Course: Automatic Control System Technology

Lecture 6: Model mechanical systems

Lecturer: UWASEKURU GISA Jean De Dieu

Model mechanical systems

Session objectives:

By the end of this session, students will be able to :

- Model Translational Mechanical Systems
- Model Rotational Mechanical Systems

Model mechanical systems

- Modeling a mechanical system means to create a mathematical or graphical representation of the system.
- A mechanical system can be represented by a differential equation, a transfer function, state space equations, a block diagram or a signal flow graph.
- In this lecture, we will focus on finding the differential equation model of mechanical systems

Model mechanical systems

- According to their motion, mechanical systems may be classified into:
  - ✓ Translational mechanical systems
  - ✓ Rotational mechanical systems.

Model translational mechanical systems

- Translational mechanical systems move along a straight line.
- These systems mainly consist of three basic elements: **mass, spring and dashpot or damper.**
- The three variables that are used to describe translational motion are **acceleration, velocity, and displacement**.

Model translational mechanical systems

- The first step in modeling a translational mechanical system is to identify all external forces acting on the system and draw a free body diagram of the system, and then,
- Apply the Newton’s second law of translational motion which states that ***the algebraic sum of external forces acting on a rigid body in a given direction is equal to the product of the mass of the body and its acceleration in the same direction.***

Model translational mechanical systems

- The Newton’s second law of translational motion can be expressed as:

$$\sum_{\textbf{External}} \textbf{forces} = \boldsymbol{Ma}$$

where **M** is the mass of the system, and **a** is the acceleration in the considered direction

Model translational mechanical systems

Model elements of translational mechanical systems:

- **Mass:** Mass is the property of a body, which stores kinetic energy.
- If a force $\boldsymbol{f(t)}$ is applied on a body having mass M, then an opposing force $\boldsymbol{F_m}$ due to mass opposes it.

Model translational mechanical systems

Model elements of translational mechanical systems(cont.):

- This opposing force is proportional to the acceleration of the body.

$$\boldsymbol{f(t) = F_m = Ma(t) = M\frac{d^2y(t)}{dt^2}}$$

$$\boldsymbol{f(t) = M\frac{dv(t)}{dt}}$$

where $\boldsymbol{a(t)}$ is the acceleration, $\boldsymbol{v(t)}$ is a linear velocity, and $\boldsymbol{y(t)}$ is the displacement of mass M.

Model translational mechanical systems

Model elements of translational mechanical systems (cont.):

- The force – mass system is illustrated in figure 1

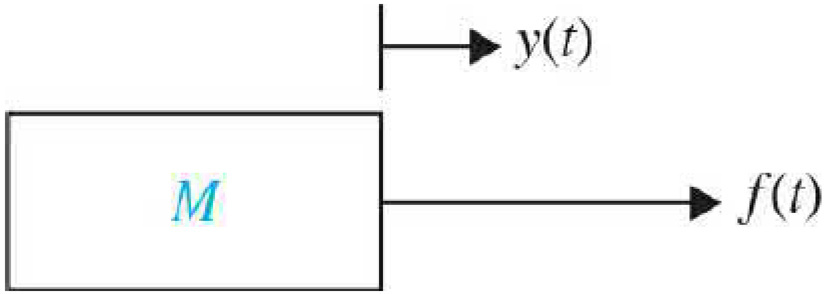

Figure 1. Force-mass system

Farid Golnaraghi & Benjamin C. Kuo (2019), Automatic Control Systems,10th Edition, McGraw-Hill Education, page 77.

Model translational mechanical systems

Model elements of translational mechanical systems(cont.):

- **Linear spring:** A linear spring is a massless element which stores potential energy.
- If a force $\boldsymbol{f(t)}$ is applied on spring K, then it is opposed by an opposing force $\boldsymbol{F_s}$ due to elasticity of spring.
- This opposing force is proportional to the displacement of the spring.

$$\boldsymbol{f(t) = F_s = Ky(t)}$$

Model translational mechanical systems

Model elements of translational mechanical systems(cont.):

- The force – spring system is illustrated in figure 2

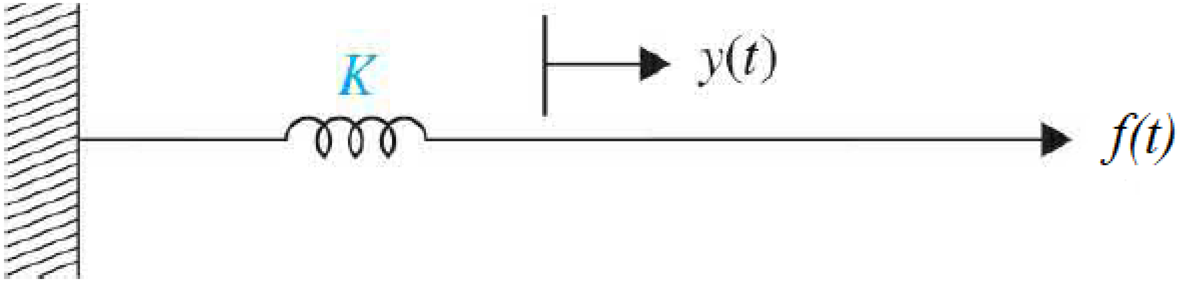

Figure 2. Force-spring system

Farid Golnaraghi & Benjamin C. Kuo (2019), Automatic Control Systems,10th Edition, McGraw-Hill Education, page 79.

Model translational mechanical systems

Model elements of translational mechanical systems(cont.):

- **Dashpot/damper:** If a force $\boldsymbol{f(t)}$ is applied on dashpot B, then it is opposed by an opposing force (Damping force) $\boldsymbol{F_d}$ due to friction of the dashpot.
- This opposing force is proportional to the velocity of the body.

$$\boldsymbol{f(t) = F_d = B\ v(t) = \frac{dy(t)}{dt}}$$

Model translational mechanical systems

- The dashpot for viscous friction is illustrated in figure 3

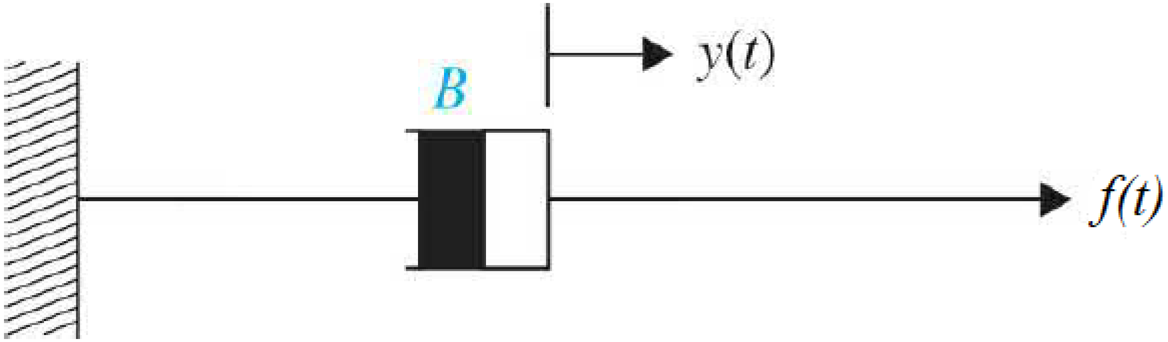

Figure 3. Dashpot for viscous friction

Farid Golnaraghi & Benjamin C. Kuo (2019), Automatic Control Systems,10th Edition, McGraw-Hill Education, page 80.

Model translational mechanical systems

Example 1: Consider the mass spring friction system shown in the figure 4 (a) and the corresponding free body diagram is shown in figure 4 (b). Determine a differential equation representing the motion of the system.

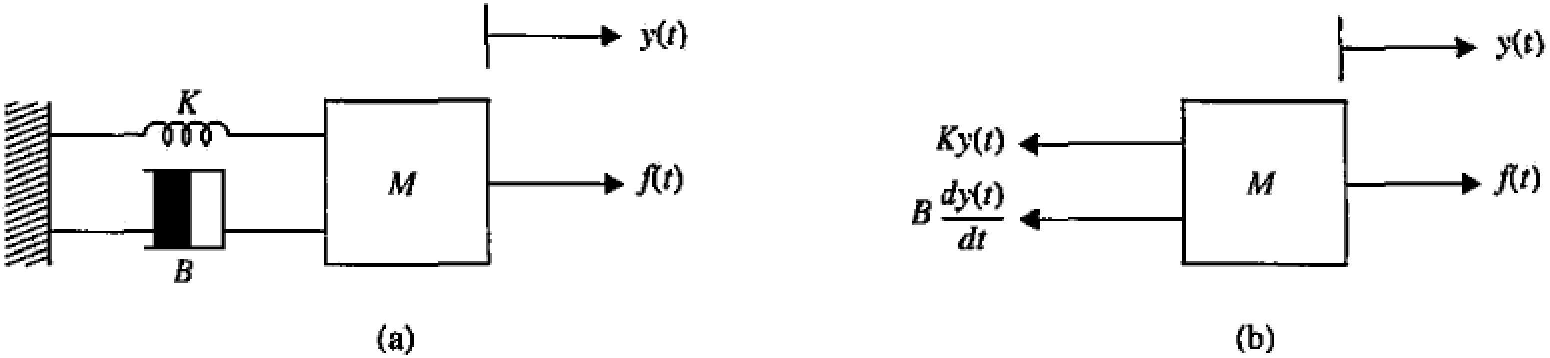

Figure 4. (a) Mass spring friction system, (a) Free body diagram

Farid Golnaraghi & Benjamin C. Kuo (2010), Automatic Control Systems,9th Edition, John Wiley&Sons, page 151.

Model translational mechanical systems

Solution:

- Applying the Newton's second law of motion on mass M, we get:

$$\sum_{\mathbf{External}} \mathbf{forces} = \boldsymbol{M}\frac{\boldsymbol{d^2y(t)}}{\boldsymbol{dt^2}}$$

$$\boldsymbol{f(t) - B\frac{dy(t)}{dt} - Ky(t) = M\frac{d^2y(t)}{dt^2}}$$

- This equation can be rearranged as:

$$\boldsymbol{M\frac{d^2y(t)}{dt^2} + B\frac{dy(t)}{dt} + Ky(t) = f(t)}$$

This is a differential equation model of the study system

Model translational mechanical systems

Example 2: Consider the two degrees of freedom mechanical system shown in figure 5 with masses M_1 and M_2 contained by three springs, while a force $f(t)$ is applied to M_2. Determine differential equations representing the motion of the system.

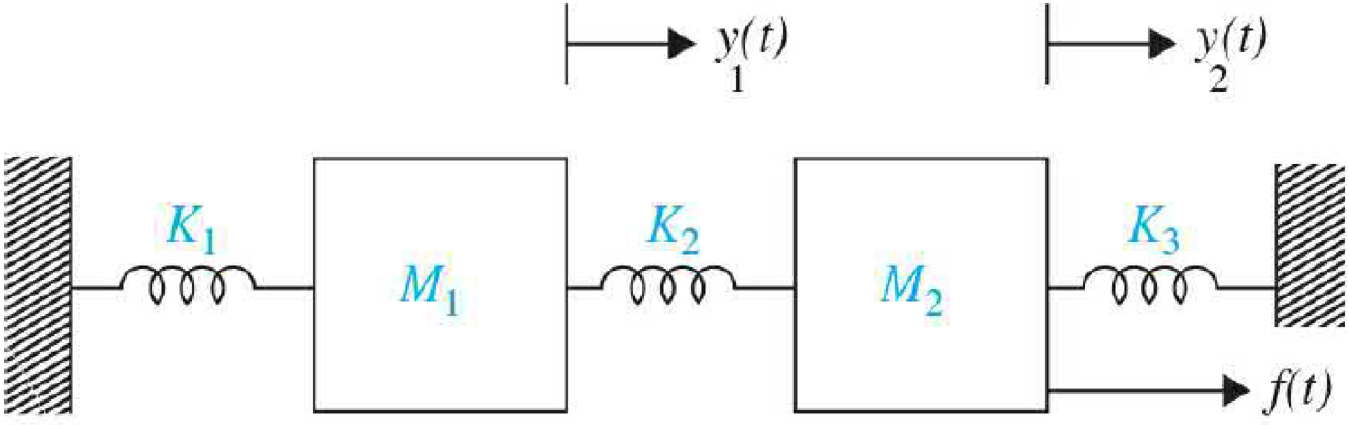

Figure 5.(a)A two-degree freedom mechanical system with 3 springs

Farid Golnaraghi & Benjamin C. Kuo (2019), Automatic Control Systems,10th Edition, McGraw-Hill Education,page 83

Model translational mechanical systems

Example 2(cont.):

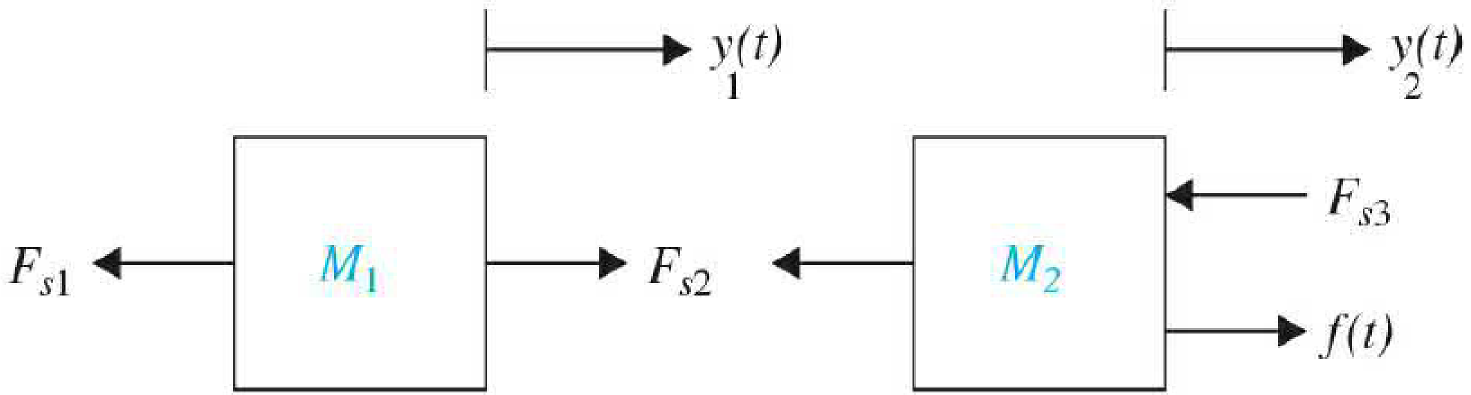

Figure 5.(b) Free body diagram of the system in Figure 5.(a)

Farid Golnaraghi & Benjamin C. Kuo (2019), Automatic Control Systems,10th Edition, McGraw-Hill Education,page 83

Model translational mechanical systems

Solution:

- Applying Newton's second law of motion to each mass, we get:

$$\sum_{\textbf{External}} \textbf{forces} = \boldsymbol{M_1 \ddot{y}_1(t)}$$

$$-K_1 y_1(t) + K_2(y_2(t) - y_1(t)) = \boldsymbol{M_1 \ddot{y}_1(t)}$$

- And,

$$\sum_{\textbf{External}} \textbf{forces} = \boldsymbol{M_2 \ddot{y}_2(t)}$$

$$-K_2(y_2(t) - y_1(t)) - K_3 y_2(t) + f(t) = \boldsymbol{M_2 \ddot{y}_2(t)}$$

Model translational mechanical systems

Solution:

- The two differential equations representing the motion of the system are:

$$M_1\ddot{y}_1(t) + (K_1 + K_2)y_1(t) - K_2y_2(t)) = 0$$

$$M_2\ddot{y}_2(t) + (K_2 + K_3)y_2(t) - K_2y_1(t) = f(t)$$

Model rotational mechanical systems

- Rotational mechanical systems move about a fixed axis.
- These systems mainly consist of three basic elements: **moment of inertia, torsional spring and dashpot/damper.**
- The three variables generally used to describe the motion of rotation are **torque, angular velocity, and angular displacement.**

Model rotational mechanical systems

- To model rotational mechanical system, you should first identify all external moments(or torques) acting on the system and draw a free body diagram of the system, and then,
- Apply Newton’s second law for rotational motion which states that ***the algebraic sum of external moments (or torques) acting on a rigid body about a fixed axis is equal to the product of the moment of inertia (J) of the body and its angular acceleration (α) about that axis.***

Model rotational mechanical systems

- The Newton's second law for rotational motion can be expressed as:

$$\sum_{\mathbf{External}} \boldsymbol{moments(or\ torques) = J\alpha(t)}$$

where **J** is the moment of inertia of the body and **α** is the angular acceleration

Model rotational mechanical systems

Model elements of rotational mechanical systems:

- **Moment of inertia:** In translational mechanical system, mass stores kinetic energy. Similarly, in rotational mechanical system, moment of inertia stores kinetic energy.
- If a torque $\boldsymbol{T(t)}$ is applied on a body having the moment of inertia J, then it is opposed by an opposing torque $\boldsymbol{T_J}$ due to the moment of inertia.

Model rotational mechanical systems

Model elements of rotational mechanical systems(cont.):

- This opposing torque is proportional to angular acceleration of the body.

$$\boldsymbol{T(t) = T_J = J\alpha(t) = J\frac{d\omega(t)}{dt} = \frac{d^2\theta(t)}{dt^2}}$$

Model rotational mechanical systems

Model elements of rotational mechanical systems(cont.) :

- The torque- inertia system is illustrated in figure 6.

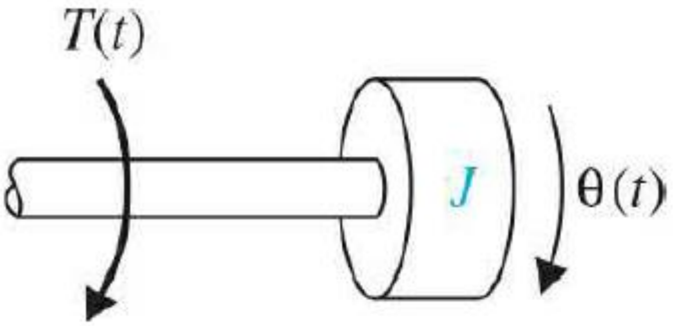

Figure 6. Torque-inertia system

where $\boldsymbol{\theta(t)}$ is the angular displacement; $\boldsymbol{\omega(t)}$, the angular velocity; and $\boldsymbol{\alpha(t)}$, the angular acceleration.

Farid Golnaraghi & Benjamin C. Kuo (2019), Automatic Control Systems,10th Edition, McGraw-Hill Education, page 86.

Model rotational mechanical systems

Model elements of rotational mechanical systems:

- **Torsional spring K:** In translational mechanical system, spring stores potential energy.Similarly, in rotational mechanical system, torsional spring stores potential energy.
- If a torque $T(t)$ is applied on torsional spring K, then it is opposed by an opposing torque T_s due to the elasticity of torsional spring.

Model rotational mechanical systems

Model elements of rotational mechanical systems(cont.):

- This opposing torque is proportional to the angular displacement of the torsional spring.

$$\boldsymbol{T(t) = T_s = K\theta(t)}$$

where $T(t)$ is the applied torque, T_s is the opposing torque due to elasticity of torsional spring, and $\theta(t)$ is the angular displacement

Model rotational mechanical systems

Model elements of rotational mechanical systems(cont.):

- **Dashpot:** If a torque $\boldsymbol{T(t)}$ is applied on dashpot B, then it is opposed by an opposing torque $\boldsymbol{T_d}$ due to the rotational friction of the dashpot.
- This opposing torque is proportional to the angular velocity of the body.

$$\boldsymbol{T(t) = T_d = B\omega(t) = B\frac{d\theta(t)}{dt}}$$

where $\omega(t)$ is the angular velocity and $\theta(t)$ is the angular displacement

Model rotational mechanical systems

Example : Consider the torque torsional spring system shown in figure 7 (a). Determine differential equations representing the motion of the system.

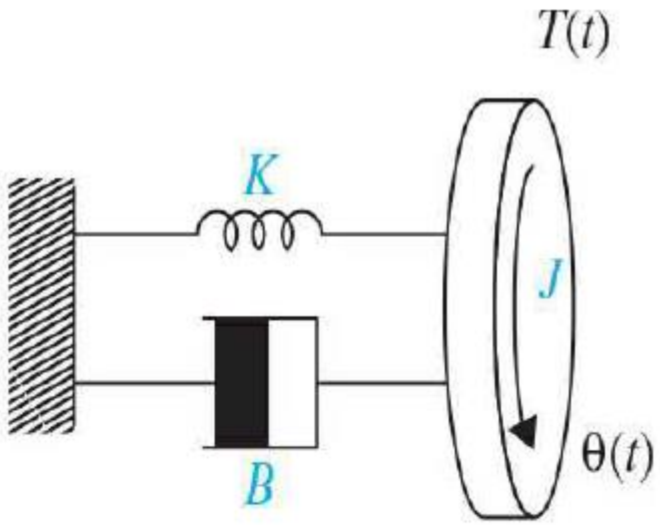

Figure 7. (a) Torque torsional spring system

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Model rotational mechanical systems

Example (cont.):

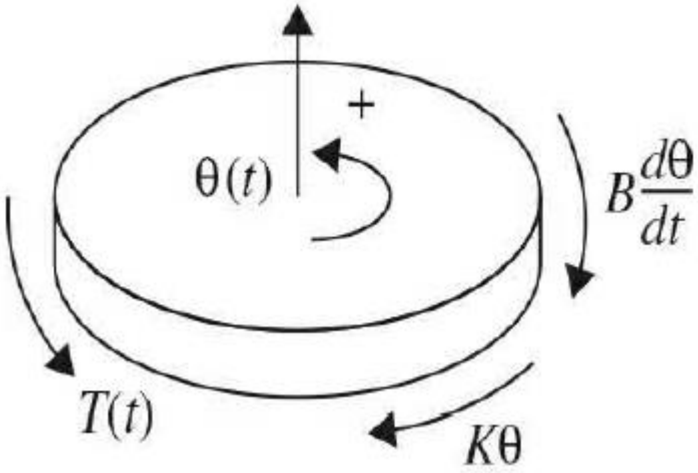

Figure 7. (b**)** Free body diagram of the system in figure 7. (a)

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Model rotational mechanical systems

Solution:

- Applying the Newton’s second law of motion, we get:

$$\sum_{\mathbf{External}} \boldsymbol{moments(or\ toques) = J\ \frac{d^2\theta(t)}{dt^2}}$$

$$\boldsymbol{T(t) - B\frac{d\theta(t)}{dt} - K\theta(t) = J\ \frac{d^2\theta(t)}{dt^2}}$$

Hence:

$$\boldsymbol{J\ \frac{d^2\theta(t)}{dt^2} + B\frac{d\theta(t)}{dt} + K\theta(t) = T(t)}$$

References

1. Farid Golnaraghi & Benjamin C. Kuo (2019), Automatic Control Systems,10th Edition, McGraw-Hill Education
2. Farid Golnaraghi & Benjamin C. Kuo (2010), Automatic Control Systems,9th Edition, John Wiley&Sons

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