Course: Automatic Control System Technology

Lecture 13: Develop a state space model of a system

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Develop a state space model of a system

Session objectives:

By the end of this session, students will be able to :

- ❖ Differentiate classical and modern control theories
- ❖ Define key terms used in state space representation
- ❖ Select state variables for state space representation
- ❖ Represent a continuous time system by state space model
- ❖ Develop the state space model of electrical network and mechanical systems by using physical variables
- ❖ Obtain the state space model using phase variables
- ❖ Convert state space model to transfer function

Differentiate classical and modern control theories

- Classical or conventional control theory for the analysis and design of control systems is based on the input-output relationship or transfer function model.
- Modern control theory for the analysis and design of control system uses a system of equations in terms of n first-order differential equations, which may be combined into a first-order vector-matrix differential equations or state space model. Katsuhiko Ogata(2009), Modern Control Engineering, 5th Edition, Prentice Hall, page 648.

Differentiate classical and modern control theories

- The transfer function model is used for the analysis and design of linear time invariant continuous time systems.
- However, the transfer function model has many limitations because it is expressed as a ratio of output to input variables, and thus the internal behavior of the system is hidden.
- The transfer function method is valid only for SISO, LTI systems with initial conditions set to zero. It is can not be used for non-linear, time varying and MIMO systems.

Palani S. (2022), Automatic Control Systems: With MATLAB, 2nd Edition, Springer, page 808.

Differentiate classical and modern control theories

- It is also difficult to handle large-scale complex systems with a transfer function model.
- Moreover, system modeling using transfer function is based on the trial and error procedure which in general will not lead to optimal control systems.
- The above described drawbacks of transfer function model are overcome by representing the system in the state space model.

Differentiate classical and modern control theories

- The state space model for continuous time systems is a differential equation model which is expressed in n first-order differential equations written in a specific format.
- The state space model is valid for linear, non-linear, time variant, time invariant, SISO and MIMO systems with zero or nonzero initial conditions.
- Unlike the transfer function model, the state space equation provide a complete description of the internal behavior of all physical variables in the system.

Define key terms used in state space representation

- **State**: The state of a dynamic system is the smallest set of variables called state variables such that the knowledge of these variables at time $t = t_0$ (initial conditions), together with the knowledge of input to the system for $t \geq t_0$, completely determines the behavior of the system at any time $t \geq t_0$.
- The state of a dynamic system refers to a **minimum number of variables(initial conditions)** that must be specified at initial time $t = t_0$ to fully describe the dynamic behavior of the system at any time $t \geq t_0$ to **any given inputs**. Palani S. (2022), Automatic Control Systems: With MATLAB, 2^{nd} Edition, Springer, page 808.

Define key terms used in state space representation

- When a given input is applied, the future state of the system for $t > t_0$ also change and we can uniquely determine these states.
- Since the states of the system vary with respect to time, they are called state variables.
- **State Variables** are a set of variables which describes the system state at any time, and from which any output variables may be computed.
- **State vector** is a vector whose elements are the state variables.
- **State space** is an n-dimensional space whose axes are the state variables.

Select state variables for state space representation

- State variables of a system are not unique. There are many choices of state variables for a given system. The following are guidelines for selection of state variables:
  - ✓ For a physical system, the number of state variables needed to represent the system must be equal to the number of energy storing elements present in the system.
  - ✓ For a LTI system represented by differential equation or transfer function, the number of state variables needed is equal to the order of the system. Here, state variables are chosen as phase variables.

Represent a continuous time system by state space model

- The state space representation/model provides the dynamics of a system as a set of coupled first-order differential equations in a set of internal variables known as state variables, together with a set of algebraic equations that combine the state variables into physical output variables.
- The philosophy of state space is based on transforming the differential equation of order *n* (highest derivative order) into an *n* equations of 1^{st} order.
- The state space model of a system consist of state equation and output equation.

Represent a continuous time system by state space model

- Both state equation and output equation of a system are function of state variables and inputs.

$$\dot{\boldsymbol{x}}_{\boldsymbol{n}}(\boldsymbol{t}) = \boldsymbol{f}(\boldsymbol{x}(\boldsymbol{t}), \boldsymbol{u}(\boldsymbol{t}))$$

$$\boldsymbol{y}(\boldsymbol{t}) = \boldsymbol{f}(\boldsymbol{x}(\boldsymbol{t}), \boldsymbol{u}(\boldsymbol{t}))$$

- $\boldsymbol{u}(\boldsymbol{t})$, $\boldsymbol{x}(\boldsymbol{t})$, and $\boldsymbol{y}(\boldsymbol{t})$ are respectively input, state, output variables of the system. These variables can also be expressed in form of vectors.
- Both state equation and output equations can simply be referred as state equations.

Represent a continuous time system by state space model

- The standard form of representing any continuous time system in state space representation/model is:

  $\dot{\boldsymbol{x}}(\boldsymbol{t}) = \boldsymbol{Ax}(\boldsymbol{t}) + \boldsymbol{Bu}(\boldsymbol{t})$………………… State equation

  $\boldsymbol{y}(\boldsymbol{t}) = \boldsymbol{Cx}(\boldsymbol{t}) + \boldsymbol{Du}(\boldsymbol{t})$………………… Output equation

- Where,
  - ✓ *A= state matrix (n x n dimension)*
  - ✓ *B= Input matrix (n x r dimension)*
  - ✓ *C= Output matrix (p x n dimension)*
  - ✓ *D= Direct transmission matrix (p x r dimension)*

Represent a continuous time system by state space model

- Where *(cont.)*,
  - ✓ $x(t)$= *state vector (n x 1 dimension)*
  - ✓ $u(t)$= *Input vector (r x 1 dimension)*
  - ✓ $y(t)$= *Output vector (p x 1 dimension)*
- Depending upon the dimensions of the vectors $x(t)$,$u(t)$, and $y(t)$, the appropriate dimensions of the matrices, A,B and C are chosen.
- In most practical applications, the direct transmission of the input $u(t)$ to the output $y(t)$ is not done and hence D=0.

Represent a continuous time system by state space model

- In forming the state equation:
  - ✓ It is to be observed that only the first derivative of *x(t)* appears on the left side of the equation and no derivative of *x(t)* appears on the right side.
  - ✓ The right side of the equation contains only the combination of state and input variables.

Represent a continuous time system by state space model

❖ The state space model of a continuous time system ca be represented in form of block diagram as show in figure 1.

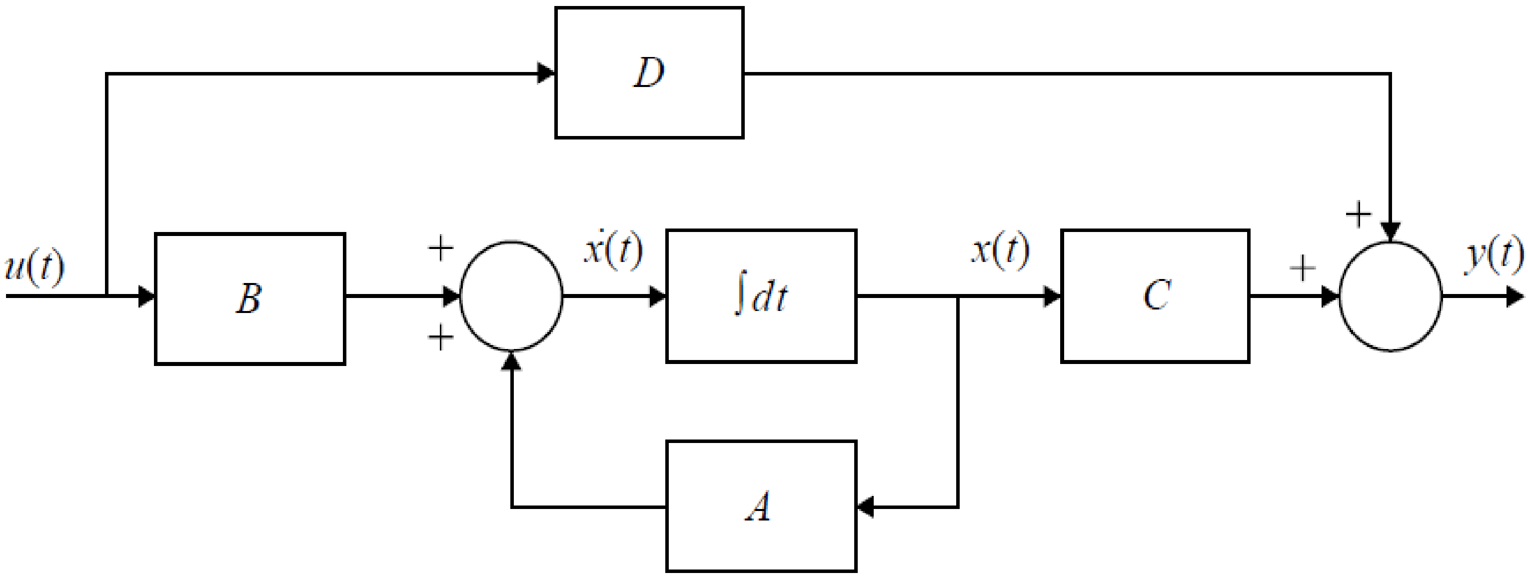


Figure 1. Block diagram representation of state space model

Palani S. (2022), Automatic Control Systems: With MATLAB, 2nd Edition, Springer, page 810.

Develop the state space model of electrical networks and mechanical systems

- **State equations for electrical circuits:**
  - ✓ **The number of independent energy-storing elements** in the electrical circuit determine the **number of state variables.**
  - ✓ The capacitor and inductor are the two energy-storing in electrical circuit.
  - ✓ The physical variables, namely the **current** through the inductor and **voltage** across the capacitor are chosen as the state variables.

Develop the state space model of electrical networks and mechanical systems

❖ **State equations for electrical circuits *(cont.)* :**

✓ Here are steps to follow while forming the state space equations for electrical circuits:

1. Choose all independent inductor currents and the capacitor voltages as the state variables.
2. The state variables and their first derivatives are expressed in terms of a set of loop circuits
3. Write loop equations and eliminate all variables other than state variables, their first derivatives and inputs.

Develop the state space model of electrical networks and mechanical systems

❖ **State equations for electrical circuits *(cont.)* :**

✓ **Example.** Develop the state space model of RLC network represented in figure 2.

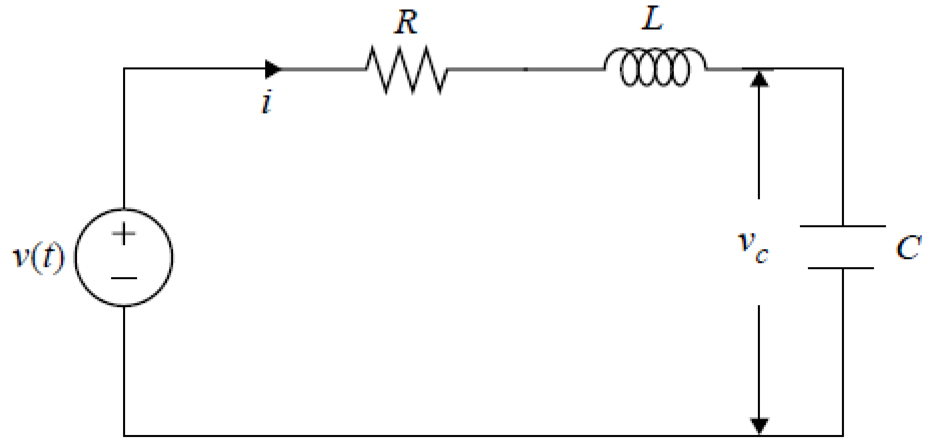


Figure 2. RLC network

Palani S. (2022), Automatic Control Systems: With MATLAB, 2nd Edition, Springer, page 817.

Develop the state space model of electrical networks and mechanical systems

❖ **State equations for electrical circuits *(cont.)* :**

✓ **Solution**

- There are 2 energy-storage elements (inductor and capacitor) in the circuit. So, the number of the state variables is equal to two.
- These state variables are the **current** flowing through the inductor, $\boldsymbol{i(t)}$ and the **voltage** across capacitor, $\boldsymbol{v_c(t)}$.
- The state vector, $x(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix}$ and the derivative of state vector is $\dot{x}(t) = \begin{bmatrix} \dot{\boldsymbol{x}}_{\boldsymbol{1}} \\ \dot{\boldsymbol{x}}_{\boldsymbol{2}} \end{bmatrix} = \begin{bmatrix} \frac{di(t)}{dt} \\ \frac{v_c(t)}{dt} \end{bmatrix}$

Develop the state space model of electrical networks and mechanical systems

- **State equations for electrical circuits *(cont.)* :**
  - ✓ **Solution**
  - From the circuit, the output voltage, $y(t)$ is equal to the voltage across the capacitor, $v_c(t)$.

    That is $y(t) = v_c(t)$
  - Applying KVL around the loop,

$$v_i(t) = Ri(t) + L\frac{di(t)}{dt} + v_c(t)$$

$$i = i_c = C\frac{dv_c}{dt}$$

Develop the state space model of electrical networks and mechanical systems

❖ **State equations for electrical circuits *(cont.)* :**

✓ **Solution**

○ Rearranging the **differential equations and output equation into standard form of state space model**, we get:

$$\frac{di(t)}{dt} = -\frac{R}{L}i(t) - \frac{1}{L}v_c(t) + \frac{1}{L}v_i(t)$$

$$\frac{dv_c}{dt} = \frac{1}{C}\ i(t)$$

$$y(t) = v_c(t)$$

Develop the state space model of electrical networks and mechanical systems

❖ **State equations for electrical circuits *(cont.)* :**

✓ **Solution**

$$\dot{x}(t) = \begin{bmatrix} \frac{di(t)}{dt} \\ \frac{v_c(t)}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} [v_i(t)]$$

$$y(t) = [0\ 1] \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix}$$

$\dot{\boldsymbol{x}}(\boldsymbol{t}) = \boldsymbol{A}\boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{B}\boldsymbol{u}(\boldsymbol{t})$…….. **State equation**

$\boldsymbol{y}(\boldsymbol{t}) = \boldsymbol{C}\boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{D}\boldsymbol{u}(\boldsymbol{t})$…….. **Output equation**

Develop the state space model of electrical networks and mechanical systems

❖ **State equations for electrical circuits *(cont.)* :**

✓ Here:

$$\dot{x}(t) = \begin{bmatrix} \dot{\boldsymbol{x}}_{\boldsymbol{1}} \\ \dot{\boldsymbol{x}}_{\boldsymbol{2}} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} [v_i(t)]$$

$$y(t) = [0\ 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \ ; \ B = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}; \ C = [0\ 1] \ ; \ D = 0$$

Develop the state space model of electrical networks and mechanical systems

- **State equations for mechanical systems :**
  - ✓ The dynamic equations of mechanical systems are written from the free body diagram,
  - ✓ The physical variables such as displacement and velocity are chosen as the states and,
  - ✓ For each state variable, the equation for its first derivative is obtained and converted into the format of state equation.

Develop the state space model of electrical networks and mechanical systems

❖ **State equations for mechanical systems *(cont.)* :**

✓ **Example:** Develop the state space model for the mechanical system shown in figure 3, where $y(t)$ is the displacement of mass M and $f(t)$ is the force applied.

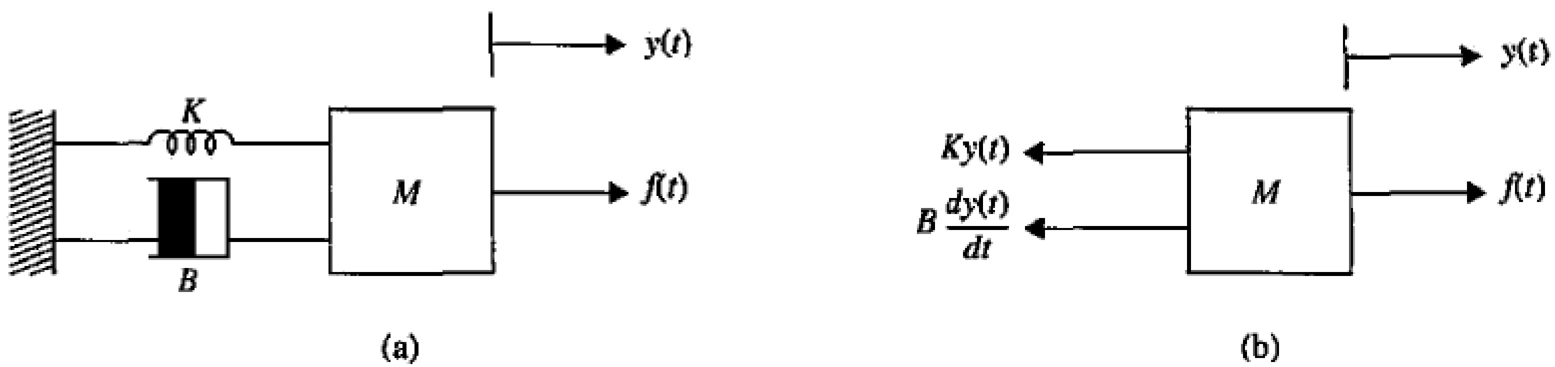


Figure 3. Mechanical system (a) and its free body diagram (b)

Farid Golnaraghi & Benjamin C. Kuo (2010), Automatic Control Systems,9th Edition, John Wiley&Sons, page 151.

Develop the state space model of electrical networks and mechanical systems

❖ **State equations for mechanical systems *(cont.)* :**

✓ **Solution**

- State variables are displacement $x_1 = \text{y}$ and velocity $x_2 = \dot{y}$

- State vector $x(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} displacement \\ velocity \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$

$$x = \begin{cases} x_1 = y \\ x_2 = \dot{y} \end{cases} \text{ and } \dot{x} = \begin{cases} \dot{x}_1 = \dot{y} = x_2 \\ \dot{x}_2 = \ddot{y} \end{cases}$$

- From the free body diagram, the dynamic equation of the system is: $\boldsymbol{M}\frac{\boldsymbol{d^2y(t)}}{\boldsymbol{dt^2}} + \boldsymbol{B}\frac{\boldsymbol{dy(t)}}{\boldsymbol{dt}} + \boldsymbol{Ky(t)} = \boldsymbol{f(t)}$

Develop the state space model of electrical networks and mechanical systems

❖ **State equations for mechanical systems *(cont.)* :**

✓ **Solution**

- By substituting the state variables $x_1 = \text{y}$ and $x_2 = \dot{y}$ in the dynamic equation of the system and taking $\dot{x}_2 = \ddot{y}$, we get: $\boldsymbol{M}\dot{x}_2 + \boldsymbol{B}x_2 + \boldsymbol{K}x_1 = \boldsymbol{f}$

  Thus, $\dot{x}_2 = -\frac{\boldsymbol{K}}{\boldsymbol{M}}x_1 - \frac{\boldsymbol{B}}{\boldsymbol{M}}\ x_2 + \frac{\boldsymbol{1}}{\boldsymbol{M}}\boldsymbol{f}$

- From the study system, the output is its displacement, $y(t) = x_1$

Develop the state space model of electrical networks and mechanical systems

❖ **State equations for mechanical systems *(cont.)* :**

✓ **Solution**

- Rearranging the above equations into the standard form of state space model, we get:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{\boldsymbol{K}}{\boldsymbol{M}} x_1 - \frac{\boldsymbol{B}}{\boldsymbol{M}} \; x_2 + \frac{\boldsymbol{1}}{\boldsymbol{M}} \boldsymbol{f}$$

$$y(t) = x_1$$

- Hence: $A = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{B}{M} \end{bmatrix}$; $B = \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}$; $C = [1\ 0]$; D=0

Develop the state space model using phase variables

- Phase variables are defined as those **particular state variables** which are obtained from one of the **system variables** and **its derivatives**.
- Usually the variables used is the **system output** and the remaining state variables are then **derivatives of the output.**
- The state model using phase variables can be easily determined if the system model is already known in the **differential equation or transfer function form**.

Develop the state space model using phase variables

A. Obtain state space model from differential equation:

❖ **Example 1 :** Construct a state model for a system characterized by the differential equation.

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y + u = 0$$

❖ **Solution**

- ✓ This differential equation relates the output *y(t)* and input *u(t)* of a system
- ✓ Let us choose *y* and its derivatives as state variables.
- ✓ The system is governed by third order differential equation, so the number of state variables required are three.

Develop the state space model using phase variables

A. Obtain state space model from differential equation:

❖ **Solution**

✓ Let the state variables $x_1, x_2\ and\ x_3$ be related to phase variables as follows:

$$x_1 = y$$

$$x_2 = \dot{y}$$

$$x_3 = \ddot{y}$$

Develop the state space model using phase variables

A. Obtain state space model from differential equation:

❖ **Solution**

✓ Put $y = x_1$, $\frac{dy}{dt} = x_2$ and $\frac{d^2y}{dt^2} = x_3$ in the given equation.

$$\dddot{y} + 6x_3 + 11x_2 + 6x_1 + u = 0$$

$$\Rightarrow \dddot{y} = \boldsymbol{-6x_1 - 11x_2 - 6x_3 - u}$$

✓ The state equation is:

$$\dot{x}_1 = \dot{y} = \boldsymbol{x_2}$$

$$\dot{x}_2 = \ddot{y} = \boldsymbol{x_3}$$

$$\dot{x}_3 = \dddot{y} = \boldsymbol{-6x_1 - 11x_2 - 6x_3\text{-}u}$$

Develop the state space model using phase variables

A. Obtain state space model from differential equation:

- **Solution**

  - ✓ Arranging the state equations in the matrix form,

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -6 & -11 & -6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ -1\end{bmatrix}[u]$$

  - ✓ From the formed phase variables, the output $y = x_1$

  - ✓ Hence, the output equation is, $y = [1\ 0\ 0]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$

Develop the state space model using phase variables

B. Obtain state space model from transfer function:

- **Example:** Obtain the state model of the system whose transfer function is given:

$$\frac{Y(s)}{U(s)} = \frac{24}{s^3 + 9s^2 + 26s + 24}$$

- **Solution:**
  - ✓ Cross-multiplying yields:

$$[s^3 + 9s^2 + 26s + 24]Y(s) = 24U(s)$$

$$s^3Y(s) + 9s^2Y(s) + 26sY(s) + 24Y(s) = 24U(s)$$

Develop the state space model using phase variables

B. Obtain state space model from transfer function:

- **Solution:**
  - ✓ Taking inverse Laplace transforms

$$\frac{d^3y(t)}{dt^3} + 9\frac{d^2y(t)}{dt^2} + 26\frac{dy(t)}{dt} + 24y(t) = 24u(t)$$

$$\dddot{y} + 9\ddot{y} + 26\dot{y} + 24y = 24u$$

  - ✓ Then, we proceed as for the previous case of differential equation.

Develop the state space model using phase variables

B. Obtain state space model from transfer function:

- **Solution:**
  - ✓ The system is governed by third order differential equation, so the number of state variables required are three.
  - ✓ Choosing the output $y(t)$ and its successive derivatives as state variables, we get:

$$x_1 = y$$

$$x_2 = \dot{y}$$

$$x_3 = \ddot{y}$$

Develop the state space model using phase variables

B. Obtain state space model from transfer function:

- **Solution:**

- The state equation is:

$$\dot{x}_1 = \dot{y} = x_2$$

$$\dot{x}_2 = \ddot{y} = x_3$$

$$\dot{x}_3 = \dddot{y} = -24x_1 - 26x_2 - 9x_3 + 24u$$

- The output equation is: $y = x_1$

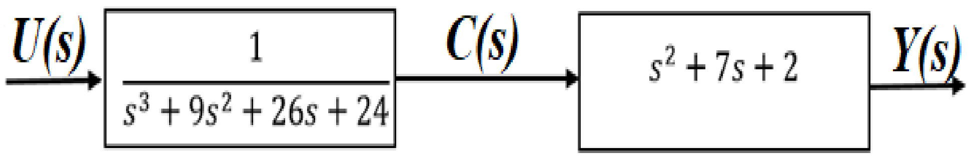
- Hence: $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix}$; $B = \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix}$; $C = [1\ 0\ 0]$; $D=0$

Develop the state space model using phase variables

B. Obtain state space model from transfer function:

- **Example 2:** Obtain the state model of the system whose transfer function is given by $\frac{Y(s)}{U(s)} = \frac{s^2+7s+2}{s^3+9s^2+26s+24}$

- **Solution**



$$\frac{C(s)}{U(s)} = \frac{1}{s^3+9s^2+26s+24}$$

$$\frac{Y(s)}{C(s)} = s^2 + 7s + 2$$

Develop the state space model using phase variables

B. Obtain state space model from transfer function:

❖ **Solution**

✓ Such that $\frac{C(s)}{U(s)} * \frac{Y(s)}{C(s)} = \frac{Y(s)}{U(s)}$

$$\frac{C(s)}{U(s)} = \frac{1}{s^3 + 9s^2 + 26s + 24}$$

✓ Cross-multiplying on both sides,

$$s^3C(s) + 9s^2C(s) + 26C(s) + 24C(s) = U(s)$$

✓ Taking inverse Laplace transform,

$$\dddot{c}(t) + 9\ddot{c}(t) + 26\dot{c}(t) + 24c(t) = u(t)$$

Develop the state space model using phase variables

B. Obtain state space model from transfer function:

❖ **Solution**

✓ Choosing the output c(t) and its successive derivatives as state variables,

$$x_1 = c$$

$$x_2 = \dot{c}$$

$$x_3 = \ddot{c}$$

✓ The state equation is:

$$\dot{x}_1 = \dot{c} = x_2$$

$$\dot{x}_2 = \ddot{c} = x_3$$

$$\dot{x}_3 = \dddot{c} = -24x_1 - 26x_2 - 9x_3 + u$$

Develop the state space model using phase variables

B. Obtain state space model from transfer function:

❖ **Solution**

✓ For output equation:

$$\frac{Y(s)}{C(s)} = s^2 + 7s + 2$$

$$Y(s) = s^2C(s) + 7sC(s) + 2C(s)$$

✓ Taking inverse Laplace transform,

$$y(t) = \ddot{c}(t) + 7\dot{c}(t) + 2c(t) = x_3 + 7x_2 + 2x_1$$

$$y(t) = 2x_1 + 7x_2 + x_3$$

$$\boldsymbol{y(t) = [2\ 7\ 1]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}$$

Develop the state space model using phase variables

B. Obtain state space model from transfer function:

- **Solution**
  - ✓ The corresponding state space model is:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + u$$
$$\boldsymbol{y(t)} = [\mathbf{2\ 7\ 1}]\begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \\ \boldsymbol{x_3} \end{bmatrix}$$

  - ✓ Hence: $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix}$; $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$; $C = [2\ 7\ 1]$; D=0

Convert state space model to transfer function

❖ Consider the standard form of a state space model below:

$\dot{x}(t) = Ax(t) + Bu(t)$…….. State equation

$y(t) = Cx(t) + Du(t)$…….. Output equation

❖ Let the initial condition be zero. Taking the Laplace transform on both sides of the above equations, we get:

$$[\boldsymbol{sI} - \boldsymbol{A}]\boldsymbol{X}(\boldsymbol{s}) = \boldsymbol{BU}(\boldsymbol{s})$$

❖ Pre-multiplying both sides by $[\boldsymbol{sI} - \boldsymbol{A}]^{-1}$, we get:

$$\boldsymbol{X}(\boldsymbol{s}) = [\boldsymbol{sI} - \boldsymbol{A}]^{-1}\boldsymbol{BU}(\boldsymbol{s})$$

Convert state space model to transfer function

- Similarly: $\boldsymbol{Y(s) = CX(s) + DU(s)}$
- Substituting $\boldsymbol{X(s)}$ by its expression, we get:

$$\boldsymbol{Y(s) = C[sI - A]^{-1}BU(s) + DU(s)}$$

$$\boldsymbol{Y(s) = \{C[sI - A]^{-1}B + D\}U(s)}$$

- The transfer function $\boldsymbol{\frac{Y(s)}{U(s)} = C[sI - A]^{-1}B + D}$
- **Example 1:** Consider the following state equations and the transfer function of the system

$$\dot{x} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \quad \text{and } y(t) = [0\ 1]x(t)$$

Convert state space model to transfer function

❖ **Solution:**

✓ The transfer function = $\frac{\boldsymbol{Y(s)}}{\boldsymbol{U(s)}} = \boldsymbol{C}[\boldsymbol{sI} - \boldsymbol{A}]^{-1}\boldsymbol{B}$ since D=0

$$[\boldsymbol{sI} - \boldsymbol{A}] = \boldsymbol{s}\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} - \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} (s+3) & -1 \\ 2 & s \end{bmatrix}$$

$$[\boldsymbol{sI} - \boldsymbol{A}]^{-1} = \frac{Adj(sI - A)}{\det(sI - A)} = \frac{1}{s(s+3)+2}\begin{bmatrix} s & 1 \\ -2 & (s+3) \end{bmatrix}$$

$$\frac{\boldsymbol{Y(s)}}{\boldsymbol{U(s)}} = \boldsymbol{C}[\boldsymbol{sI} - \boldsymbol{A}]^{-1}\boldsymbol{B} = [\mathbf{0} \ \ \mathbf{1}]\frac{\begin{bmatrix} s & 1 \\ -2 & (s+3) \end{bmatrix}}{s(s+3)+2}\begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix}$$

Convert state space model to transfer function

- **Solution:**

✓ $\frac{\boldsymbol{Y(s)}}{\boldsymbol{U(s)}} = \boldsymbol{C[sI - A]^{-1}B} = \frac{1}{s(s+3)+2}[\mathbf{0}\ \ \mathbf{1}]\begin{bmatrix} s & 1 \\ -2 & (s+3) \end{bmatrix}\begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix}$

$$= \frac{1}{s(s+3)+2}[\mathbf{-2}\ \ (\boldsymbol{s}+\mathbf{3})]\begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix} = -\frac{\mathbf{2}}{\boldsymbol{s}(\boldsymbol{s}+\mathbf{3})+\mathbf{2}}$$

Example 2: Consider the following state space model of RLC circuit to find the transfer function of the circuit.

$$\dot{x}(t) = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} x(t) + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u(t)$$

$$y(t) = [0\ 1]x(t)$$

Convert state space model to transfer function

❖ **Solution:**

✓ The transfer function = $\frac{\boldsymbol{Y(s)}}{\boldsymbol{U(s)}} = \boldsymbol{C}[\boldsymbol{sI} - \boldsymbol{A}]^{-\mathbf{1}}\boldsymbol{B}$ since D=0

$$[\boldsymbol{sI} - \boldsymbol{A}] = \begin{bmatrix} \left(s + \frac{R}{L}\right) & \frac{1}{L} \\ -\frac{1}{C} & s \end{bmatrix}$$

$$[\boldsymbol{sI} - \boldsymbol{A}]^{-\mathbf{1}} = \frac{Adj(sI - A)}{\det(sI - A)} = \frac{1}{s\left(s + \frac{R}{L}\right) + \frac{1}{LC}} \begin{bmatrix} s & -\frac{1}{L} \\ \frac{1}{C} & \left(s + \frac{R}{L}\right) \end{bmatrix}$$

$$\frac{\boldsymbol{Y(s)}}{\boldsymbol{U(s)}} = \boldsymbol{C}[\boldsymbol{sI} - \boldsymbol{A}]^{-\mathbf{1}}\boldsymbol{B} = \frac{1}{s\left(s + \frac{R}{L}\right) + \frac{1}{LC}} [\mathbf{0} \;\; \mathbf{1}] \begin{bmatrix} s & -\frac{1}{L} \\ \frac{1}{C} & \left(s + \frac{R}{L}\right) \end{bmatrix} \begin{bmatrix} \frac{\mathbf{1}}{\boldsymbol{L}} \\ \mathbf{0} \end{bmatrix}$$

Convert state space model to transfer function

❖ **Solution:**

$$\frac{\boldsymbol{Y(s)}}{\boldsymbol{U(s)}} = \frac{1}{s\left(s+\frac{R}{L}\right)+\frac{1}{LC}}\begin{bmatrix}\boldsymbol{\frac{1}{C}} & \left(\boldsymbol{s+\frac{R}{L}}\right)\end{bmatrix}\begin{bmatrix}\boldsymbol{\frac{1}{L}} \\ \boldsymbol{0}\end{bmatrix} = \frac{\frac{1}{LC}}{s\left(s+\frac{R}{L}\right)+\frac{1}{LC}}$$

$$\frac{\boldsymbol{Y(s)}}{\boldsymbol{U(s)}} = \frac{\frac{1}{LC}}{s\left(\frac{sL+R}{L}\right)+\frac{1}{LC}} = \frac{\frac{1}{LC}}{\frac{sC(sL+R)+1}{LC}}$$

✓ Finally : $\frac{\boldsymbol{Y(s)}}{\boldsymbol{U(s)}} = \frac{\boldsymbol{1}}{\boldsymbol{LCs^2+RCs+1}}$, which is the transfer function of R-L-C series circuit.

References

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THANK YOU