Course: Automatic Control System Technology

Lecture 9: Develop a control system signal flow graph model

Lecturer: UWASEKURU GISA Jean De Dieu

Develop a control system signal flow graph model

Session objectives:

By the end of this session, students will be able to :

- ❖ Define a Signal Flow Graph(SFG)
- ❖ Describe elements of SFG
- ❖ Describe SFG manipulation rules
- ❖ Define SFG terms
- ❖ Apply Mason’s gain formula for SFG
- ❖ Convert the Block diagram to SFG of the system
- ❖ Construct the SFG of a dynamic system

Define a Signal Flow Graph

- A **Signal flow graph (SFG)** is a graphical means of portraying the input-output relationships among of a set of linear algebraic equations.
- It describes how a signal **gets modified as it travels from input to output** of the system.

Farid Golnaraghi & Benjamin C. Kuo (2019), Automatic Control Systems, 10^{th} Edition, McGraw-Hill Education, page 327.

Define a Signal Flow Graph

- ❖ The SFG was introduced by S. J. Mason for the cause-and effect (input-output) representation of linear systems that are modeled by algebraic equations.
- ❖ The **overall transfer function** can be obtained very easily by using **Mason's gain formula**.

Describe basic elements of SFG

- The basic elements of SFG are **nodes** and **branches.**
- A **node** is a point that represents a variable.
- A **branch** is a line segment which joins two nodes according to the cause-and-effect equation
- The branches have associated **gains** and **directions.**
- A signal can transmit through a branch only in the direction of the arrow.

Describe basic elements of SFG

❖ The basic signal flow graph is represented in figure 1.

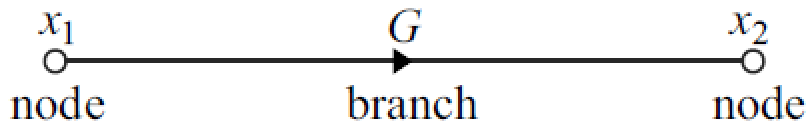

Figure 1. Basic signal flow graph

✓ $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ are variables and,

✓ The line drawn between $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ is the branch,

✓ G is the gain or the transmission function,

✓ The arrow indicates the direction of signal flow.

Palani S. (2022), Automatic Control Systems: With MATLAB, 2^{nd} Edition, Springer, page 110.

Describe SFG manipulation rules

❖ **The Addition Rule:** the value of the variable represented by a node is equal to the sum of all signals entering that node.

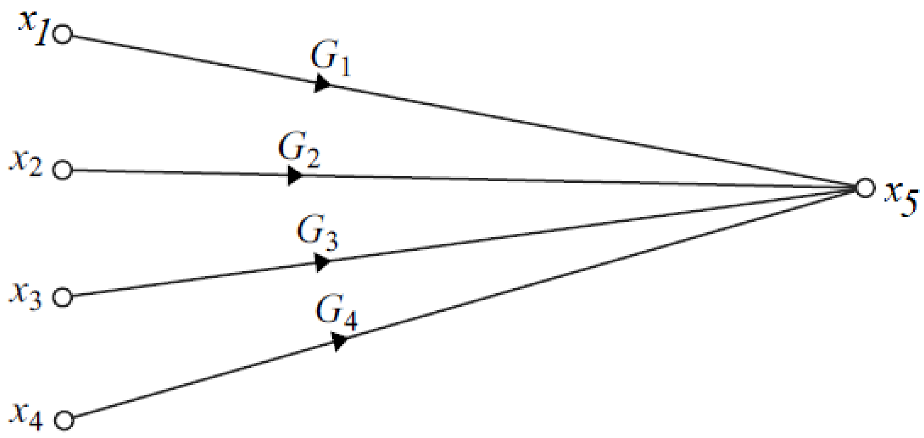

Figure 2. Illustration of addition rule of SFG

$$\boldsymbol{x_5 = G_1 x_1 + G_2 x_2 + G_3 x_3 + G_4 x_4}$$

Palani S. (2022), Automatic Control Systems: With MATLAB, 2nd Edition, Springer, page 110.

Describe SFG manipulation rules

❖ **The Transmission Rule:** the value of the variable represented by a node is transmitted on every branch leaving that node.

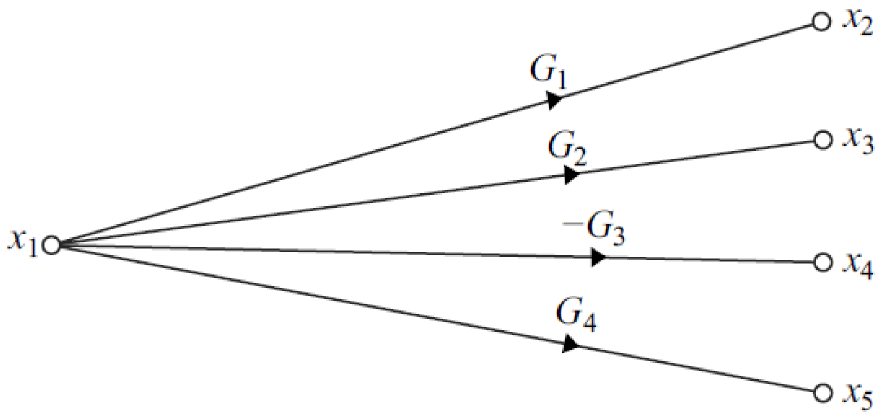

Figure 3. Illustration of transmission rule of SFG

$$x_2 = G_1 x_1\ ,\ \ x_3 = G_2 x_1\ ,\ \ x_4 = -G_3 x_1,\ \ \ x_5 = G_4 x_1$$

Palani S. (2022), Automatic Control Systems: With MATLAB, 2nd Edition, Springer, page 111.

Describe SFG manipulation rules

❖ **Multiplication Rule:** The series (cascade) connection of unidirectional branches can be replaced by a single branch with gain equal to the product of the branch gains.

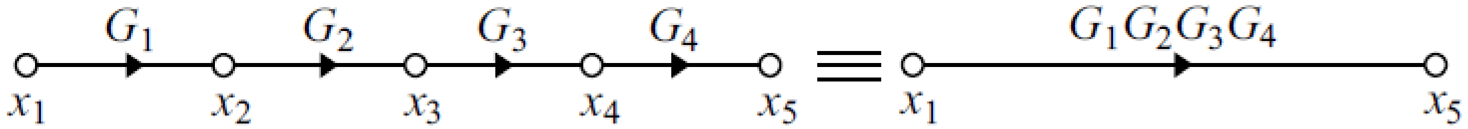

Figure 4. Illustration of multiplication rule of SFG

Palani S. (2022), Automatic Control Systems: With MATLAB, 2nd Edition, Springer, page 111.

Define SFG Terms

Let define each term of the SFG by using the example of signal flow graph shown in Figure 5.

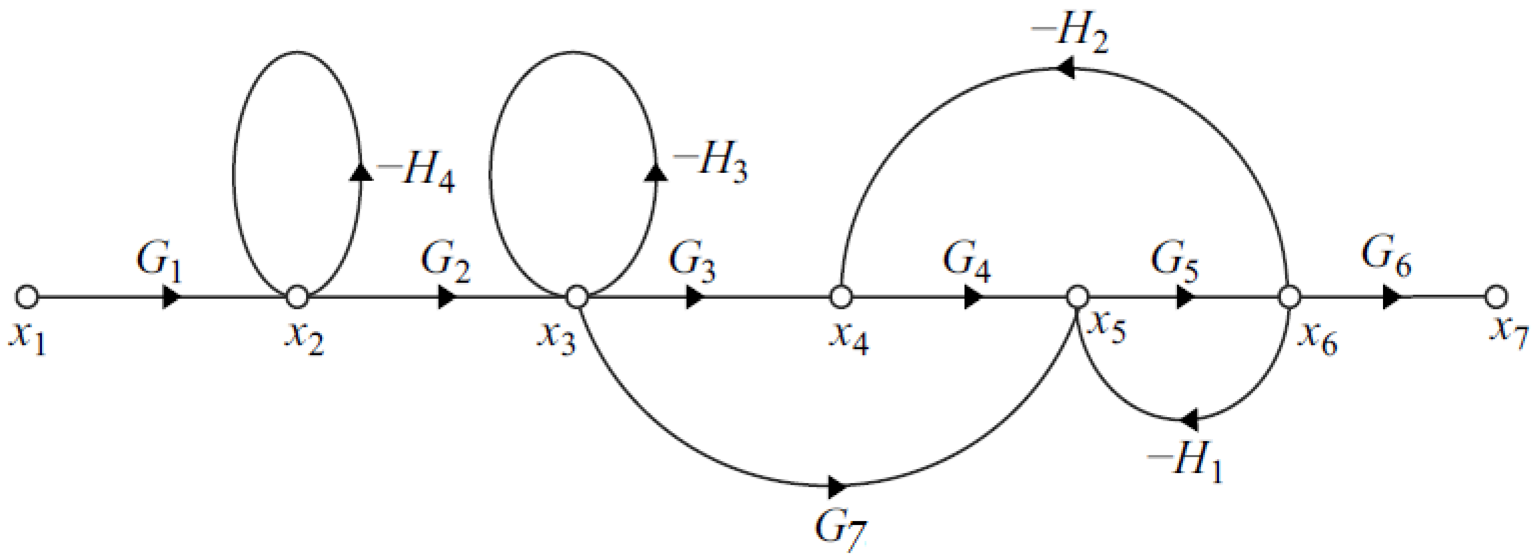

Figure 5. Illustration of terminologies of SFG

Palani S. (2022), Automatic Control Systems: With MATLAB, 2nd Edition, Springer, page 112.

Define SFG Terms

- **Node:** It is a dot o or a point that is used to represent a variable of the system within a signal flow graph. In the figure 5; $\boldsymbol{x_1}$, $\boldsymbol{x_2}$, $\boldsymbol{x_3}$, $\boldsymbol{x_4}$, $\boldsymbol{x_5}$, $\boldsymbol{x_6}$ and $\boldsymbol{x_7}$ are nodes.
- **Input node:** It is a node which has only outgoing branches. It is also called a **source node**. In the figure 5; the node $\boldsymbol{x_1}$ is an input node.
- **Output node:** It is a node which has only incoming branches. It is also called a **sink node**. In the figure 5; the node $\boldsymbol{x_7}$ is an output node.

Define SFG Terms

- **Intermediate node:** It is a node with both input and output nodes. It is also called a **mixed node**. In the figure 5; $\boldsymbol{x_2}$, $\boldsymbol{x_3}$, $\boldsymbol{x_4}$, $\boldsymbol{x_5}$, and $\boldsymbol{x_6}$ are intermediate nodes.
- **Branch:** It is a line segment which links two nodes. A signal travels along a branch from one node to another node, in the direction of the branch and gets multiplied by the gain of the branch. In the figure 5; $\boldsymbol{x_1}$-$\boldsymbol{x_2}$ is a branch with a branch gain $\boldsymbol{G_1}$. Similarly, $\boldsymbol{x_2}$-$\boldsymbol{x_3}$ is a branch with a branch gain $\boldsymbol{G_2}$, etc.

Define SFG Terms

- **Path:** It is a continuous unidirectional succession of branches along which no node is passed more than once. In the figure 5; x_1 to x_2 is a path, x_1-x_2-x_3-x_4-x_5-x_6-x_7 is a path, x_5-x_6-x_5 is a path, and so on.
- **Forward path:** It is a path from the input node to the output node. It is a traversal from input node to the output node, through branches, in the direction of branches such that no node is traversed more than once. In the figure 5; x_1-x_2-x_3-x_4-x_5-x_6-x_7 is a forward path. Similarly x_1-x_2-x_3-x_5-x_6-x_7 is also a forward path.

Define SFG Terms

- **Loop:** It is a path starting and ending on the same node. In the figure 5; $\boldsymbol{x_4}$-$\boldsymbol{x_5}$-$\boldsymbol{x_6}$-$\boldsymbol{x_4}$ is a loop. Similarly $\boldsymbol{x_5}$-$\boldsymbol{x_6}$-$\boldsymbol{x_5}$ is a loop. Also $\boldsymbol{x_2}$-$\boldsymbol{x_2}$ and $\boldsymbol{x_3}$-$\boldsymbol{x_3}$ are loops.
- **Non touching loops:** They are loops which do not have any common node. In the figure 5; loops $\boldsymbol{x_2}$-$\boldsymbol{x_2}$ and $\boldsymbol{x_3}$-$\boldsymbol{x_3}$ are non touching. Similarly $\boldsymbol{x_2}$-$\boldsymbol{x_2}$, $\boldsymbol{x_3}$-$\boldsymbol{x_3}$, and $\boldsymbol{x_4}$-$\boldsymbol{x_5}$-$\boldsymbol{x_6}$-$\boldsymbol{x_4}$ do not touch each other.

Define SFG Terms

- **Forward path gain:** It is the product of gains of branches forming a forward path. In the figure 5; for the forward path $\boldsymbol{x_1}$ - $\boldsymbol{x_2}$ - $\boldsymbol{x_3}$ - $\boldsymbol{x_4}$ - $\boldsymbol{x_5}$ - $\boldsymbol{x_6}$ - $\boldsymbol{x_7}$, the forward path gain is : $\boldsymbol{G_1 G_2 G_3 G_4 G_5 G_6}$. Similarly for the forward path $\boldsymbol{x_1}$-$\boldsymbol{x_2}$-$\boldsymbol{x_3}$-$\boldsymbol{x_5}$-$\boldsymbol{x_6}$-$\boldsymbol{x_7}$, the forward path gain is $\boldsymbol{G_1 G_2 G_7 G_5 G_6}$. .
- **Loop gain:** It is the product of gains of branches forming a loop. In the figure 5; for the loop $\boldsymbol{x_4}$-$\boldsymbol{x_5}$-$\boldsymbol{x_6}$-$\boldsymbol{x_4}$, the loop gain is: $\boldsymbol{-G_4 G_5 H_2}$. Similarly for the loop $\boldsymbol{x_5}$-$\boldsymbol{x_6}$-$\boldsymbol{x_5}$, the loop gain is:$\boldsymbol{-G_5 H_1}$

Apply Mason's gain formula for SFG

❖ The Mason's gain formula is given by:

$$T = \frac{1}{\Delta}\sum_{k} P_k \Delta_k$$

Where:

T= Transfer function of the system

P_k= Gain of the k^{th} forward path

k = Number of forward path in SFG

Δ = determinant of the graph

Farid Golnaraghi & Benjamin C. Kuo (2019), Automatic Control Systems,10th Edition, McGraw-Hill Education, page 339.

Apply Mason's gain formula for SFG

- Δ = 1 – (sum of individual loop gains) + (sum of gain products of all possible combinations of two non-touching loops) – (sum of gain products of all possible combinations of three non-touching loops) + …,
- $\Delta = 1 - \sum_m P_{m1} + \sum_m P_{m2} + \sum_m P_{m3} + \ldots + (-1)^r \sum_m P_{mr}$
- P_{mr} = gain product of 'm' possible combinations of 'r' non touching loops
- Δ_k= 1 – gain of that part of SFG that is non-touching with k^{th} forward path.

Apply Mason's gain formula for SFG

Example: Find the transfer function $T(s) = \frac{C(s)}{R(s)}$ by using Mason's gain formula for the system represented the signal flow graph shown in Figure 6.

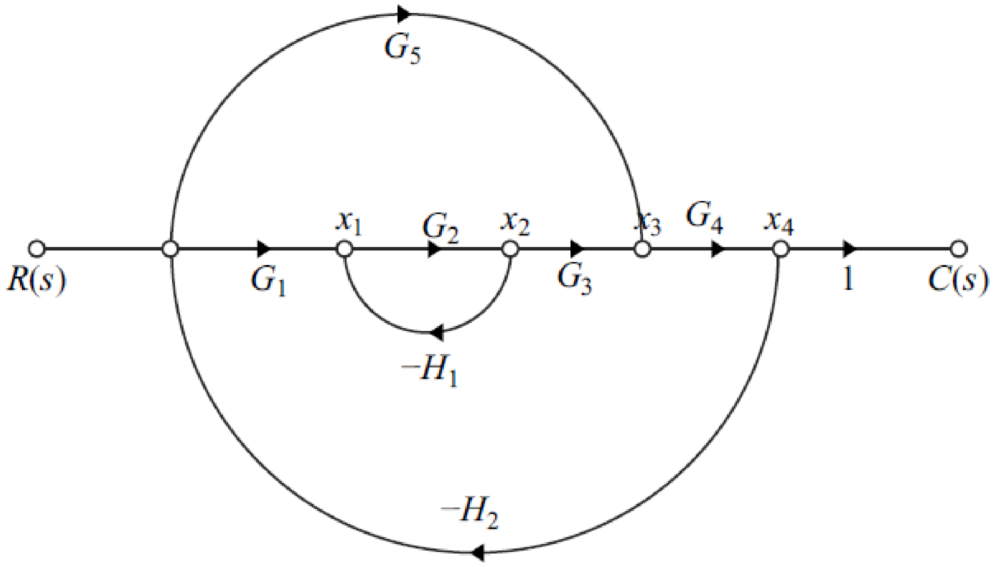

Figure 6. Example SFG

Palani S. (2022), Automatic Control Systems: With MATLAB, 2nd Edition, Springer, page 117.

Apply Mason’s gain formula for SFG

Solution:

Step 1: Forward path gain: P_k

✓ Number of forward path: k=2

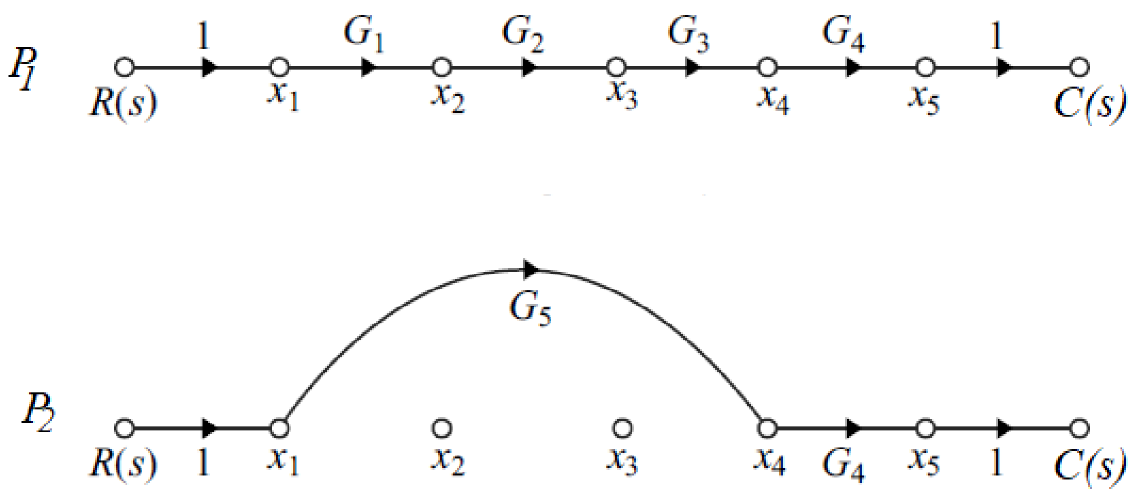

Figure 6. (a) Forward paths P_1 and P_2

✓ Thus: $P_1 = G_1 G_2 G_3 G_4$

$$P_2 = G_5 G_4$$

Apply Mason's gain formula for SFG

Step 2: Individual loop gains: P_{m1}

✓ There 3 individual loops in the SFG:

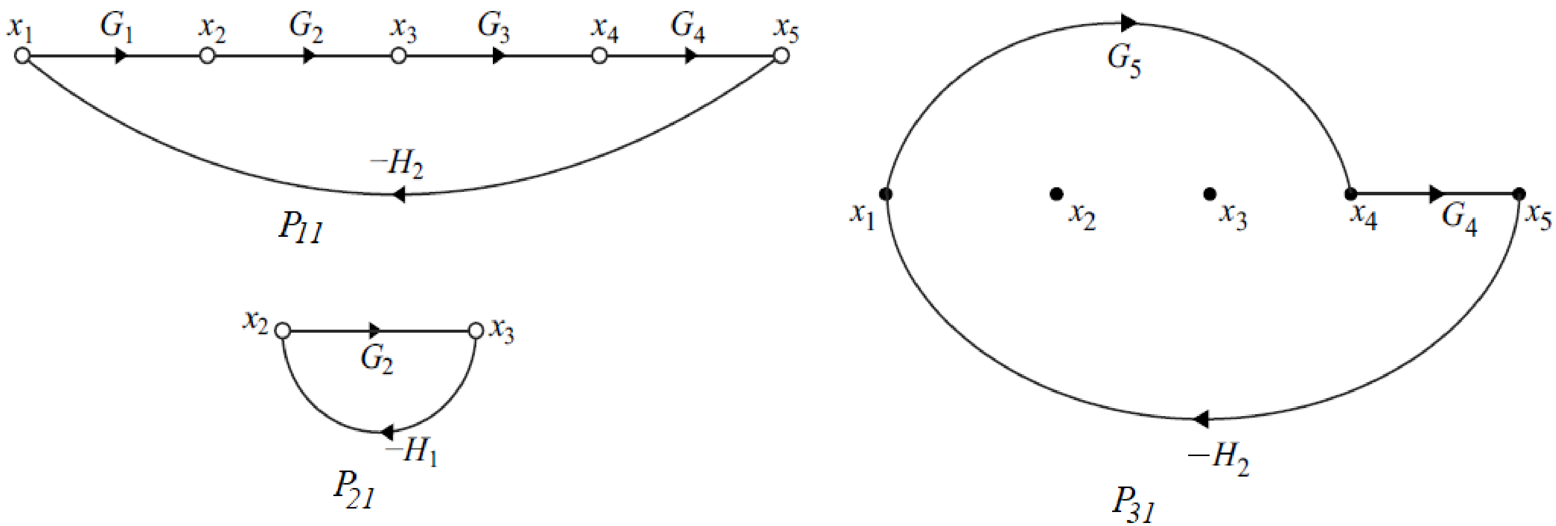

Figure 6. (b) Individual loops P_{11}, P_{21} and P_{31}

✓ Thus: $P_{11} = -G_1G_2G_3G_4H_2$

$P_{21} = -G_2H_1$

$P_{31} = -G_5G_4H_2$

Apply Mason's gain formula for SFG

Step 3: Non-touching loops product gains: P_{m2} and P_{m3}

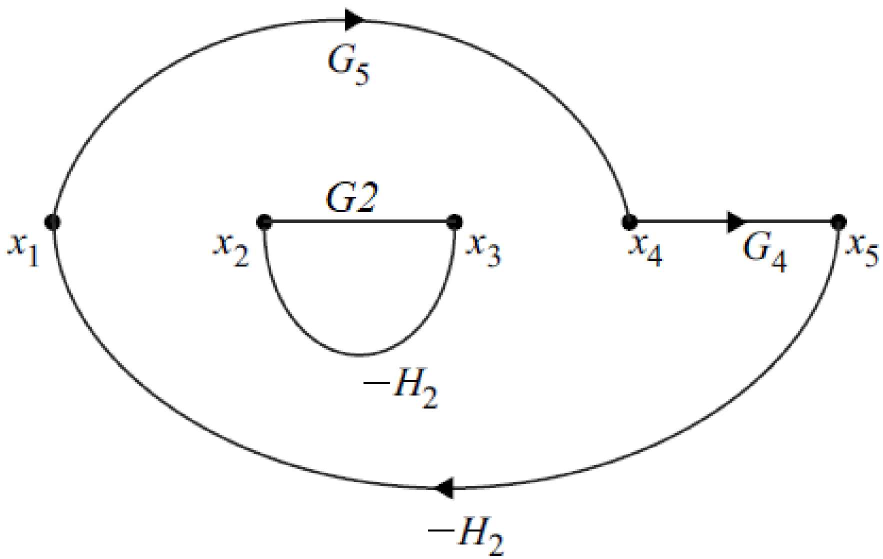

Figure 6. (c) Non touching loops

- ✓ There is only 1 possibility to combine 2 loops without a common node: $P_{21}$and P_{31}Thus: $\boldsymbol{P_{12}} = G_2H_1G_5G_4H_2$
- ✓ There are not 3 non touching loop in SFG: $P_{m3} = 0$

Apply Mason's gain formula for SFG

Step 4: Determinants: Δ and Δ_k, k=1 or 2

$$\Delta = 1 - \sum_m P_{m1} + \sum_m P_{m2}$$

$$\Delta = 1 - (-G_1G_2G_3G_4H_2 - G_2H_1 - G_5G_4H_2) + G_2H_1G_5G_4H_2$$

$$\Delta = 1 + G_1G_2G_3G_4H_2 + G_2H_1 + G_5G_4H_2 + G_2H_1G_5G_4H_2$$

$$\Delta_k = 1 - \text{the gain of non touching part on } k^{th} \text{ forward path}$$

- ✓ All individual loops (P_{11}, P_{21}, and P_{31}) touch the forward path P_1. Hence $\Delta_1 = 1 - 0 = 1$
- ✓ The individual loop P_{21} does not touch the forward path P_2. Hence $\Delta_2 = 1 - P_{21} = 1 - (-G_2H_1)$

$$\Delta_2 = 1 + G_2H_1$$

Apply Mason's gain formula for SFG

Step 5: Determine $T(s) = \frac{C(s)}{R(s)}$ by applying Mason's gain formula:

$$T(s) = \frac{\sum P_k \Delta_k}{\Delta} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

$$T(s) = \frac{G_1G_2G_3G_4 * 1 + G_5G_4(1 + G_2H_1)}{1 + G_1G_2G_3G_4H_2 + G_2H_1 + G_5G_4H_2 + G_2H_1G_5G_4H_2}$$

Convert the Block diagram to SFG of the system

- Instead of applying Block diagram reduction techniques to determine the transfer function, we can use Mason's gain formula.
- However, it becomes easier if an equivalent SFG is drawn for the block diagram before applying the Mason's gain formula.

Convert the Block diagram to SFG of the system

- To convert a block diagram to SFG you can proceed as follows:
  - ✓ Represent input, output, output of summing points and take-off points of the block diagram as **nodes** in signal flow graph.
  - ✓ Represent the blocks of the block diagram as **branches** in signal flow graph.
  - ✓ Represent the transfer functions inside the blocks of block diagram as **gains** of the branches in signal flow graph.

Convert the Block diagram to SFG of the system

- To convert a block diagram to SFG you can proceed as follows (cont.):
  - ✓ Connect the nodes as per the block diagram. If there is connection between two nodes (but there is no block in between), then represent the gain of the branch as **on**e. For example, between summing points, between summing point and takeoff point, between input and summing point, between take-off point and output.

Convert the Block diagram to SFG of the system

❖ **Example:** Convert the block diagram shown in figure 7 (a) into signal flow graph.

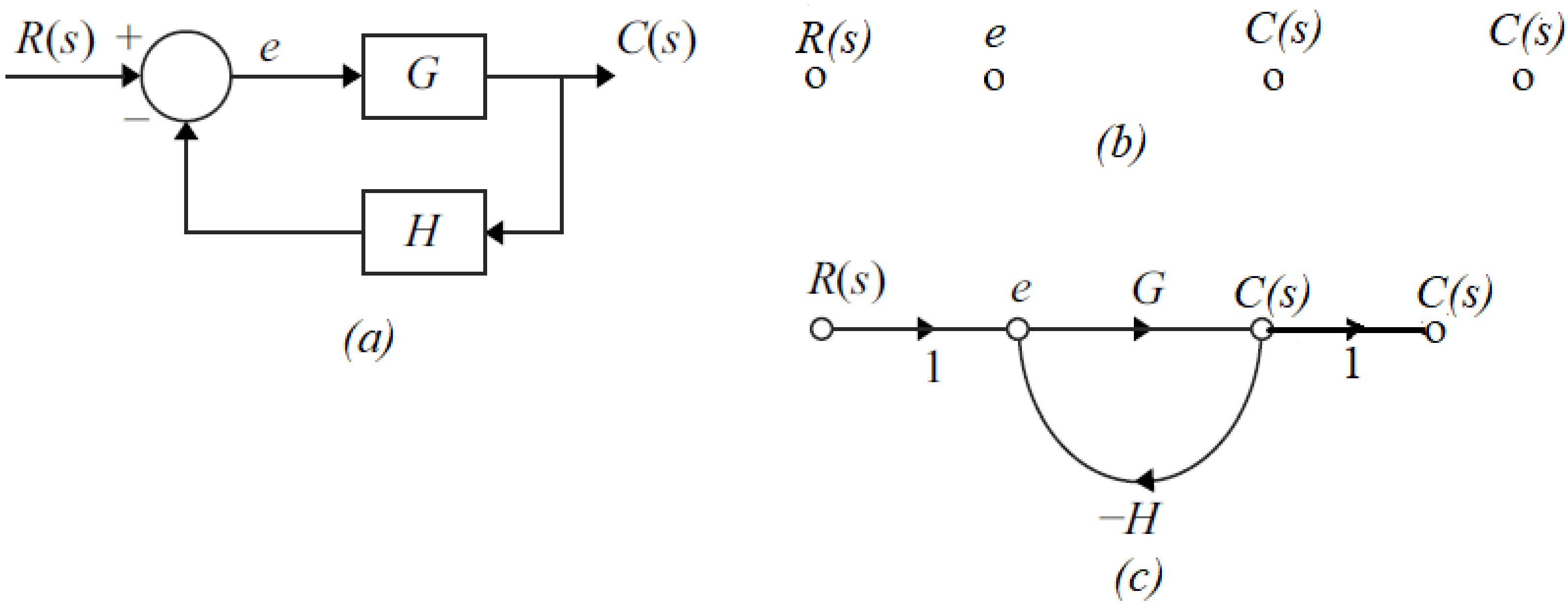

Figure 7. (a) Block diagram (b) Block diagram variables converted to nodes, and (c) Equivalent SFG

Convert the Block diagram to SFG of the system

- ❖ First represent input **R(s),** output **C(s)**, output of summing point **e**, and take off point **C(s)** by nodes (figure 7 (b)).
- ❖ Next using branches interconnect the nodes while ensuring the branches directions match the signal directions in the block diagram (figure 7 (c)).
- ❖ Then label each branch with the appropriate gain corresponding to the block transfer function.
- ❖ Make sure to incorporate the negative sign into gain ***-H(s)*.**

Convert the Block diagram to SFG of the system

- Determine $\frac{C(s)}{R(s)}$ from both block diagram and signal flow graph.

Solution:

- The converted block diagram as shown in figure 7 (a) is a non unit negative feedback and it can be reduced as follow:

$$\frac{C(s)}{R(s)} = \frac{G}{1 + GH}$$

Convert the Block diagram to SFG of the system

- $\frac{C(s)}{R(s)}$ from SFG shown in **figure 7 (c)** :

  - ✓ Step 1: Forward path gain P_k

    Number of forward path: k=1

    P_1 =G

  - ✓ Step 2: Individual loop gain P_{11}

    $P_{11} = -GH$

  - ✓ Step 3: Non touching loop gain $P_{m2} = 0$

Convert the Block diagram to SFG of the system

- ✓ Step 4: Determinant of SFG: $\Delta \; and \; \Delta_k$

$$\Delta = 1 - P_{11} = 1 - (-GH)$$

$$\Delta = 1 + GH$$

$\Delta_1 = 1 - 0 = 1$, because there is no part of the SFG which does not touch the forward path

Step 5: Determinant of SFG: $\Delta \; and \; \Delta_k$

$$T(s) = \frac{C(s)}{R(s)} = \frac{\sum P_k \Delta_k}{\Delta} = \frac{P_1 \Delta_1}{\Delta}$$

$$T(s) = \frac{G}{1 + GH}$$

Construct the SFG of a dynamic system

❖ To develop a SFG for a physical system, follow these steps:

1. First, write the governing equations that describe the relationship between the system's input, output, and intermediate variables (use physical laws).
2. Then, take the Laplace transforms of these equations, assuming zero initial conditions.
3. And, represent each Laplace-transformed equation individually in SFG form.
4. Finally, assemble those basic signal flow graph into a complete SFG

Construct the SFG of a physical system

❖ **Example:** Consider the electrical network shown in figure 8 (a) to construct its corresponding signal flow graph, and (b) Find the transfer function of the network $\frac{V_O(s)}{V_i(s)}$.

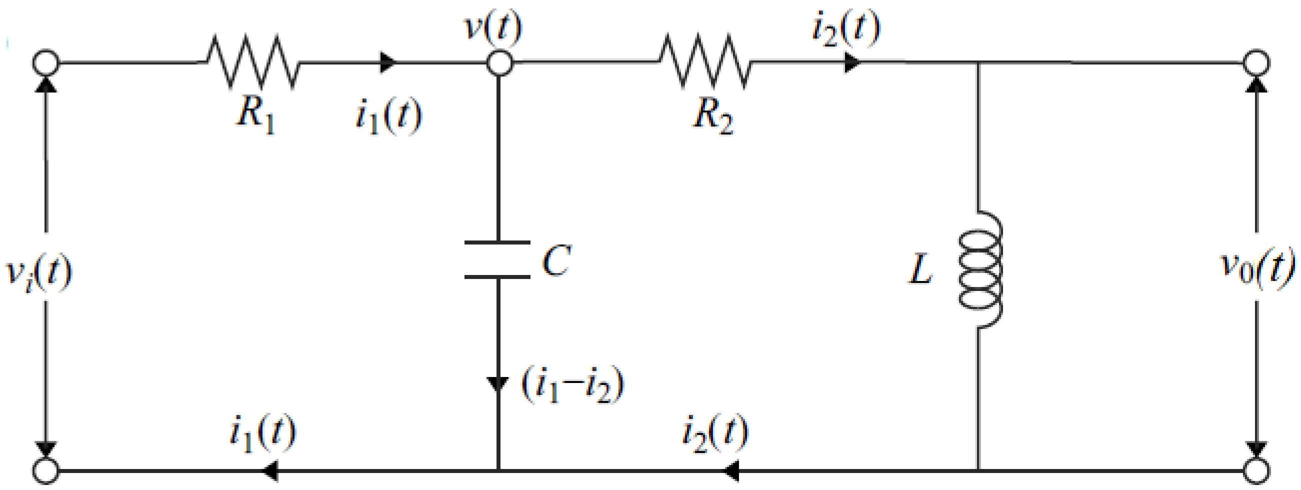

Figure 8. (a) Example of electrical network

Palani S. (2022), Automatic Control Systems: With MATLAB, 2nd Edition, Springer, page 131.

Construct the SFG of a physical system

❖ Write the equations describing the dynamic behavior of the network.

$$v_i(t) = R_1 i_1(t) + v(t)$$

$$v(t) = \frac{1}{c}\int_0^t (i_1(\tau) - i_2(\tau))\, d\tau$$

$$v(t) = R_2 i_2 + L\frac{di_2(t)}{dt}$$

$$v_0(t) = L\frac{di_2(t)}{dt}$$

Construct the SFG of a physical system

❖ Laplace transformed equation:

$$I_1(s) = \frac{1}{R_1}\left(V_i(s) - V_o(s)\right) = \frac{1}{R_1}V_i(s) - \frac{1}{R_1}V_o(s) \quad (\boldsymbol{eq1})$$

$$V(s) = \frac{1}{Cs}\left(I_1(s) - I_2(s)\right) = \frac{1}{Cs}I_1(s) - \frac{1}{Cs}I_2(s) \quad (\boldsymbol{eq2})$$

$$\mathrm{V}(s) = R_2 I_2(s) + LsI_2(s)$$

$$I_2(s) = \frac{1}{(R_2 + Ls)}V(s) \quad (\boldsymbol{eq3})$$

$$V_O(s) = LsI_2(s) \quad (\boldsymbol{eq4})$$

Construct the SFG of a physical system

❖ Represent each equation in SFG form

✓ $I_1(s) = \frac{1}{R_1} V_i(s) - \frac{1}{R_1} V_o(s) \ \ (eq1)$

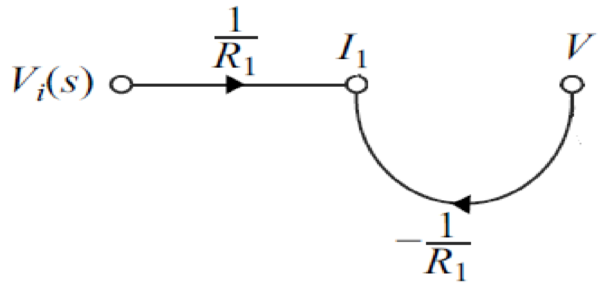

Figure 8. (b) *eq1* in SFG form

✓ $V(s) = \frac{1}{Cs} I_1(s) - \frac{1}{Cs} I_2(s) \ (eq2)$

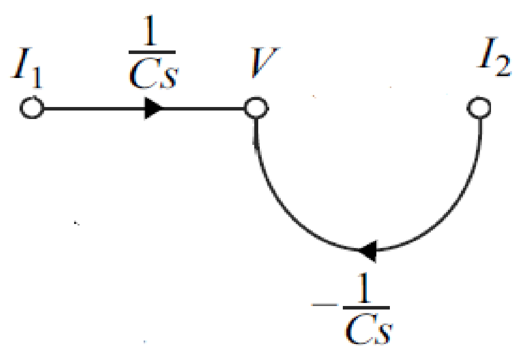

Figure 8. (c) *eq2* in SFG form

Construct the SFG of a physical system

❖ Represent each equation in SFG form

✓ $I_2(s) = \frac{1}{(R_2 + Ls)} V(s) \quad (eq3)$

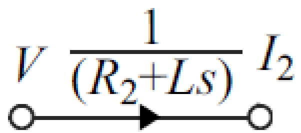

Figure 8. (d) *eq3* in SFG form

✓ $V_O(s) = LsI_2(s) \quad (eq4)$

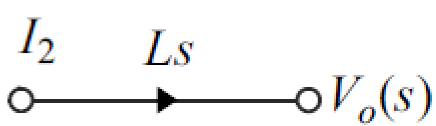

Figure 8. (e) *eq4* in SFG form

Construct the SFG of a physical system

- Now, appropriately assemble basic signal flow graphs in figure 8 (b), (c), (d), and (e) into a complete SFG as shown in figure 9.

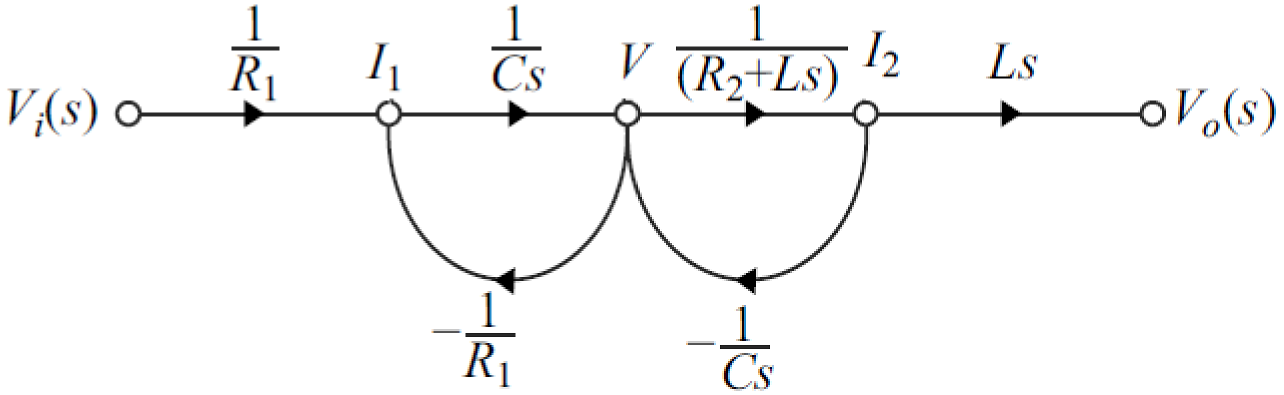

Figure 9. SFG of the network in figure 8 (a)

Palani S. (2022), Automatic Control Systems: With MATLAB, 2nd Edition, Springer, page 131.

Construct the SFG of a physical system

- Now, apply Mason's gain formula to the SFG shown in figure 9 to $\frac{\boldsymbol{V_O(s)}}{\boldsymbol{V_i(s)}}$
- Step 1: Forward path gain (k=1)

$$P_1 = \frac{1}{R_1}\frac{1}{Cs}\frac{1}{R_2+Ls}Ls = \frac{Ls}{R_1Cs(R_2+Ls)}$$

- Step 2: Individual loop gains

$$P_{11} = -\frac{1}{R_1Cs} \quad , P_{21} = -\frac{1}{(R_2+Ls)Cs}$$

- Step 3: Non touching loop gains: $P_{m2} = 0$, because the two loops are touching in node *V.*

Construct the SFG of a physical system

- Step 4: Determinant Δ $and\ \Delta_k$

$$\Delta = 1 - \sum P_{m1} = 1 - (P_{11} + P_{21})$$

$$\Delta = 1 + \frac{1}{R_1 Cs} + \frac{1}{(R_2 + Ls)Cs}$$

$$\Delta = \frac{R_1 Cs(R_2 + Ls) + (R_2 + Ls) + R_1}{R_1 Cs(R_2 + Ls)}$$

$$\Delta_1 = 1$$

- Step 5: $\mathrm{T(s)} = \frac{\boldsymbol{V_O(s)}}{\boldsymbol{V_i(s)}} = \frac{P_1 \Delta_1}{\Delta}$

$$\mathrm{T(s)} = \frac{Ls}{R_1 Cs(R_2 + Ls) + (R_2 + Ls) + R_1}$$

References

1. Palani S. (2022), Automatic Control Systems: With MATLAB, 2nd Edition, Springer.
2. Farid Golnaraghi & Benjamin C. Kuo (2019), Automatic Control Systems,10th Edition, McGraw-Hill Education.
3. Nise, Norman S. (2015), Control Systems Engineering,7th Edition,John Wiley & Sons.

THANK YOU