Course: Automatic Control System Technology

Lecture 12: Perform the system stability analysis

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Perform the system stability analysis

Session objectives:

By the end of this session, students will be able to :

- ❖ Define stability
- ❖ Define absolute and relative stability
- ❖ Define and prove BIBO stability
- ❖ Demonstrate BIBO stability based on the location of closed loop poles
- ❖ Apply Routh-Hurwitz stability criterion for the linear system

Define stability

- **Stability:** It is a concept that describes whether the system will be able to follow the input command.
- A system is said to be stable, if its output is under control and it is unstable if its output is out of control.

Farid Golnaraghi & Benjamin C. Kuo (2010), Automatic Control Systems,9th Edition, John Wiley&Sons, page 9.

Define absolute and relative stability

- **Absolute stability:** refers to the condition whether the given system is stable or not.
- **Relative stability:** If the system is stable, the closeness of stability is measured by relative stability.
- That is, once the system is found to be stable, it is of interest to determine how stable it is , and this degree of stability is a measure of relative stability.

Palani S. (2022), Automatic Control Systems: With MATLAB, 2nd Edition, Springer, page 504.

Define and prove BIBO stability

- Let $u(t)$, $y(t)$, and $h(t)$ be the input, output, and the impulse response of a linear time-invariant system, respectively.
- With zero initial conditions, the system is said to be **BIBO (bounded input bounded output) stable**, or simply stable, if its output $\boldsymbol{y(t)}$ is bounded to a bounded input $\boldsymbol{u(t)}$.
- As per lecture 2, the output $\boldsymbol{y(t)}$ is defined by the convolution integral $u(t)$ and $h(t)$ as follows:

$$\boldsymbol{y(t) = h(t) * u(t) = \int_0^\infty u(t-\tau)h(\tau)d\tau}$$

Define and prove BIBO stability

- Taking the absolute value of both sides of the equation, we get:

$$|\boldsymbol{y}(\boldsymbol{t})| = \left|\int_{\boldsymbol{0}}^{\infty} \boldsymbol{u}(\boldsymbol{t}-\boldsymbol{\tau})\boldsymbol{h}(\boldsymbol{\tau})\boldsymbol{d\tau}\right| \leq \int_{\boldsymbol{0}}^{\boldsymbol{t}} |\boldsymbol{u}(\boldsymbol{t}-\boldsymbol{\tau})|\,|\boldsymbol{h}(\boldsymbol{\tau})|\,\boldsymbol{d\tau}$$

Or

$$|\boldsymbol{y}(\boldsymbol{t})| \leq \int_{\boldsymbol{0}}^{\boldsymbol{t}} |\boldsymbol{u}(\boldsymbol{t}-\boldsymbol{\tau})|\,|\boldsymbol{h}(\boldsymbol{\tau})|\,\boldsymbol{d\tau}$$

- If $u(t)$ is bounded $|\boldsymbol{u}(\boldsymbol{t})| \leq \boldsymbol{A}$,where $\boldsymbol{A}$ is a finite positive number. Then:

$$|\boldsymbol{y}(\boldsymbol{t})| \leq \boldsymbol{A}\int_{\boldsymbol{0}}^{\infty} |\boldsymbol{h}(\boldsymbol{\tau})|\boldsymbol{d\tau}$$

Define and prove BIBO stability

- Thus, if y(t) is to be bounded, $|\boldsymbol{y}(\boldsymbol{t})| \leq \boldsymbol{B}$, where $\boldsymbol{B}$ is a finite positive number. The following expression must hold:

$$\boldsymbol{A}\int_{\mathbf{0}}^{\infty} |\boldsymbol{h}(\boldsymbol{\tau})| d\boldsymbol{\tau} \leq \boldsymbol{B} < \infty$$

$$\int_{\mathbf{0}}^{\infty} |\boldsymbol{h}(\boldsymbol{\tau})| d\boldsymbol{\tau} \leq \frac{\boldsymbol{B}}{\boldsymbol{A}} < \infty$$

- The expression is the area of impulse response curve. This means that for a system to be BIBO stable, the area of impulse response curve should be finite in time interval: $\mathbf{0} \leq \boldsymbol{t} < \infty$
- That is shown in figure 1 (a) and (b) for unstable system.
- A system is BIBO stable when its impulse response is summable

Define and prove BIBO stability

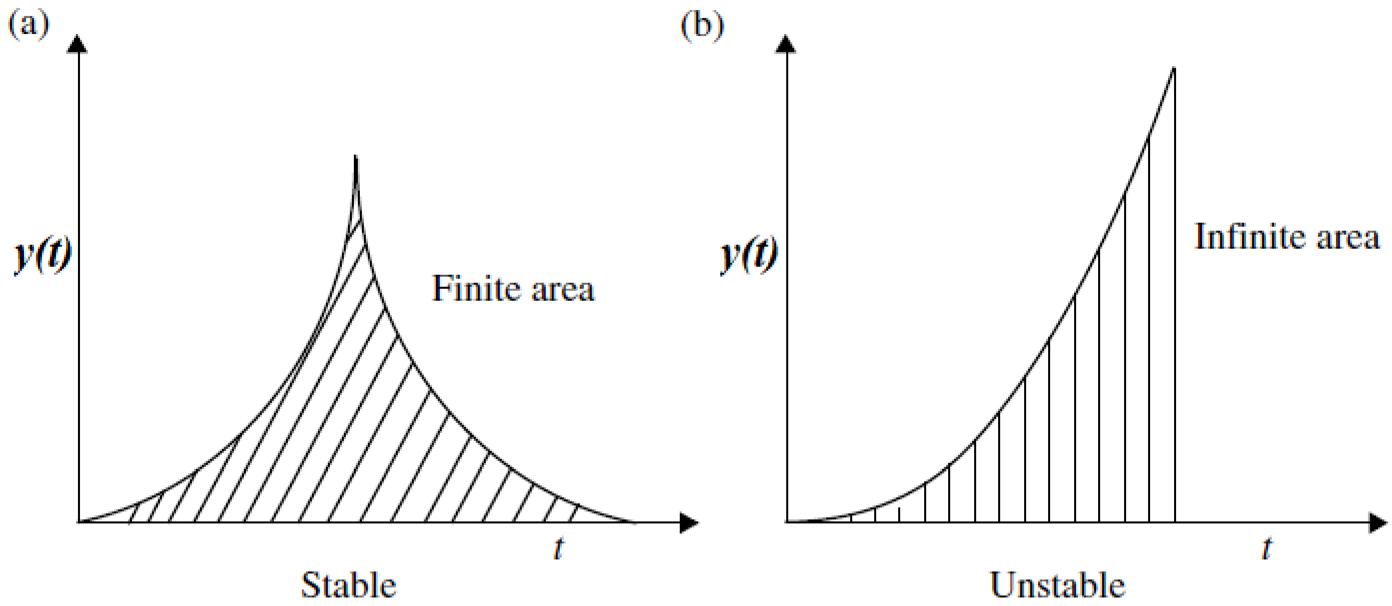

Figure 1:Impulse response curve with (a) Finite area, (b) Infinite area

Palani S. (2022), Automatic Control Systems: With MATLAB, 2nd Edition, Springer, page 506.

Demonstrate BIBO stability based on the location of closed loop poles

- It is not always possible to test the system with impulse input and measure the output with respect to time to assess the BIBO stability requirement.
- However, the BIBO stability can be easily determined based on the location of closed loop poles of the system in s-plane.
- By definition, the transfer function ***H(s)*** of a system is the Laplace transform of its impulse response ***h(t)***:

$$\boldsymbol{H(s) = \int_0^{\infty} h(t) e^{-st} dt}$$

Demonstrate BIBO stability based on the location of closed loop poles

❖ Taking the absolute value of both sides of the equation, we get:

$$|\boldsymbol{H}(\boldsymbol{s})| = \left|\int_{0}^{\infty} \boldsymbol{h}(\boldsymbol{t})\boldsymbol{e}^{-\boldsymbol{st}}\boldsymbol{dt}\right| \leq \int_{0}^{\infty} |\boldsymbol{h}(\boldsymbol{t})|\,|\boldsymbol{e}^{-\boldsymbol{st}}|\,\boldsymbol{dt}$$

$$|\boldsymbol{H}(\boldsymbol{s})| \leq \int_{0}^{\infty} |\boldsymbol{h}(\boldsymbol{t})||\boldsymbol{e}^{-\boldsymbol{\sigma t}}|\boldsymbol{dt}$$

❖ Because $\left|\boldsymbol{e}^{-\boldsymbol{st}}\right| = |\boldsymbol{e}^{-\boldsymbol{\sigma t}}|$, where σ is the real part of $s = \sigma + j\omega$, when s assumes a value of a pole of $H(s)$, $H(s) = \infty$, and the above equation become:

$$\infty \leq \int_{0}^{\infty} |\boldsymbol{h}(\boldsymbol{t})||\boldsymbol{e}^{-\boldsymbol{\sigma t}}|\boldsymbol{dt}$$

Demonstrate BIBO stability based on the location of closed loop poles

- If one or more closed loop poles are in the right-half s-plane or on the $j\omega$ – axis, $\sigma \geq 0$, then:

$$|e^{-\sigma t}| \leq M = 1$$

- And finally:

$$\infty \leq \int_0^\infty |h(t)||M dt \leq \int_0^\infty |h(t)| dt$$

$$\int_0^\infty |h(t)| dt \geq \infty$$

- Which violates the BIBO stability requirement .

Demonstrate BIBO stability based on the location of closed loop poles

❖ Therefore, according to the location of closed loop poles:

- ✓ A system is BIBO stable if all of its closed loop poles are located in the left half s-plane (LHP).
- ✓ A system is BIBO unstable when at least one of its closed loop poles lies in the right half s-plane (RHP) or on the $j\omega$ axis.

Demonstrate BIBO stability based on the location of closed loop poles

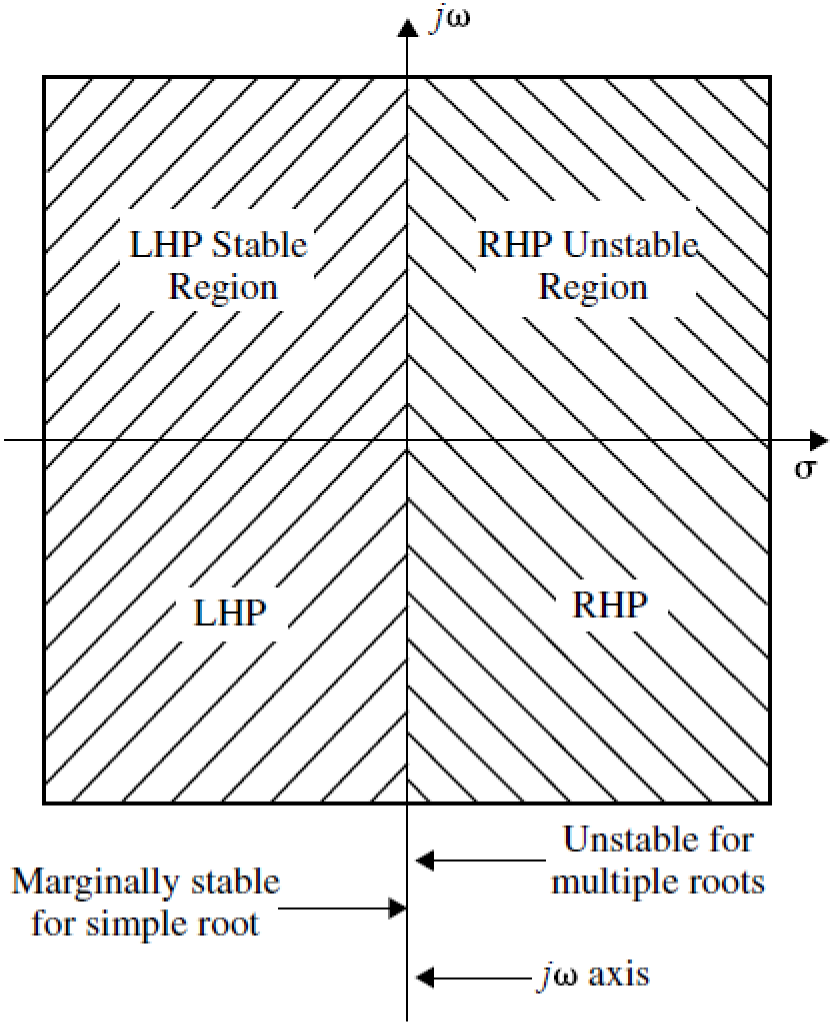

Figure 2: Location of closed loop poles in s-plane and stability

Palani S. (2022), Automatic Control Systems: With MATLAB, 2nd Edition, Springer, page 512.

Demonstrate BIBO stability based on the location of closed loop poles

- In figure 2, the stable region, unstable region and marginally stable region in the s-plane are sketched and shown.
- The closed loop poles in LHP indicate asymptotic stability, poles in RHP region indicates instability and the presence of single pole in the $j\omega$ axis indicates marginal stability.
- The repeated poles in the $j\omega$ axis indicates instability.

Demonstrate BIBO stability based on the location of closed loop poles

❖ **To summarize:**

✓ A LTI system is asymptotically stable if all the closed loop poles are in the LHP.

✓ A LTI system is unstable either at least one pole is in the RHP or there are repeated poles in the $j\omega$ axis.

✓ A LTI system is marginally stable if there are no poles in the RHP and there are unrepeated poles in the $j\omega$ axis.

✓ An asymptotically stable system is also BIBO stable.

✓ An asymptotically unstable system or a marginally stable system is BIBO unstable system.

Apply Routh-Hurwitz stability criterion for the linear system

- In the previous slides, we have demonstrated that a control system is **stable if and only if all closed-loop poles lie in the left-half s-plane.**
- Ruth-Harwitz criterion is a time domain approach to determine the number of roots of the characteristic polynomial with constant real coefficients with respect to the LHP, RHP and the $j\omega$ axis of s-plane.
- Routh's stability criterion tells us whether or not there are unstable roots in a polynomial equation without solving them.

Apply Routh-Hurwitz stability criterion for the linear system

- Routh-Hurwitz stability criterion consists of **one necessary condition** and **one sufficient condition** for stability.
- If any control system doesn't satisfy the necessary condition, then we can conclude that the control system is unstable.
- But, if the control system satisfies the necessary condition, then it may or may not be stable. So, the sufficient condition is helpful for knowing whether the control system is stable or not.
- When Routh's stability criterion is applied to a control system, information about absolute stability can be obtained directly from the coefficients of the characteristic polynomial.

Apply Routh-Hurwitz stability criterion for the linear system

- Most linear closed-loop systems have a closed-loop transfer function of the form:

$$\boldsymbol{G(s)} = \frac{\boldsymbol{Y(s)}}{\boldsymbol{U(s)}} = \frac{\boldsymbol{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}}{\boldsymbol{a_o s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}}$$

- Where $\boldsymbol{a_o}, \boldsymbol{a_1},\ldots, \boldsymbol{a_n}$ and $\boldsymbol{b_o}, \boldsymbol{b_1},\ldots, \boldsymbol{b_m}$ are constants and $n > m$
- The Routh's stability criterion enables us to determine the number of closed-loop poles that lie in the right-half s-plane without having to factor the denominator polynomial.
- The polynomial may have parameters that MATLAB can not handle

Apply Routh-Hurwitz stability criterion for the linear system

❖ The characteristic polynomial in s is of the following form:

$$a_0 s^n + a_1 s^{n-1} + \cdots a_{n-1} s + a_n = 0$$

❖ **Necessary Condition for Routh-Hurwitz Stability:**

✓If any of the coefficients are zero or negative in the presence of at least one positive coefficient, a root or roots exist that are imaginary or that have positive real parts.

✓Therefore, in such a case, the system is not stable. And if we are interested in only the absolute stability, there is no need to follow the procedure further.

Apply Routh-Hurwitz stability criterion for the linear system

- ✓ **The necessary condition but not sufficient for stability** is that the coefficients of the characteristic polynomial should all be present and all have a positive sign.
- ❖ **The necessary and sufficient Condition for Routh-Hurwitz Stability:**
- ✓ If the necessary conditions are satisfied, we can proceed further to establish the sufficient condition required for the closed loop system to be stable. For this, **Routh's array is prepared** as explained below.

Apply Routh-Hurwitz stability criterion for the linear system

- ✓ The coefficients of the characteristic polynomial are arranged in rows and columns with even powers of s in one row and odd powers of s in the next row.
  - If the highest degree of the polynomial is even, then the even row is started first followed by the odd row.
  - If the highest degree of the polynomial is odd, the odd row components are started first followed by even power components.
- ✓ By this, the first two rows of the Routh array are formed, from which subsequent rows are formed as explained below.

Apply Routh-Hurwitz stability criterion for the linear system

Routh-Hurwitz Array/Table

s^n	a_0	a_2	a_4	$\cdots$
s^{n-1}	a_1	a_3	a_5	$\cdots$
s^{n-2}	b_1	b_2	b_3	$\cdots$
s^{n-3}	c_1	c_2	c_3	$\cdots$
$\vdots$	$\vdots$	$\vdots$		
s^2	d_1	d_2		
s^1	e_1			
s^0	f_1			

Palani S. (2022), Automatic Control Systems: With MATLAB, 2nd Edition, Springer, page 512.

Apply Routh-Hurwitz stability criterion for the linear system

- ✓ For n^{th} degree characteristic polynomial, there will be $(\boldsymbol{n}+\mathbf{1})$ Routh's rows.
- ✓ The first two rows are prepared from the coefficients of the given characteristic equation. The coefficients of the remaining $(\boldsymbol{n}-\mathbf{1})$ rows are evaluated as follows:

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} \qquad c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1} \qquad c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1} \qquad c_3 = \frac{b_1 a_7 - a_1 b_4}{b_1}$$

$$\vdots \qquad\qquad \vdots$$

Apply Routh-Hurwitz stability criterion for the linear system

- ✓ The process is continued until the s^{0th} row is completed. The complete array of coefficients looks like a triangle.
- ✓ After preparing the Routh's array following the procedure described above, the Routh's stability criterion is applied to the characteristic equation/polynomial.
- ✓ The Routh-Hurwitz criterion states that the number of roots of the characteristic equation with positive real parts is equal to the number of changes in sign of the first column of the Routh Array.

Apply Routh-Hurwitz stability criterion for the linear system

The necessary and sufficient condition for which all roots of the characteristic polynomial lie in the left-half s-plane is that all the coefficients of the characteristic polynomial be positive and all terms in the first column of the array have positive signs.

Apply Routh-Hurwitz stability criterion for the linear system

- ✓ ***Note:*** For simplifying the numerical calculation, the entire row may be multiplied or divided by a positive number by which the stability condition is unaffected.

❖ **Example 1:**

- ✓ For the following characteristic polynomial, determine whether the closed loop system is stable. Use Routh-Hurwitz stability criterion.

  1. $F(s) = s^5 + 4s^4 - 8s^3 + 11s^2 + 6s + 4$
  2. $F(s) = s^6 + 7s^5 + 4s^3 + 2s^2 + 10s + 4$

Apply Routh-Hurwitz stability criterion for the linear system

- ❖ **Example 1 (*cont.*):**
- ✓ **Solution**

1. $F(s) = s^5 + 4s^4 - 8s^3 + 11s^2 + 6s + 4$

- In the above characteristic polynomial,-8 is one coefficient with negative sign.
- This indicates that there is at least one root of the characteristic equation in RHP and hence the system is unstable. There is no need to prepare Routh's array.

Apply Routh-Hurwitz stability criterion for the linear system

- **Example 1 (*cont.*):**

✓ **Solution**

$$2.\ F(s) = s^6 + 7s^5 + 4s^3 + 2s^2 + 10s + 4$$

- In the above characteristic polynomial, the coefficient of s^4 is zero. Hence, all the roots will not be in LHP.
- Some roots may be in the $j\omega$ axis or in RHP. Hence, the system is not stable.
- Here also, the necessary condition for Routh-Hurwitz stability criterion is not satisfied and the system is unstable. There is no need to prepare the Routh's array.

Apply Routh-Hurwitz stability criterion for the linear system

❖ **Example 2:**

✓ Find stability of the system given by $G(s) = \frac{K}{s(s+1)}$ and $H(s) = 1$ using Routh-Hurwitz stability criterion.

✓ The closed loop transfer function :

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{\frac{K}{s(s+1)}}{1 + \frac{K}{s(s+1)}} = \frac{K}{s^2 + s + K}$$

✓ Characteristic equation: $s^2 + s + K = 0$

s^2	1	K
s^1	1	0
s^0	K	

The system is always stable for K>0 because there is no sign change in first column of Routh's array.

Apply Routh-Hurwitz stability criterion for the linear system

❖ **Special cases:**

1. The first element in any one row of the Routh Table is zero, but the other elements are not
2. The elements in one row of the Routh Table are all zero.

❖ **First special case: The first element of a row in Ruth Table is zero:**

- ✓ If a zero appears in the first position of a row but not all elements of the row are zeros, the elements in the next row will all become infinite, and the Routh Test breaks down.

Apply Routh-Hurwitz stability criterion for the linear system

- **First special case (*cont.*):**
  - In this case, one may replace the zero element in the Routh Table by an arbitrary small positive number ε and then proceed with the Routh Test or replace $s = \frac{1}{z}$ and reconstruct Routh array with new coefficient of z
  - **Example:** Using Routh-Hurwitz, examine stability of the system defined by its characteristic equation:

$$s^5 + 2s^4 + 4s^3 + 8s^2 + 3s + 1 = 0$$

Apply Routh-Hurwitz stability criterion for the linear system

- **First special case (*cont.*):**
  - **Routh Table**

s^5	1	4	3
s^4	2	8	1
s^3	0	$\frac{5}{2}$	0
s^2	∞		
s^1			
s^0			

Here, the criterion fails. Apply one of the above two techniques to remove this difficulty.

Apply Routh-Hurwitz stability criterion for the linear system

❖ **First special case (*cont.*):**

Method 1. Replace 0 by ε and complete the array with ε. Finally examine the sign change by taking $\varepsilon \to 0$

	With ε			With $\varepsilon = 0$		
s^5	1	4	3	**1**	4	3
s^4	2	8	1	**2**	8	1
s^3	ε^+	$\frac{5}{2}$	0	$\mathbf{0^+}$	$\frac{5}{2}$	0
s^2	$\frac{8\varepsilon - 5}{\varepsilon}$	1		$-\infty$	1	
s^1	$\frac{\frac{5}{2}\frac{8\varepsilon - 5}{\varepsilon} - \varepsilon}{\frac{8\varepsilon - 5}{\varepsilon}}$	0		**5/2**		
s^0	1			**1**		

In the table With $\varepsilon = 0$, there is 2 sign changes.Thus the system is unstable.

Apply Routh-Hurwitz stability criterion for the linear system

❖ **First special case (*cont.*):**

Method 2. Replace $s = \frac{1}{z}$ and reconstruct Routh array with new coefficient of z.

System characteristic equation $s^5 + 2s^4 + 4s^3 + 8s^2 + 3s + 1 = 0$

becomes

$$\frac{1}{z^5} + \frac{2}{z^4} + \frac{4}{z^3} + \frac{8}{z^2} + \frac{3}{z} + 1 = 0$$

$$\Longrightarrow z^5 + 3z^4 + 8z^3 + 4z^2 + 2z + 1 = 0$$

Apply Routh-Hurwitz stability criterion for the linear system

❖ **First special case (*cont.*):**

✓ **Routh Table**

s^5	1	8	2
s^4	3	4	1
s^3	$\frac{20}{3}$	$\frac{5}{3}$	0
s^2	$\frac{13}{4}$	1	
s^1	-0.38		
s^0	1		

There are two sign changes in first column elements of this array. Therefore, the system is unstable.

Apply Routh-Hurwitz stability criterion for the linear system

- **Second special case: All elements in one row of the Routh Table are zeros**
  - In the second case, when all the elements in one row of the Routh Table are zeros, the test breaks down.
  - The equation that is formed by using the coefficients of the row just above the row of zeros is called the auxiliary equation.

Apply Routh-Hurwitz stability criterion for the linear system

❖ **Second special case (*cont.*) :**

✓ Routh Test may be carried on by performing the following steps:

1. Take the derivative of the auxiliary equation with respect to s.
2. Replace the row of zeros with the coefficients of the resultant equation obtained by taking the derivative of the auxiliary equation.
3. Carry on the Routh Test in the usual manner with the newly formed table.

Apply Routh-Hurwitz stability criterion for the linear system

❖ **Second special case (*cont.*) :**

✓ **Example:** Examine stability of the system whose characteristic equation is: $s^5 + 2s^4 + 2s^3 + 4s^2 + 4s + 8 = 0$

✓ **Routh table**

s^5	1	2	4
s^4	2	4	8
s^3	0	0	0
s^2			
s^1			
s^0			

Here, the criterion fails. To remove the difficulty we use the auxiliary equation from the coefficients of the row just above the row of zeros, calculate its derivative and use coefficients found after derivative to construct the Routh table.

Apply Routh-Hurwitz stability criterion for the linear system

❖ **Second special case (*cont.*) :**

✓The auxiliary equation is: $A(s) = 2s^4 + 4s^2 + 8$

✓Calculate its first derivative: $\frac{dA(s)}{ds} = 8s^3 + 8s$

✓Use coefficients found after derivative to construct the Routh table

s^5	1	2	4
s^4	2	4	8
s^3	8	8	0
s^2	2	8	0
s^1	-24	0	
s^0	8		

There are two sign changes in elements of the first column of this array. Therefore, the system is unstable.

References

1. Palani S. (2022), Automatic Control Systems: With MATLAB, 2nd Edition, Springer.
2. Farid Golnaraghi & Benjamin C. Kuo (2010), Automatic Control Systems,9th Edition, John Wiley&Sons.

THANK YOU