Course: Automatic Control System Technology

Lecture 4: Demonstrate the application of Laplace transform to solve Linear Differential Equations

Lecturer: UWASEKURU GISA Jean De Dieu

Demonstrate the application of Laplace transform to solve linear Differential Equations

Session objectives:

By the end of this session, students will be able to :

- ❖Describe a linear ordinary differential equation(ODE)
- ❖Recall Laplace transform properties used in solving linear ODEs
- ❖Describe procedures followed to solve linear differential equations
- ❖Solve some examples of Linear Ordinary differential equations

Describe a Linear ordinary differential equation

- A wide range of systems in engineering are represented mathematically by differential equations.
- Differential equations generally involve derivatives (or integrals) of the **dependent variables** with respect to the **independent variable** (usually time).
- This lecture deals with Linear ODEs only as we will be modeling LTI systems.

Describe a Linear ordinary differential equation

❖ In general, a linear differential equation of n^{th}- order is written as:

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = b_m\frac{d^m u(t)}{dt^m} +$$

$$b_{m-1}\frac{d^{m-1}u(t)}{dt^{m-1}} + \cdots + b_1\frac{du(t)}{dt} + b_0 u(t)$$

Where $n > m, a_{n-1}, . . , a_1 ,\ a_0\ and\ b_m , b_{m-1},.., b_1, b_0$ are real constants. Farid Golnaraghi & Benjamin C. Kuo (2019), Automatic Control Systems,10th Edition, McGraw-Hill Education, page 174-175

Describe a Linear ordinary differential equation

- In control system theory, the terms u(t) and y(t) , of the equation presented previously, are known as the input and the output of the system, respectively.
- Thus, the input - output relationship of a SISO LTI system whose input u(t) and output y(t) is described by n^{th} order linear ordinary differential equation as presented on the previous slide.

Describe a Linear ordinary differential equation

- In control system analysis, the output y(t) is obtained by solving the ODE describing the dynamic behavior of the system in time domain, but this is difficult and time consuming .
- To simplify these computations, Laplace transform is used to convert that ODE into its algebraic equivalent equation in the s-domain
- The output of the system is found in s-domain as Y(s) and y(t) is determined by applying inverse Laplace transform to Y(s).

Recall Laplace transform properties used in solving linear ODEs

❖ The Laplace Transform can greatly simplify the solution of linear ordinary differential equations

❖ For this, we will need two main properties of Laplace transform (Linearity and Differentiation), and the knowledge of inverse Laplace transform techniques

✓**Linearity:** $\mathcal{L}\{\alpha_1 f_1(t) + \alpha_2 f_2(t)\} = \alpha_1 F_1(s) + \alpha_2 F_2(s)$

Recall Laplace transform properties used in solving linear ODEs

✓ **Differentiation (cont.)**:

For first order derivative

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

For second order derivative

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

For third order derivative

$$\mathcal{L}\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

Recall Laplace transform properties used in solving linear ODEs

✓ **Differentiation (cont.):**

For fourth order derivative

$$\mathcal{L}\{f^{(4)}(t)\} = s^4F(s) - s^3f(0) - s^2f'(0) - sf''(0) - f'''(0)$$

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For $\boldsymbol{n^{th}}$ order derivative

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

Farid Golnaraghi & Benjamin C. Kuo (2010), Automatic Control Systems,9th Edition, John Wiley&Sons, page 54.

Describe procedures followed to solve linear differential equations

The procedures:

- ✓ The first step is to take the Laplace transform of both sides of the differential equation in t to convert it into an algebraic equation in s using Laplace transform table.
- ✓ Solve and find Y (s).

Describe procedures followed to solve linear differential equations

The procedures(cont.):

- ✓ Simplify the expression of Y (s) using the method of partial fractions.
- ✓ Recall the inverse Laplace transforms using the linearity property and Laplace transform table.

Farid Golnaraghi & Benjamin C. Kuo (2019), Automatic Control Systems,10th Edition, McGraw-Hill Education, page 199.

Solve some examples of Linear Ordinary differential equations:Examples

Example 1:

Given the following first order differential equation:

$$y' + y = 3e^{2t} \text{ , where } y(0) = 4$$

Find y(t) using Laplace transforms.

Solution:

✓ Take Laplace transform of both sides of the equation, by using Laplace transform properties and table:

$$sY(s) - y(0) + Y(s) = 3 * \frac{1}{s-2}$$

Solve some examples of Linear Ordinary differential equations: Examples

Example 1 (cont.):

✓ Substitute **y(0)=4** (initial condition); and solve to find Y(s):

$$sY(s) - 4 + Y(s) = 3 * \frac{1}{s-2}$$

$$(s+1)Y(s) = 3 * \frac{1}{s-2} + 4$$

$$(s+1)Y(s) = \frac{3 + 4(s-2)}{s-2}$$

Solve some examples of Linear Ordinary differential equations: Examples

Example 1 (cont.):

- ✓ Solve to find Y(s):

$$(s+1)Y(s) = \frac{3+4(s-2)}{s-2}$$

$$(s+1)Y(s) = \frac{3+4s-8}{s-2}$$

$$(s+1)Y(s) = \frac{4s-5}{s-2}$$

$$Y(s) = \frac{4s-5}{(s-2)(s+1)}$$

Solve some examples of Linear Ordinary differential equations: Examples

Example 1 (cont.):

- ✓ We can apply inverse Laplace transform on Y(s) to get y(t), but the inverse of Y(s) can not be found from Laplace transform table directly. Hence, we have to perform partial fraction to expand Y(s):
- ✓ Perform partial fractions to expand **Y(s)**:

$$\boldsymbol{Y(s) = \frac{4s-5}{(s-2)(s+1)}}$$, Y(s) has real and simple poles: 2 and -1

Solve some examples of Linear Ordinary differential equations: Examples

Example 1 (cont.):

✓ Hence, it can be expanded as follows:

$$Y(s) = \frac{4s-5}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$$

With:

$$A = \cancel{(s-2)}\frac{4s-5}{\cancel{(s-2)}(s+1)}\Bigg|_{s=2} = \frac{4s-5}{(s+1)}\Bigg|_{s=2} = \frac{4*2-5}{2+1} = \frac{3}{3}$$

$$A = 1$$

Solve some examples of Linear Ordinary differential equations: Examples

Example 1 (cont.):

✓ Hence, it can be expanded as follows:

$$\boldsymbol{Y(s) = \frac{4s-5}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}}$$

And,

$$\boldsymbol{B = \cancel{(s+1)}\frac{4s-5}{(s-2)\cancel{(s+1)}}\bigg|_{s=-1} = \frac{4s-5}{(s-2)}\bigg|_{s=-1}}$$

$$\boldsymbol{= \frac{4*(-1)-5}{-1-2} = \frac{-9}{-3} = 3}$$

$\boldsymbol{B} = 3$

Solve some examples of Linear Ordinary differential equations: Examples

Example 1 (cont.):

- ✓ So, the expanded form of Y(s) is:

$$\boldsymbol{Y(s) = \frac{1}{s-2} + \frac{3}{s+1}}$$

- ✓ Now, y(t):solution of the given differential equation; can be obtained from Y(s) by applying inverse Laplace transform, linearity property and use of Laplace transform table:

- ✓ $\boldsymbol{y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{\frac{1}{s-2} + \frac{3}{s+1}\}}$

Solve some examples of Linear Ordinary differential equations: Examples

Example 1 (cont.):

- ✓ $\boldsymbol{y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{\frac{1}{s-2}\} + \mathcal{L}^{-1}\{\frac{3}{s+1}\}}$
- ✓ $\boldsymbol{y(t) = e^{2t} + 3e^{-t}}$

Solve some examples of Linear Ordinary differential equations:Examples

Example 2:

Given the following second order differential equation:

$$y'' + y' = 5\cos(2t)\text{ , where } y(0) = 0;\ y'(0)=0$$

Find y(t) using Laplace transforms.

Solution:

✓ Take Laplace transform of both sides of the equation, by using Laplace transform properties and table:

$$s^2Y(s) - sy(0) - y'(0) + sY(s) - y(0) = 5\frac{s}{s^2+4}$$

Solve some examples of Linear Ordinary differential equations:Examples

Example 2:

✓ Substitute $\boldsymbol{y}(\mathbf{0}) = \mathbf{0}$; $\boldsymbol{y}'$**(0)=0:** initial conditions; and solve to find Y(s):

$$\boldsymbol{s^2Y(s) - 0 - 0 + sY(s) - 0 = 5\frac{s}{s^2+4}}$$

$$\boldsymbol{s^2Y(s) + sY(s) = 5\frac{s}{s^2+4}}$$

$$\boldsymbol{(s^2+s)Y(s) = 5\frac{s}{s^2+4}}$$

Solve some examples of Linear Ordinary differential equations: Examples

Example 2(cont.):

- ✓ Solve to find Y(s):

$$Y(s) = 5\frac{s}{(s^2+s)(s^2+4)}$$

$$Y(s) = \frac{5\cancel{s}}{\cancel{s}(s+1)(s^2+4)}$$

$$Y(s) = \frac{5}{(s+1)(s^2+4)}$$

Solve some examples of Linear Ordinary differential equations:Examples

Example 2 (cont.):

- ✓ Perform partial fractions to expand **Y(s)**:

$$\boldsymbol{Y(s) = \frac{5}{(s+1)(s^2+4)}},$$ Y(s) has simple pole: **-1** and pure imaginary conjugate poles: **-j2** and **+j2**

- ✓ Hence, it can be expanded as follows:

$$\boldsymbol{Y(s) = \frac{5}{(s+1)(s^2+4)} = \frac{A_0}{s+1} + \frac{A_1 s + A_2}{s^2+4}}$$

Solve some examples of Linear Ordinary differential equations:Examples

Example 2 (cont.):

✓ Putting on common denominator:

$$Y(s) = \frac{5}{(s+1)(s^2+4)} = \frac{A_0(s^2+4)+(A_1 s+A_2)(s+1)}{(s+1)(s^2+4)}$$

$$Y(s) = \frac{5}{(s+1)(s^2+4)} = \frac{A_0 s^2+4A_0+A_1 s^2+A_1 s+A_2 s+A_2}{(s+1)(s^2+4)}$$

$$Y(s) = \frac{5}{(s+1)(s^2+4)} = \frac{(A_0+A_1)s^2+(A_1+A_2)s+4A_0+A_2}{(s+1)(s^2+4)}$$

Solve some examples of Linear Ordinary differential equations:Examples

Example 2 (cont.):

- ✓ Thus: $\mathbf{5} = (\boldsymbol{A_0} + \boldsymbol{A_1})\boldsymbol{s^2} + (\boldsymbol{A_1} + \boldsymbol{A_2})\boldsymbol{s} + \mathbf{4}\boldsymbol{A_0} + \boldsymbol{A_2}$
- ✓ The unknowns $A_0,\ A_1 and\ A_2$ are found by equating coefficients of powers of s in the above equation:

$$\Rightarrow \begin{cases} A_0 + A_1 = 0 \quad eq1 \\ A_1 + A_2 = 0 \quad eq2 \\ 4A_0 + A_2 = 5 \quad eq3 \end{cases}$$

Solve some examples of Linear Ordinary differential equations:Examples

Example 2 (cont.):

✓From $eq1$: $A_1 = -A_0$ and substituting into eq2, we have:

$$\begin{cases} -A_0 + A_2 = 0 \quad (-1) \\ 4A_0 + A_2 = 5 \quad (1) \end{cases}$$

$5A_0 = 5$; A_0=1

A_2=A_0; A_2=1

$A_1 = -A_0$; A_1=-1

✓ So, the expanded form of Y(s) is : $\boldsymbol{Y(s)} = \frac{\mathbf{1}}{\boldsymbol{s}+\mathbf{1}} + \frac{-\boldsymbol{s}+\mathbf{1}}{\boldsymbol{s}^{\mathbf{2}}+\mathbf{4}}$

Solve some examples of Linear Ordinary differential equations:Examples

Example 2 (cont.):

- ✓ Which is the same as: $\boldsymbol{Y(s)} = \frac{\mathbf{1}}{\boldsymbol{s+1}} - \frac{\boldsymbol{s}}{\boldsymbol{s^2+4}} + \frac{\mathbf{1}}{\boldsymbol{s^2+4}}$
- ✓ Now, apply inverse Laplace transform, linearity property and use of Laplace transform table to find y(t) which is the solution of the given differential equation:
- ✓ $\boldsymbol{y(t)} = \mathcal{L}^{-1}\{\boldsymbol{Y(s)}\} = \mathcal{L}^{-1}\{\frac{\mathbf{1}}{\boldsymbol{s+1}} - \frac{\boldsymbol{s}}{\boldsymbol{s^2+4}} + \frac{\mathbf{1}}{\boldsymbol{s^2+4}}\}$

Solve some examples of Linear Ordinary differential equations:Examples

Example 2 (cont.):

- ✓ $y(t) = \mathcal{L}^{-1}\{\frac{1}{s+1}\} - \mathcal{L}^{-1}\{\frac{s}{s^2+4}\} + \mathcal{L}^{-1}\{\frac{1}{s^2+4}\}$
- ✓ $y(t) = \mathcal{L}^{-1}\{\frac{1}{s+1}\} - \mathcal{L}^{-1}\{\frac{s}{s^2+4}\} + \frac{1}{2}\mathcal{L}^{-1}\{\frac{2}{s^2+4}\}$
- ✓ $y(t) = e^{-t} - cos(2t) + \frac{1}{2}sin(2t)$

Solve some examples of Linear Ordinary differential equations:Examples

Example 3 : Consider a system represented by the differential equation $y''(t) + 4y'$(t)+3y(t)=2 r(t) ; where the initial conditions are y(0)=1, y'(0)=0 , and r(t)=1, t $\geq$ 0

Find y(t) using Laplace transforms.

Richard C.Dorf & Robert H.Bishop(2022),Modern Control Systems, 14th Edition, Pearson Education, page 98.

Solve some examples of Linear Ordinary differential equations:Examples

Example 3 (cont.): Solution:

- ✓ Take Laplace transform of both sides of the equation, by using Laplace transform properties and table:

$$s^2Y(s) - sy(0) - y'(0) + 4[sY(s) - y\,(0)] + 3Y(s) = \frac{2}{s}$$

- ✓ Substitute **y(0)=1, y'(0)=0**: initial conditions; and solve to find Y(s):

- ✓ $s^2Y(s) - s + 4[sY(s) - 1] + 3Y(s) = \frac{2}{s}$

Solve some examples of Linear Ordinary differential equations:Examples

Example 3 (cont.): Solution:

- ✓ Solve to find Y(s):
- ✓ $s^2Y(s) - s + 4sY(s) - 4 + 3Y(s) = \frac{2}{s}$
- ✓ $(s^2+4s+3)Y(s) = \frac{2}{s} + s + 4 = \frac{s^2+4s+2}{s}$
- ✓ $Y(s) = \frac{s^2+4s+2}{s(s^2+4s+3)} = \frac{s^2+4s+2}{s(s+1)(s+3)}$

Solve some examples of Linear Ordinary differential equations:Examples

Example 3 (cont.): Solution:

✓ Perform partial fractions to expand **Y(s)**:

$$Y(s) = \frac{s^2+4s+2}{s(s+1)(s+3)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3}$$

$$\textbf{A} = \cancel{s} * \frac{s^2+4s+2}{\cancel{s}(s+1)(s+3)}\Bigg|_{s=0} = \frac{s^2+4s+2}{(s+1)(s+3)}\Bigg|_{s=0} = \frac{0+0+2}{(0+1)(0+3)} = \frac{2}{3}$$

$$\textbf{B} = \cancel{(s+1)} * \frac{s^2+4s+2}{s\cancel{(s+1)}(s+3)}\Bigg|_{s=-1} = \frac{s^2+4s+2}{s(s+3)}\Bigg|_{s=-1} = \frac{1-4+2}{(-1)(-1+3)} = \frac{1}{2}$$

Solve some examples of Linear Ordinary differential equations:Examples

Example 3 (cont.): Solution:

- ✓ Perform partial fractions to expand **Y(s)**:

C= $(s+3) * \frac{s^2+4s+2}{s(s+1)\cancel{(s+3)}}\bigg|_{s=-3} = \frac{s^2+4s+2}{s(s+1)}\bigg|_{s=-3} = \frac{9-12+2}{(-3)(-3+1)} = \frac{-1}{6}$

- ✓ So, the expanded form of Y(s) is :

$$Y(s) = \frac{2/3}{s} + \frac{1/2}{s+1} + \frac{-1/6}{s+3} = \frac{2}{3} * \frac{1}{s} + \frac{1}{2} * \frac{1}{s+1} - \frac{1}{6} * \frac{1}{(s+3)}$$

Solve some examples of Linear Ordinary differential equations:Examples

Example 3 (cont.): Solution:

- ✓ Now, apply inverse Laplace transform, linearity property and use of Laplace transform table to find y(t) which is the solution of the given differential equation

$$\mathbf{y(t)} = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left\{\frac{2}{3} * \frac{1}{s} + \frac{1}{2} * \frac{1}{s+1} - \frac{1}{6} * \frac{1}{(s+3)}\right\}$$

$$\mathbf{y(t)} = \frac{2}{3} + \frac{1}{2}e^{-t} - \frac{1}{6}e^{-3t}, t \geq 0$$

References

1. Richard C.Dorf & Robert H.Bishop(2022),Modern Control Systems, 14th Edition, Pearson Education.
2. Farid Golnaraghi & Benjamin C. Kuo (2019), Automatic Control Systems,10th Edition, McGraw-Hill Education,.
3. Farid Golnaraghi & Benjamin C. Kuo (2010), Automatic Control Systems,9th Edition, John Wiley&Sons.

THANK YOU