation models** of electrical networks with simple passive elements such as resistors, inductors, and capacitors.
- ❖ Differential equation model is a time domain mathematical model of control system.

Define modeling a system

- Other system models such as transfer function model, block diagram model, signal flow graph model and state space model will be taught in other lectures

Model passive electrical elements

❖ The basic passive electrical elements: resistors, inductors, and capacitors.

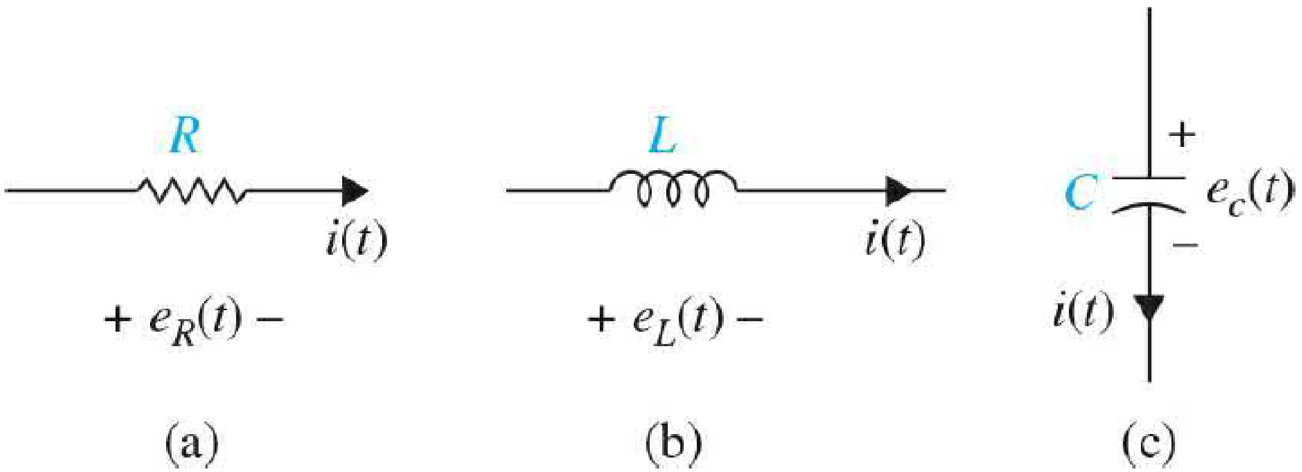

Figure 1. Basic passive electrical elements.
(a) A resistor, (b) An inductor, (c) A capacitor

Farid Golnaraghi & Benjamin C. Kuo (2019), Automatic Control Systems, 10th Edition, McGraw-Hill Education, page 103.

Model passive electrical elements

- ***Resistors.*** Ohm's law states that the voltage drop, $e_R(t)$, across a resistor R is proportional to the current $i(t)$ going through the resistor.

$$\boldsymbol{e_R(t) = Ri(t)}$$

Model passive electrical elements

- ***Inductors*.** The voltage drop, $e_L(t)$, across an inductor L is proportional to the variation of current $i(t)$ going through the inductor with respect to time. Thus,

$$\boldsymbol{e_L(t) = L\frac{di(t)}{dt}}$$

Model passive electrical elements

❖ ***Capacitor*.** The voltage drop, $e_C(t)$, across a capacitor C is proportional to the integral of current $i(t)$ going through the capacitor with respect to time. Therefore:

$$\boldsymbol{e_C(t) = \frac{1}{C}\int_0^t i(\tau)d\tau}$$

Model passive electrical elements

- ❖ By differentiating both sides of this equation representing the voltage drop across the capacitor , we get the current in the capacitor because the derivative is the inverse of integral.

$$\frac{de_C(t)}{dt} = \frac{1}{C} i(t) \text{ , which gives } i(t) = c\,\frac{de_C(t)}{dt}$$

Model Electrical networks

- ❖ Follow these two steps to find a differential equation model of electrical networks
  - ✓ ***Apply Kirchhoff's laws to electrical circuit.***
  - ✓ Get the differential equation in terms of ***input and output*** by eliminating the ***intermediate variable(s)***

Model Electrical networks

- ❖ Kirchhoff's laws state:
  - ✓ ***Current law :***The algebraic summation of all currents entering a node is zero.
  - ✓ ***Voltage law :*** The algebraic sum of all voltage drops around a complete closed loop is zero.

Model Electrical networks

Example 1: Consider the RLC network shown in the figure 2. Find the differential equation of the system. The **output** is the **voltage** $\boldsymbol{e_C(t)}$ **across the capacitor.**

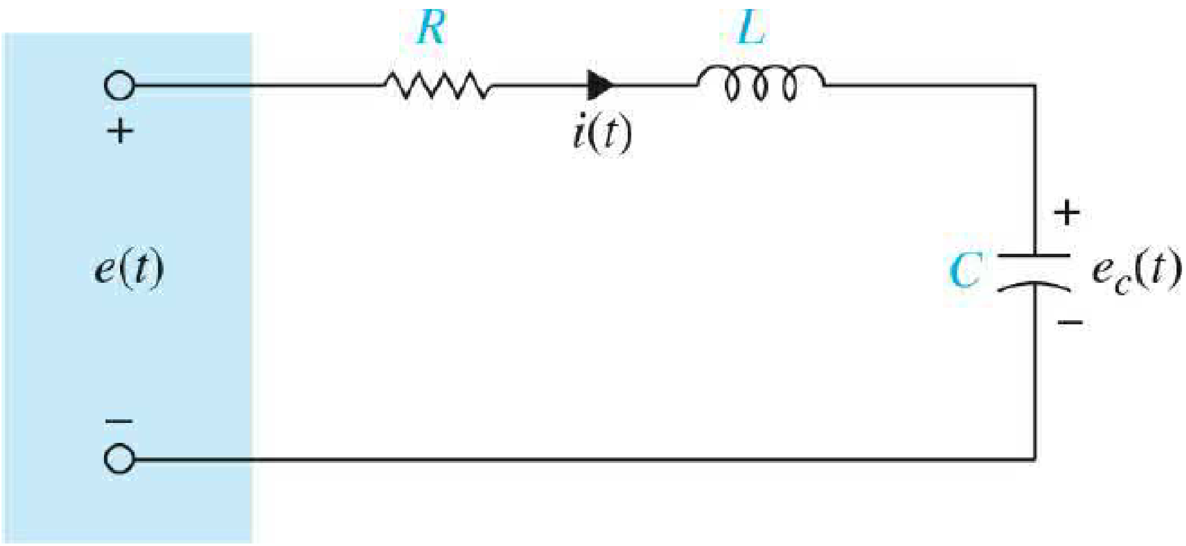

Figure 2. RLC circuit

Farid Golnaraghi & Benjamin C. Kuo (2019), Automatic Control Systems,10th Edition, McGraw-Hill Education, page 104.

Model Electrical networks

- Applying Kirchhoff's voltage law to the RLC circuit shown in figure 2, we get:

$$\boldsymbol{e(t) = e_R(t) + e_L(t) + e_C(t)}$$

where $e_R(t) = Ri(t)$, and $e_L(t) = L\frac{di(t)}{dt}$

- Hence, we have:

$$\boldsymbol{e(t) = Ri(t) + L\frac{di(t)}{dt} + e_C(t)} \text{ ----- eq1}$$

Model Electrical networks

- **Note:** $\boldsymbol{e(t)}$ and $\boldsymbol{e_C(t)}$ are respectively input and output voltage. Thus, the differential equation of the RLC circuit will be expressed in terms of those variables only. The current $i(t)$ is the intermediate variable and has to be eliminated from the differential equation.
- The voltage $e_C(t) = \frac{1}{C}\int_0^t i(\tau)\, d\tau$ and hence the current in the capacitor is $\boldsymbol{i(t) = C\frac{de_C(t)}{dt}}$

Model Electrical networks

- By substituting the current $\boldsymbol{i(t)}$ in the eq1, we get the differential equation of the RLC network:

$$\boldsymbol{e(t) = Rc\frac{de_C(t)}{dt} + LC\ \frac{d^2 e_C(t)}{dt^2} + e_C(t)}$$

- This equation can be rearranged from the highest derivative of the output as follows:

$$\boldsymbol{LC\ \frac{d^2 e_C(t)}{dt^2} + Rc\frac{de_C(t)}{dt} + e_C(t) = e(t)}$$

Model Electrical networks

Example 2: Consider the RC circuit shown in figure 3. Find the differential equation of the system. The **output** is the **voltage** $\boldsymbol{e_C(t)}$ **across the capacitor.**

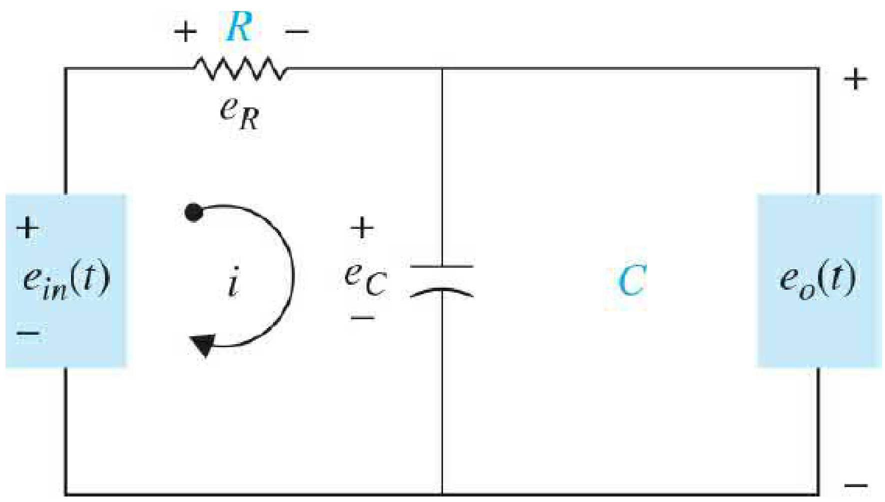

Figure 3. RC circuit

Farid Golnaraghi & Benjamin C. Kuo (2019), Automatic Control Systems, 10^{th} Edition, McGraw-Hill Education, page 106.

Model Electrical networks

- Applying Kirchhoff's voltage law to the RC circuit shown in figure 3, we get:

$$\boldsymbol{e_{in}(t) = e_R(t) + e_C(t)}$$

where $e_R(t) = Ri(t)$

- Hence, we have:

$$\boldsymbol{e_{in}(t) = Ri(t) + e_C(t)} \text{ ---- eq2}$$

Model Electrical networks

- From the figure 3., the output voltage $e_C(t) = e_o(t)$ and then, the current in the capacitor is $i(t) = C\frac{de_C(t)}{dt} = C\frac{de_o(t)}{dt}$ and by substituting these variables in the equation eq2, we find the differential equation of RC network:

$$e_{in}(t) = RC\frac{de_o(t)}{dt} + e_o(t)$$

- This equation can be rearranged from the highest derivative of the output as follows:

$$RC\frac{de_o(t)}{dt} + e_o(t) = e_{in}(t)$$

Model Electrical networks

Example 3: Consider the RC circuit shown in figure 4. Find the differential equation of the system. The **output** is the **voltage $\boldsymbol{e_R(t)}$ across the Resistor**.

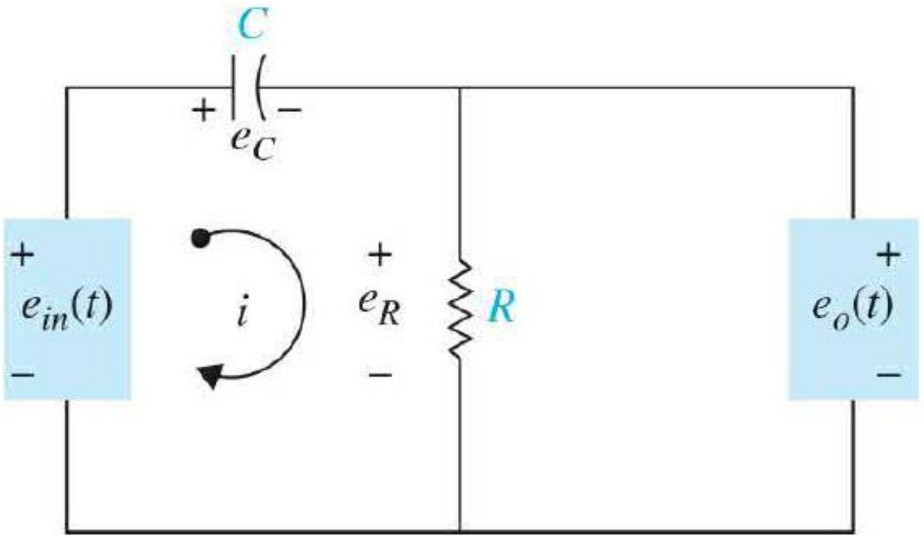

Figure 4. RC circuit

Farid Golnaraghi & Benjamin C. Kuo (2019), Automatic Control Systems,10th Edition, McGraw-Hill Education, page 108.

Model Electrical networks

- Again, by applying Kirchhoff's voltage law to the RC circuit shown in figure 4, we get:

$$\boldsymbol{e_{in}(t) = e_C(t) + e_R(t)}$$

With: $e_R(t) = e_0(t)$, and $e_C(t) = \frac{1}{C}\int_0^t i(\tau)\, d\tau$

- Hence, we have:

$$\boldsymbol{e_{in}(t) = \frac{1}{C}\int_0^t i(\tau)\, d\tau + e_o(t)} \text{ ---- eq3}$$

Model Electrical networks

- The output voltage $\boldsymbol{e_R(t) = Ri(t) = e_o(t)}$,

  and $i(t) = \frac{e_R(t)}{R} = \frac{e_0(t)}{R}$

- By substituting the current $\boldsymbol{i(t)}$ in the eq3, we get the differential equation of the RC network with the output take across the resistor:

$$\boldsymbol{e_{in}(t) = \frac{1}{C}\int_0^t \frac{e_0(\tau)}{R} d\tau + e_o(t)}$$

Model Electrical networks

- Rearranging the previous equation, we get:

$$e_{in}(t) = \frac{1}{RC}\int_0^t e_o(\tau)\,d\tau + e_o(t) \text{ ----eq4}$$

- To solve this equation, we can differentiate it with respect to time:

$$\dot{e}_{in}(t) = \frac{1}{RC}e_o(t) + \dot{e}_o(t)$$

- This equation can be written from the highest derivative of the output as follows:

$$\dot{e}_o(t) + \frac{1}{RC}e_o(t) = \dot{e}_{in}(t)$$

Model Electrical networks

Example 4: Consider the voltage divider shown in figure 5. Find the mathematical model of the system. The **output** is the **voltage** $\boldsymbol{e_{R_2}(t)}$ **across the resistor** $\boldsymbol{R_2}$.

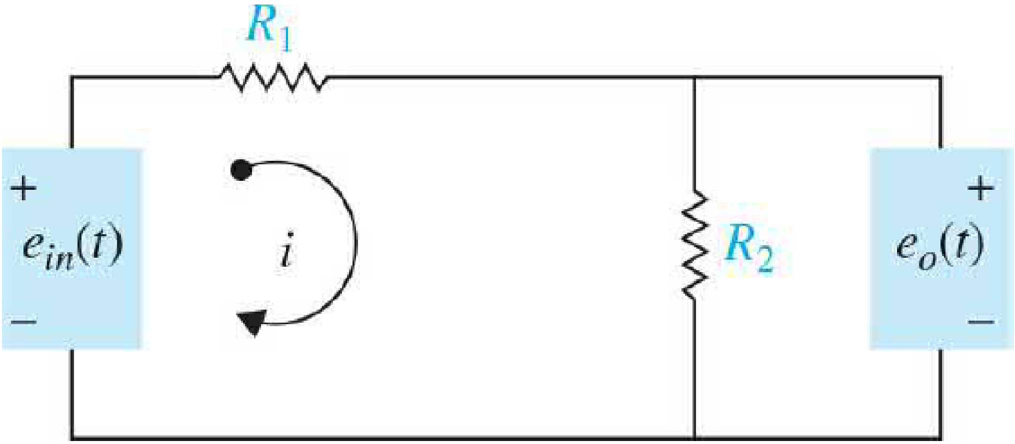

Figure 5. A voltage divider

Farid Golnaraghi & Benjamin C. Kuo (2019), Automatic Control Systems,10th Edition, McGraw-Hill Education, page 109.

Model Electrical networks

- From figure 5, we can write:

$$e_{in}(t) = e_{R_1}(t) + e_{R_2}(t)$$

where $e_{R_2}(t) = e_o(t)$, and $e_{R_1}(t) = R_1 i(t)$

- Hence, we have:

$$e_{in}(t) = R_1 i(t) + e_o(t) \text{------eq4}$$

- The current $i(t)$ through the resistor R_2 is the intermediate variable and it has to be eliminated from the above equation.

Model Electrical networks

- From the figure 5, it is clear that $\boldsymbol{e_{R_2}(t) = e_o(t) = R_2 i(t)}$,

  hence $\boldsymbol{i(t) = \frac{e_o(t)}{R_2}}$.

- Substituting $\boldsymbol{i(t) = \frac{e_o(t)}{R_2}}$ in the equation eq4, we get:

$$\boldsymbol{e_{in}(t) = R_1 \frac{e_o(t)}{R_2} + e_o(t)}$$

Model Electrical networks

- Rearranging the previous equation gives the following equation for voltage divider:

$$\boldsymbol{e_{in}(t) = \left(\frac{R_1}{R_2} + 1\right) e_o(t)}$$

$$\boldsymbol{e_o(t) = \frac{R_1}{R_1 + R_1} e_{in}(t)}$$

References

1. Farid Golnaraghi & Benjamin C. Kuo (2019), Automatic Control Systems,10th Edition, McGraw-Hill Education
2. Farid Golnaraghi & Benjamin C. Kuo (2010), Automatic Control Systems,9th Edition, John Wiley&Sons

THANK YOU