Course: Automatic Control System Technology

Lecture 10: Conduct the first-order system time response analysis

Lecturer: UWASEKURU GISA Jean De Dieu

Conduct the first-order system time response analysis

Session objectives:

By the end of this session, students will be able to :

- ❖ Describe a system response
- ❖ Analyze the unit step time response of a first-order system
- ❖ Analyze the unit ramp time response of a first-order system
- ❖ Analyze the unit impulse time response of a first-order system

Describe a system response

- When concerned with dynamic systems, **it is important to know how the output of the system will change with respect to time as result of input change.**
- In the control system analysis, a reference input signal is applied to a system and the performance of the system is evaluated by studying its response.
- **Response of the system:** It is the output of the system. The response may be a natural (free) response or a forced response.

Describe a system response

- **Natural or free response of the system**: It is the response of the system to zero input and nonzero initial conditions.
- **Forced response of the system:** It is the response of the system to nonzero input and zero initial conditions.
- In general, the total response of the system is the sum of the response due to the initial condition alone (free response) and the response due to the input (forced response).

Allen J. Stubberud Ivan J. Williams Joseph J. DiStefano (1994), Schaum's Outline of Feedback and Control Systems, 2nd , McGraw-Hill,page 56.

Describe a system response

- Although the natural or free response of a system always decays to zero, the forced response has no such restriction.
- In fact, the **forced response of the system will take the same form** as the **forcing function**, as time goes to infinity.
- Control system analysis deals with systems subjected to the **input excitation with zero initial conditions**. Hence, system with **forced response**

Describe a system response

- The analysis of a control system may be done with reference to time (time domain analysis) or frequency (frequency domain analysis).
- Thus, we define **time response** and **frequency response**.
- **Time response:** The variation of the output with respect to time.
- **Frequency response:** The steady-state response of a system to a sinusoidal input.
- This lecture deals with the **analysis of time response** only.

Describe a system response

- After applying an input to the control system, the output takes some time to **reach the steady or stable condition.**
- The response of a control system during this stage is known as **transient response** whereas its output in its steady state is known as **steady state response**.
- Hence, any system's **time response** has two parts; **transient response** and **steady response**.

Describe a system response

- **Transient response:** The part of time response that **goes to zero or disappears** after large interval of time.
- **Steady state response**: The part of time response **that remains** after transient response. It is that part of the response which is fixed when time t tends to ∞.
- Thus the system response $c(t)$ may be written as

  $\boldsymbol{c(t) = c_{tr}(t) + c_{ss}(t)}$. Where $\boldsymbol{c_{tr}(t)}$ is the transient response and $\boldsymbol{c_{ss}(t)}$ is the steady-state response

Katsuhiko Ogata(2009), Modern Control Engineering, 5th Edition, Prentice Hall, page 169.

Describe a system response

- ❖ In this lecture, we are going to analyze how a first-order system responds to a given input, standard or test input, over time, focusing on how the output evolves from the initial state to steady state.
- ❖ We are going to apply test signals to a first order system, derive the output/ response in the time domain, examine how the output behaves over time and evaluate how fast and accurately the system responds.

Describe a system response

- The typical or standard test inputs used to judge or analyze the behavior of the system's response include impulse, step, ramp, and parabolic signals.
- Test signals may be a sudden shock (impulse input), a sudden change (step input), a linear change with time (ramp input) or faster changes with time (parabola input).

Analyze the unit step time response of a first-order system

❖ Consider the first-order system shown in Figure 1 (a). $R(s)$, $C(s)$, $E(s)$, and T respectively represent input signal, output signal , error signals and constant.

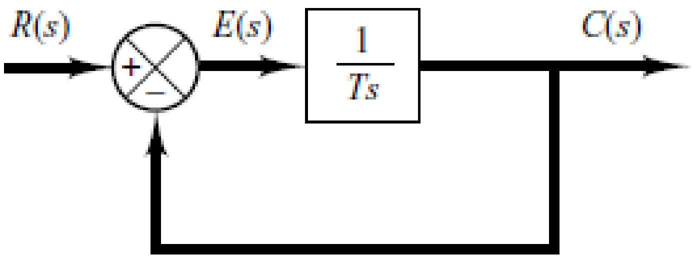

Figure 1. (a) Block diagram of a first-order system

Katsuhiko Ogata(2009), Modern Control Engineering, 5th Edition, Prentice Hall, page 161.

Analyze the unit step time response of a first-order system

- A reduced block diagram, as shown in figure 1 (b), is obtained by applying block diagram reduction techniques as follows:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{\frac{1}{Ts}}{1 + \frac{1}{Ts} * 1} = \frac{1}{Ts + 1}$$

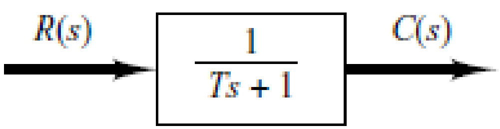

Figure 1. (b) A reduced block diagram of a first-order system

Analyze the unit step time response of a first-order system

- The input-output relationship, for a first order system, is given by:

$$\frac{C(s)}{R(s)} = \frac{1}{Ts + 1}$$

Where T(Tau) is the time constant

- Unit step time response of a first-order system is an output or response of the system when the driving input is a unit step signal or function: $r(t) = 1\,, t \geq 0,\ \ and\ r(t) = 0,$ elsewhere.
- The Laplace transform of a unit step function, $\boldsymbol{R(s)} = \frac{\mathbf{1}}{\boldsymbol{s}}$

Analyze the unit step time response of a first-order system

❖ Hence, the system response in s-domain is:

$$\boldsymbol{C(s)} = \frac{\boldsymbol{R(s)}}{\boldsymbol{Ts+1}}$$

$$\boldsymbol{C(s)} = \frac{\boldsymbol{1}}{\boldsymbol{s(Ts+1)}}$$

❖ Expand $\boldsymbol{C(s)}$ into partial fractions:

$$C(s) = \frac{\boldsymbol{1}}{\boldsymbol{s(Ts+1)}} = \frac{\mathrm{A}}{s} + \frac{\mathrm{B}}{\mathrm{Ts+1}}$$

$$\mathrm{A} = \left.\frac{\boldsymbol{1}}{\boldsymbol{Ts+1}}\right|_{s=0} = \frac{1}{1} = 1 \quad \text{and} \quad \mathrm{B} = \left.\frac{\boldsymbol{1}}{\boldsymbol{s}}\right|_{s=-\frac{1}{T}} = \frac{1}{-\frac{1}{T}} = -T$$

❖ Thus, $C(s) = \frac{1}{s} + \frac{-\mathrm{T}}{\mathrm{Ts+1}} = \frac{1}{s} - \frac{1}{s+\frac{1}{T}}$

Analyze the unit step time response of a first-order system

- Taking inverse Laplace transform of $C(s)$, we obtain:

$$\boldsymbol{c(t) = 1 - e^{-\frac{t}{T}}, t > 0}$$

- $c(t)$ is a unit step response of first-order system
- $c(t)$ displays two components:
  - ✓ the part that goes to zero after large interval of time $\boldsymbol{c_{tr}(t)}$ and,
  - ✓ The part that remains constant when the time becomes very large or simply when t tends to infinite $\boldsymbol{c_{ss}(t)}$.

Analyze the unit step time response of a first-order system

❖ Hence;

✓ the transient response: $\boldsymbol{c_{tr}(t)} = -e^{-\frac{t}{T}},\ t \geq 0$

✓ and the steady state response: $\boldsymbol{c_{ss}(t)} = 1\ ,\ t \geq 0$

❖ Initial value of the response $\boldsymbol{c(0) = \lim_{t\to 0} c(t)}$

$$c(0) = \lim_{t\to 0}\left(\boldsymbol{1 - e^{-\frac{t}{T}}}\right) = 1 - \boldsymbol{e^{0} = 1 - 1 = 0}$$

❖ Final value of the response $\boldsymbol{c(\infty) = \lim_{t\to\infty} c(t)}$

$$c(\infty) = \lim_{t\to\infty}\left(\boldsymbol{1 - e^{-\frac{t}{T}}}\right) = \boldsymbol{1 - e^{-\infty} = 1 - 0 = 1}$$

Analyze the unit step time response of a first-order system

- The initial value and final value theorems applied to $\boldsymbol{c(t)}$ shows that it **initially starts from zero** and **finally becomes unity**.

- At t=T, $c(T) = 1 - e^{-\frac{T}{T}} = 1 - e^{-1} = 0.632$, that is 63.2% of the final value $c(\infty) = 1$

- Practically, this means that the response $\boldsymbol{c(t)}$ reaches 63.2% of its final value after **one time constant** *T*.

- Here, it important to note that **the smaller** the time constant *T*, **the faster** the system response.

Analyze the unit step time response of a first-order system

- On previous slide, we demonstrated that in **one time constant**, the exponential response curve has gone from 0 to 63.2% of its final value.
- In the same manner, it can be demonstrated that in two time constants *($t=2T$)*, the response reaches 86.5% of the final value.
- At $t=3T$, $4T$, and $5T$, the response reaches, respectively, 95%, 98.2%, and 99.3% of the final value.

Analyze the unit step time response of a first-order system

- Thus, for $t \geq 4T$, the response remains within a band of $\pm$ 2% of the final value.
- The time required for the response curve to rise, reach, and remains within a range of $\pm$5% or $\pm$2% of its final value is called the **settling time $\boldsymbol{t_s}$**.
- Hence, we have:

$$\text{Settling time } \boldsymbol{t_{s5\%} = 3T}$$

$$\text{Settling time } \boldsymbol{t_{s5\%} = 4T}$$

Analyze the unit step time response of a first-order system

- Now, calculate the slope of the tangent line at origin ($t=0$) on the response curve $\boldsymbol{c(t)}$:

$$\left.\frac{\boldsymbol{dc(t)}}{\boldsymbol{dt}}\right|_{\boldsymbol{t=0}} = \left.\frac{\boldsymbol{1}}{\boldsymbol{T}}\boldsymbol{e}^{-\frac{\boldsymbol{t}}{\boldsymbol{T}}}\right|_{\boldsymbol{t=0}} = \frac{\boldsymbol{1}}{\boldsymbol{T}}$$

- This means that the output of a first order system would reach the final value at $t = T$ if it maintained its initial speed of response.
- The slope of the response curve $c(t)$ decreases monotonically from $\frac{1}{T}$ at $\mathrm{t} = 0$ to zero at $t = +\infty$, refer to figure 2.

Analyze the unit step time response of a first-order system

- The steady state error signal $e(t)$ is then $e(t) = r(t) - c(t)$

  $e(t) = \boldsymbol{e}^{-\frac{\boldsymbol{t}}{\boldsymbol{T}}},\ for\ t \geq 0$

- The steady state error : $\boldsymbol{e_{ss}} = \lim_{\boldsymbol{t}\to\infty} \boldsymbol{e(t)} = \boldsymbol{0}$

- Thus, the system tracks or follows the unity step input with zero steady state error.

Analyze the unit step time response of a first-order system

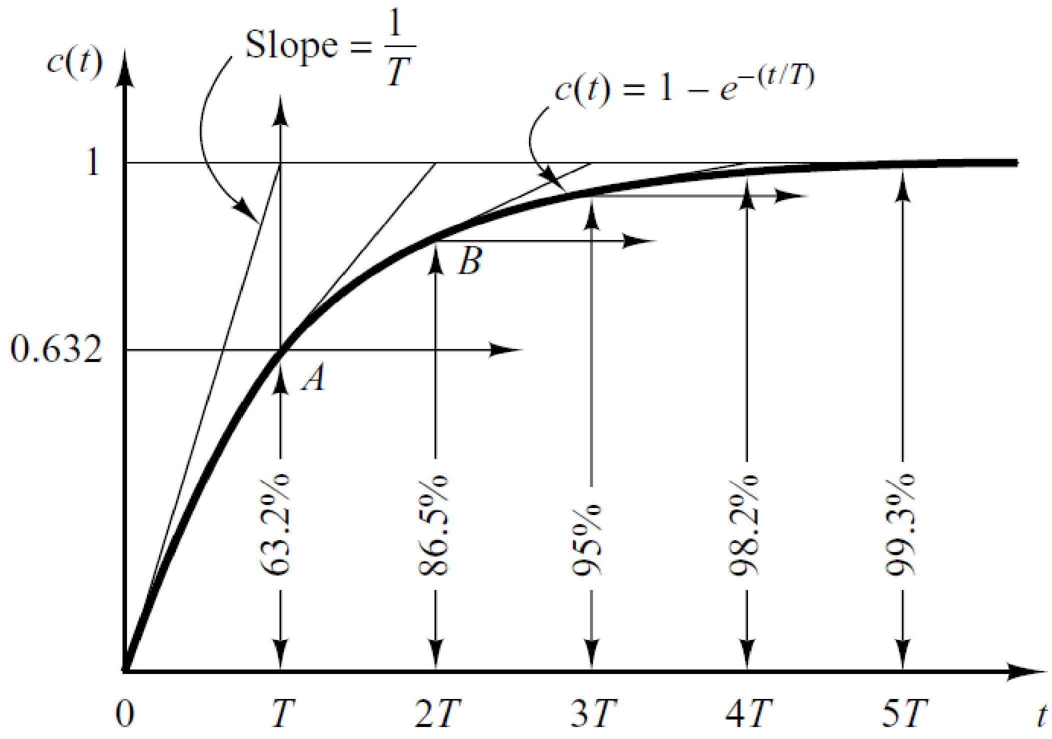

Figure 2. First-order system unit step time response

Katsuhiko Ogata(2009), Modern Control Engineering, 5th Edition, Prentice Hall, page 162.

Analyze the unit ramp time response of a first-order system

- ❖ Unit ramp time response of a first order system is an output or response of the system when the driving input is a unit ramp signal or function.
- ❖ The Laplace transform of unit ramp function, $\boldsymbol{R(s) = \frac{1}{s^2}}$
- ❖ Hence, the system response in s-domain is:

$$\boldsymbol{C(s) = \frac{R(s)}{Ts+1} = \frac{1}{s^2(Ts+1)}}$$

Analyze the unit ramp time response of a first-order system

❖ Expand $\boldsymbol{C(s)}$ into partial fractions:

$$\boldsymbol{C(s) = \frac{1}{s^2(Ts+1)} = \frac{A_1}{s} + \frac{A_2}{s^2} + \frac{B}{Ts+1}}$$

$$\boldsymbol{B} = \left.\frac{1}{s^2}\right|_{s=-\frac{1}{T}} = T^2$$

$$\boldsymbol{A_2} = \left.\frac{1}{(Ts+1)}\right|_{s=0} = 1$$

$$\boldsymbol{A_1} = \left.\frac{-\mathrm{T}}{(Ts+1)^2}\right|_{s=0} = -T$$

Analyze the unit ramp time response of a first-order system

- The expanded form of $C(s) = -\frac{T}{s} + \frac{1}{s^2} + \frac{T^2}{Ts+1}$
- Taking inverse Laplace transform of $C(s)$, we obtain:

$$c(t) = -Tu(t) + tu(t) + Te^{-\frac{t}{T}}u(t)$$

$$c(t) = \left(t - T + Te^{-\frac{t}{T}}\right)u(t) = t - T\left(1 - e^{-\frac{t}{T}}\right), t \geq 0$$

- Note that any **causal signal** can be written using a **unit step function $\boldsymbol{u(t)}$**. Thus, the use $u(t)$ to represent c(t).

Analyze the unit ramp time response of a first-order system

❖ Hence;

✓ The transient response: $\boldsymbol{c_{tr}(t)} = Te^{-\frac{t}{T}}$, ≥ 0 because this term disappears when t approaches large values of t

✓ And the steady state response: $\boldsymbol{c_{ss}(t)} = t - T, \geq 0$ because this term remains for large values of t.

❖ The error signal $e(t)$ is then $e(t) = r(t) - c(t)$

$$e(t) = T\left(1 - e^{-\frac{t}{T}}\right) \; because \; r(t) = t, \; for \; t \geq 0$$

Analyze the unit ramp time response of a first-order system

- Thus, $e_{ss} = e(\infty) = \lim_{t \to 0} e(t)$

$$e_{ss} = e(\infty) = \lim_{t \to 0} T\left(1 - e^{-\frac{t}{T}}\right) = T(1 - e^{-\infty})$$

$$e_{ss} = e(\infty) = T$$

- The error is equal to T for sufficiently large values of t.
- The **smaller the time constant T**, the **smaller the steady-state error for a unit ramp** input.
- The unit ramp input and the system output are shown in figure 2.

Analyze the unit ramp time response of a first-order system

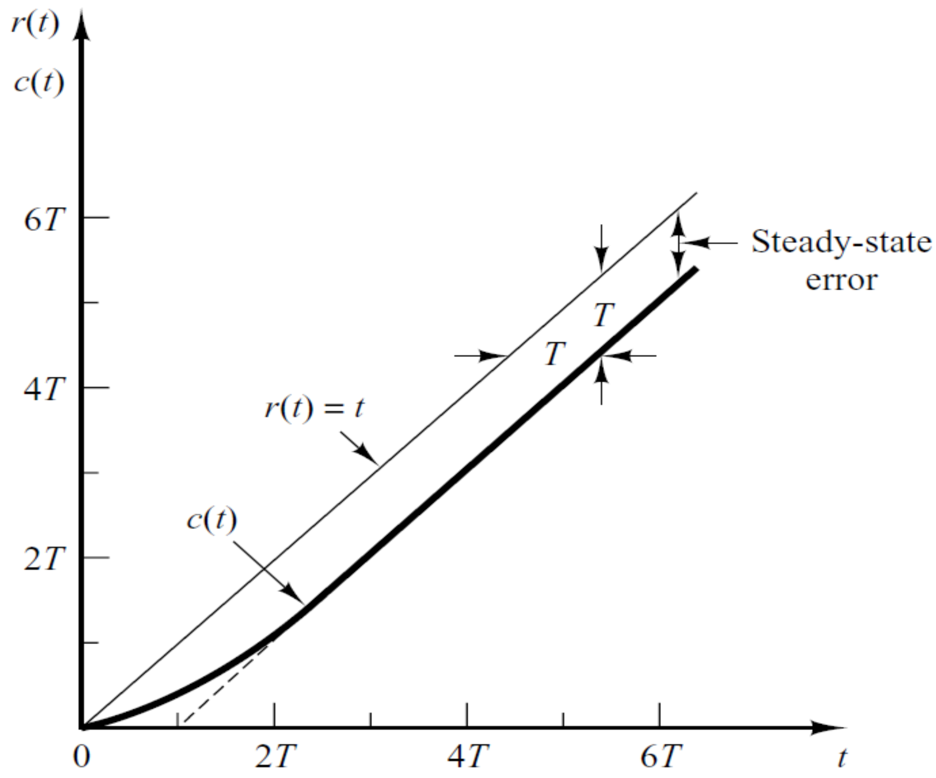

Figure 3. First-order system unit ramp time response

Katsuhiko Ogata(2009), Modern Control Engineering, 5th Edition, Prentice Hall, page 163.

Analyze the unit impulse time response of a first-order system

- Unit impulse time response of a first order system is an output or response of the system when the driving input is a unit impulse signal or function.
- The Laplace transform of a unit-impulse input, $\boldsymbol{R(s) = 1}$.
- The output of the system can be obtained as

$$C(s) = \frac{1}{(Ts + 1)}$$

Analyze the unit impulse time response of a first-order system

- Taking the inverse Laplace transform, we obtain

$$c(t) = \frac{1}{T} e^{-\frac{t}{T}}, t \geq 0$$

- $c(t)$ tends to zero when $t \to \infty$. Thus, the impulse response consists only of transient term, refer to figure 4.
- Impulse response is also called system response or zero input response or natural response or free response.
- The steady state error can not be defined for impulse response because it has only transient term.

Analyze the unit impulse time response of a first-order svstem

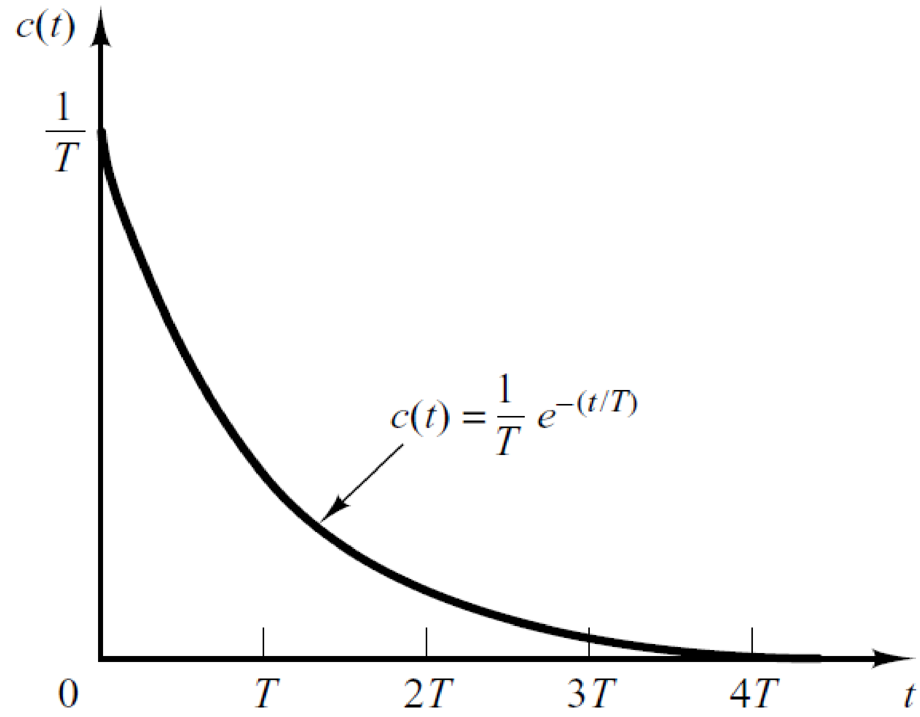

Figure 4. First-order system unit impulse time response

Katsuhiko Ogata(2009), Modern Control Engineering, 5th Edition, Prentice Hall, page 163.

References

1. Palani S. (2022), Automatic Control Systems: With MATLAB, 2nd Edition, Springer.
2. Katsuhiko Ogata(2009), Modern Control Engineering, 5th Edition, Prentice Hall.
3. Allen J. Stubberud Ivan J. Williams Joseph J. DiStefano (1994), Schaum's Outline of Feedback and Control Systems, 2nd , McGraw-Hill.

THANK YOU