Course: Regulation and control

Lecture 2: Modelling physical systems II

Lecturer: Chalachew Werku

Revision

On the last lecture we have

- Defined regulation and control
- Studied why we build control system
- Explained the key components of every control loop
- Differentiate between the two main control loops
- Mathematically model simple electrical and mechanical systems and combined them to model an electro-mechanical device.

Example

Consider a DC motor with the following parameters:

$$L_a = 0.005\ H, R_a = 2\Omega\ , k_b = k_t = 0.5\ V \cdot s/rad$$

$$J = 0.1\ kg \cdot m^2, B = 0.2\ Nm \cdot \frac{s}{rad}$$

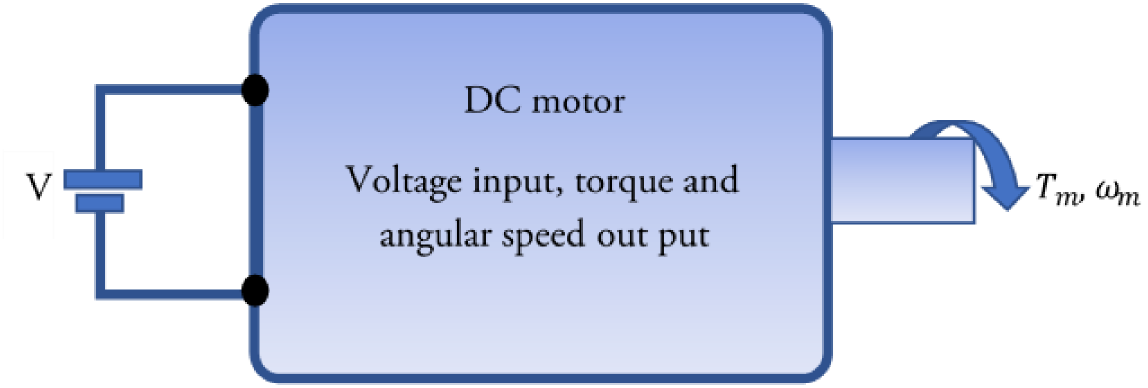


Fig 1: DC motor layout [1]

Example cont....

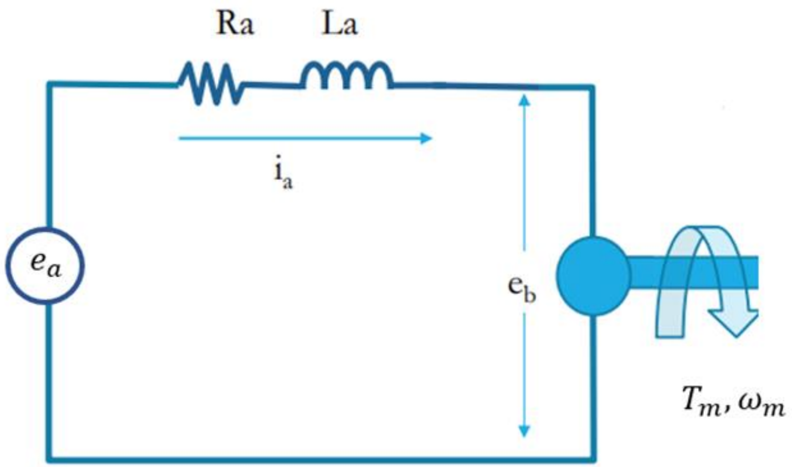


Fig 2: DC motor schematics [2]

- First, the electrical equation. Starting with KVL:

$$V_a = i_a R_a + L_a \frac{di_a}{dt} + e_b$$

- Substituting the given values and

$$e_b = k_b \omega$$

$$V_a = 2i_a + 0.005\frac{di_a}{dt} + 0.5\omega$$

Example cont….

- Now for the mechanical part

$$T_m = J\frac{d\omega}{dt} + B\omega$$

$$T_m = \ K_t i_a$$

- Equating both equations

➔ $T_m = J\frac{d\omega}{dt} + B\omega = K_t i_a$

- Substituting the given values

$$0.5i_a = 0.1\frac{d\omega}{dt} + 0.2\omega$$

We have got two equations

$$V_a = 2i_a + 0.005\frac{di_a}{dt} + 0.5\omega$$

$$0.5i_a = 0.1\frac{d\omega}{dt} + 0.2\omega$$

Example cont....

- what is the motor's steady-state speed if we apply a constant voltage, say 12 Volts?

$$V_a = 2i_a + 0.005\frac{di_a}{dt} + 0.5\omega$$

$$0.5i_a = 0.1\frac{d\omega}{dt} + 0.2\omega$$

at steady-state, all derivatives are zero.

So, $\frac{di_a}{dt} = 0 \; and \frac{d_\omega}{dt} = 0$

This gives us

$$V_a = 2i_a + 0.5\omega$$

$$0.5i_a = 0.2\omega$$

$$\omega = \omega_{ss} = 9.23\frac{rad}{sec}$$

Objectives of this lecture

- Model ideal gear systems and analyze their kinematic and kinetic relationships to reflect load inertia and damping to the motor shaft.
- Interpret torque-speed characteristics of a DC motor to determine key motor constants (k_t, k_b).
- Apply electro-mechanical analogies (Force-Voltage and Force-Current) to translate mechanical systems into equivalent electrical circuits for unified analysis and simulation.

Gear systems

- Gears are transformers of mechanical power.
- They allow us to tradeoff between torque and speed in a predictable, quantifiable way.
- Why we need to model gears?
  - To reflect load properties into motor shaft.
  - To predict system performance using simulation.

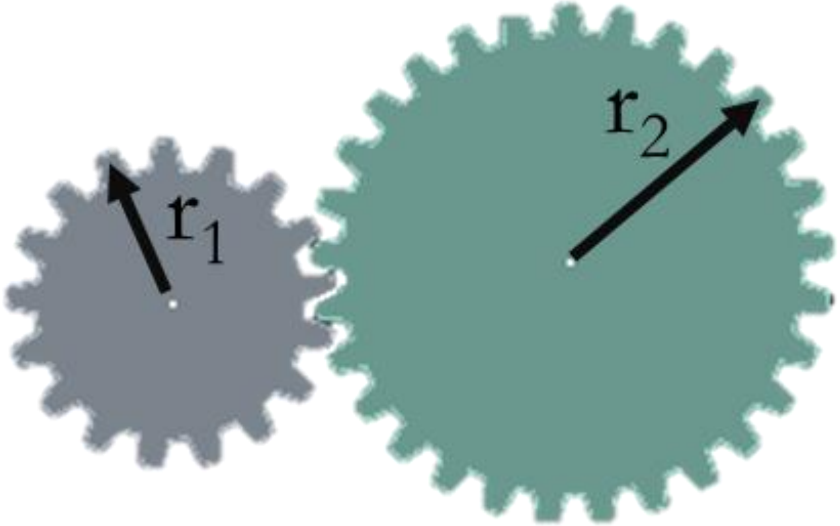


Fig 3: Gear train [3]

The Two Fundamental Laws of Gears

For an ideal gearbox:

- Speed/ kinematic Relationship $v = \omega_1 r_1 = \omega_2 r_2$

$$\frac{\omega_{motor}}{\omega_{load}} = N \quad \textit{(The motor is N times faster than the load)}$$

- Torque/ kinetic Relationship: $P_1 = T_1 \omega_1 = P_2 = T_2 \omega_2$

$$\frac{T_{load}}{T_{motor}} = N \quad \textit{(The load torque is N times greater than the motor torque)}$$

- Where the Gear Ratio N is defined as:

$$N = n_{load}/n_{motor}$$

NB: where subscript "1" is motor and "2" is load.

Load reflected to the motor side

We have two gears, a small one connected to our motor shaft. We'll call this Gear 1. It has a number of teeth "n_1". Meshed with it is a larger Gear 2, connected to the load shaft. It has "n_2" teeth.

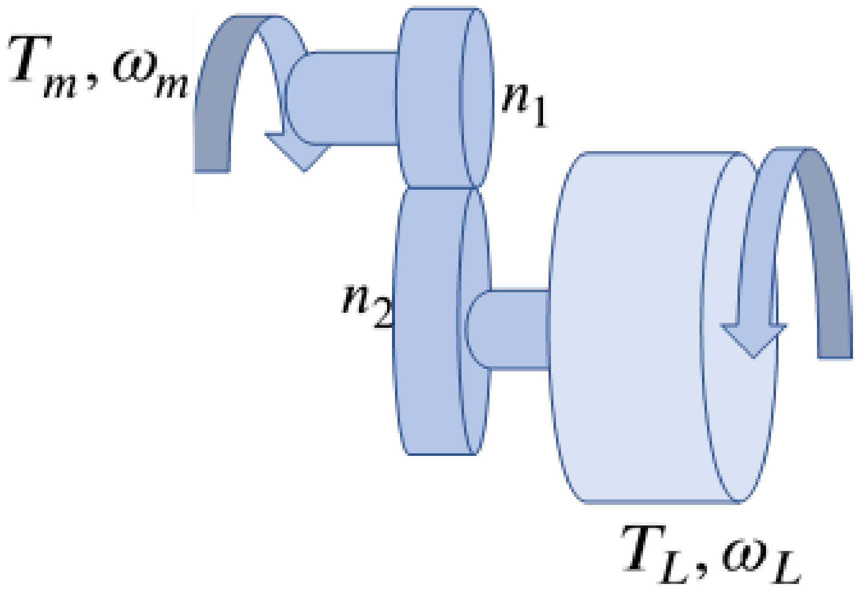


Fig 4: Layout of motor and load connected via gear [4]

Load reflected to the motor side cont....

- From newton 2nd law for rotational motion we have

$$\sum T = J\alpha$$

Torque produced by motor "T_m" used to overcome motor friction "B_m", to accelerate the inertial mass (rotor) of the motor "J_m" and to drive the load through gear 2

$$T_m - B_m\omega_m - T_1 = J_m\alpha_m$$

$$T_m = J_m\alpha_m + B_m\,\omega_m + T_1$$

Where

T_m- motor torque

T_1-torque on gear 1

$$T_1 = T_L\frac{n_1}{n_2}$$

Load reflected to the motor side cont....

$$T_m = J_m\alpha_m + B_m\,\omega_m + T_L\frac{n_1}{n_2}$$

But amount of torque required on the load side

$$T_L = J_L\alpha_L + B_L\omega_L$$

$$T_m = J_m\alpha_m + B_m\,\omega_m + \frac{n_1}{n_2}(J_L\alpha_L + B_L\omega_L)$$

Load motions also need to reflected to the motor side

$$\frac{\omega_L}{\omega_m} = \frac{\alpha_L}{\alpha_m} = \frac{n_1}{n_2}$$

Load reflected to the motor side cont….

$$T_m = J_m\alpha_m + B_m\,\omega_m + \frac{n_1}{n_2}(J_L\frac{n_1}{n_2}\alpha_m + B_L\frac{n_1}{n_2}\omega_m)$$

$but\ N = \dfrac{n_2}{n_1}$

$$\Rightarrow T_m = (J_m + \frac{J_L}{N^2})\,\alpha_m + (B_m\omega_m + \frac{B_L}{N^2})\omega_m$$

➔ $T_m = J_{total}\alpha_m + B_{total}\omega_m$

Load reflected to the motor side summary

- Therefore, the Total Effective Inertia the motor actually has to accelerate is:

$$J_{Total} = J_m + J_L / N^2$$

- And the Total Effective Damping is:

$$B_{Total} = B_m + B_L / N^2$$

A gear ratio of 10:1 means the motor feels only one hundredth of the load's inertia.

Predicting system performance

- Our motor's equation now becomes:

$$T_m = J_{Total}\frac{d\omega}{dt} + B_{Total}\omega$$

- This equation, combined with our electrical equation, allows us to simulate the entire system's response to any input.

Predicting system performance cont....

By simulating the system now we can answer critical questions like

- *Transient Response:* How long will it take for the load to reach its desired speed? Will it vibrate or oscillate?
- *Steady-State Performance:* What is the final operating speed and torque at the load shaft under different conditions?
- *Power Requirements:* Will the motor overheat under this new reflected load, or is it operating within a safe and efficient range?

Example derive mathematical equation

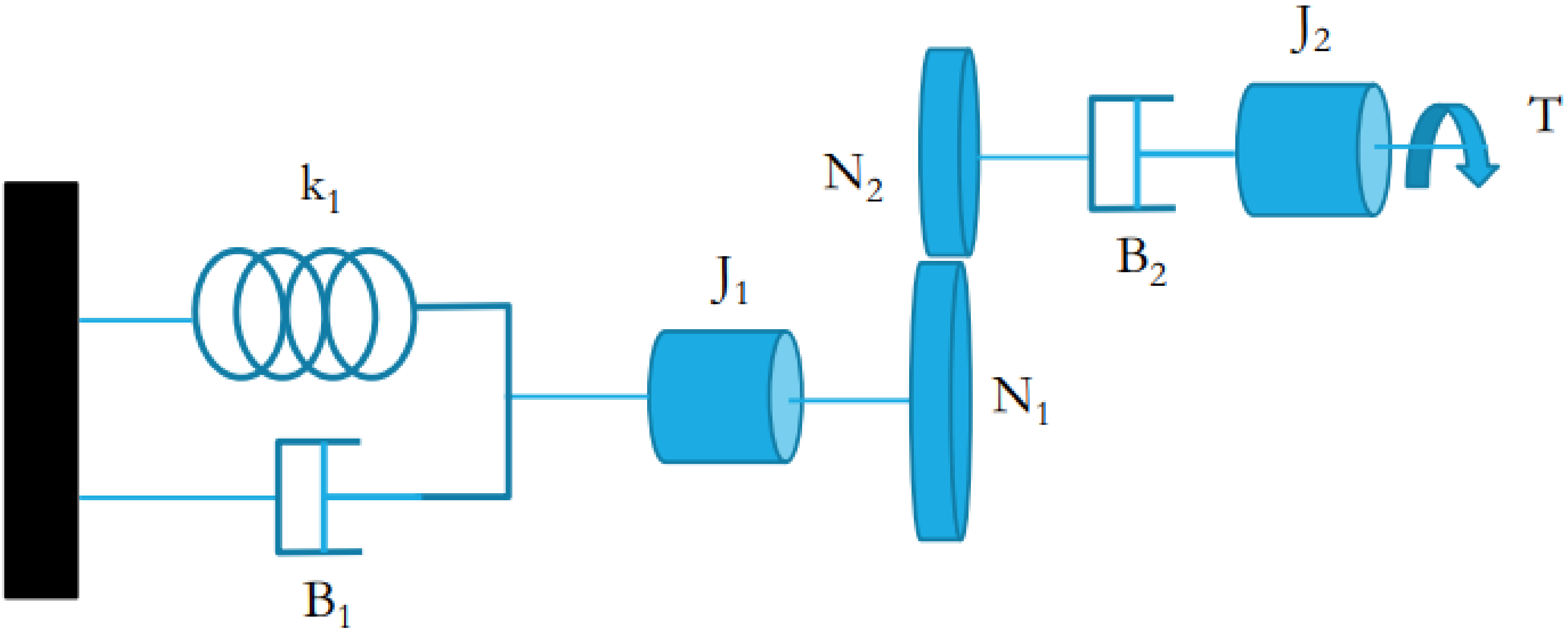
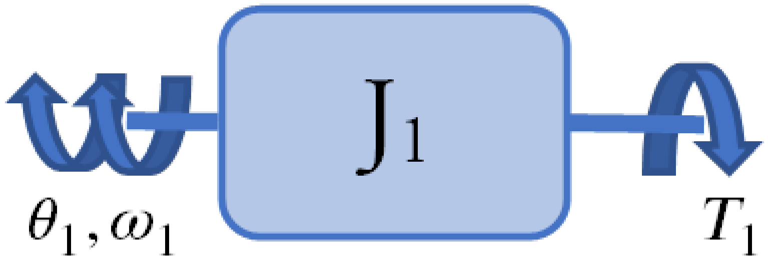


Fig 5: Rotational mechanical system [5]

Example derive mathematical equation cont….

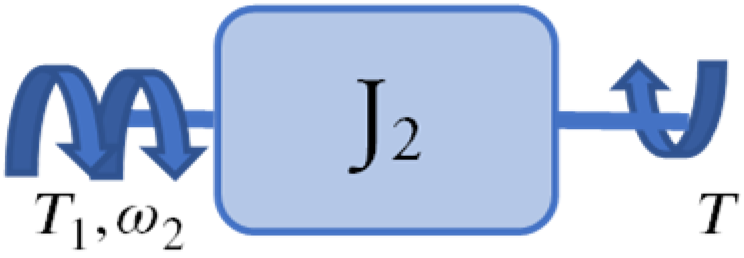
For mass 1



$$T_1 - k_1\theta_1 - B_1\omega_1 = J_1\alpha_1$$

$$T_1 = k_1\theta_1 + B_1\omega_1 + J_1\alpha_1$$

For mass 2



$$T - T_2 - B_2\omega_2 = J_2\alpha_2$$

$$T = T_2 + B_2\omega_2 + J_2\alpha_2$$

Example derive mathematical equation cont….

- But from kinetic and kinematic relationship we have

$$\frac{T_1}{T_2} = \frac{N_1}{N_2} = \frac{\alpha_2}{\alpha_1} = \frac{\omega_2}{\omega_1} = \frac{\theta_2}{\theta_1}$$

- This gives us

$$T_2 = \frac{N_2}{N_1} T_1$$

$$T = J_2\alpha_2 + B_2\omega_2 + \frac{N_2}{N_1} T_1$$

- Substituting the value of T_1 into the equation

$$T = J_2\alpha_2 + B_2\omega_2 + \frac{N_2}{N_1}(J_1\alpha_1 + B_1\omega_1 + k_1\theta_1)$$

Example derive mathematical equation cont....

Referring loads into the motor side

$$\alpha_1 = \alpha_2 \frac{N_2}{N_1} \ldots \omega_1 = \omega_2 \frac{N_2}{N_1} \ldots \theta_1 = \theta_2 \frac{N_2}{N_1} \ldots$$

$$T = J_2\alpha_2 + B_2\omega_2 + \frac{N_2}{N_1}\left(J_1\alpha_2\frac{N_2}{N_1} + B_1\omega_2\frac{N_2}{N_1} + k_1\theta_2\frac{N_2}{N_1}\right)$$

$$T = \left(J_2 + J_1\left(\frac{N_2}{N_1}\right)^2\right)\alpha_2 + \left(B_2 + B_1\left(\frac{N_2}{N_1}\right)^2\right)\omega_2 + \left(\frac{N_2}{N_1}\right)^2 \mathrm{k}_1\theta_2$$

Example derive mathematical equation of lever

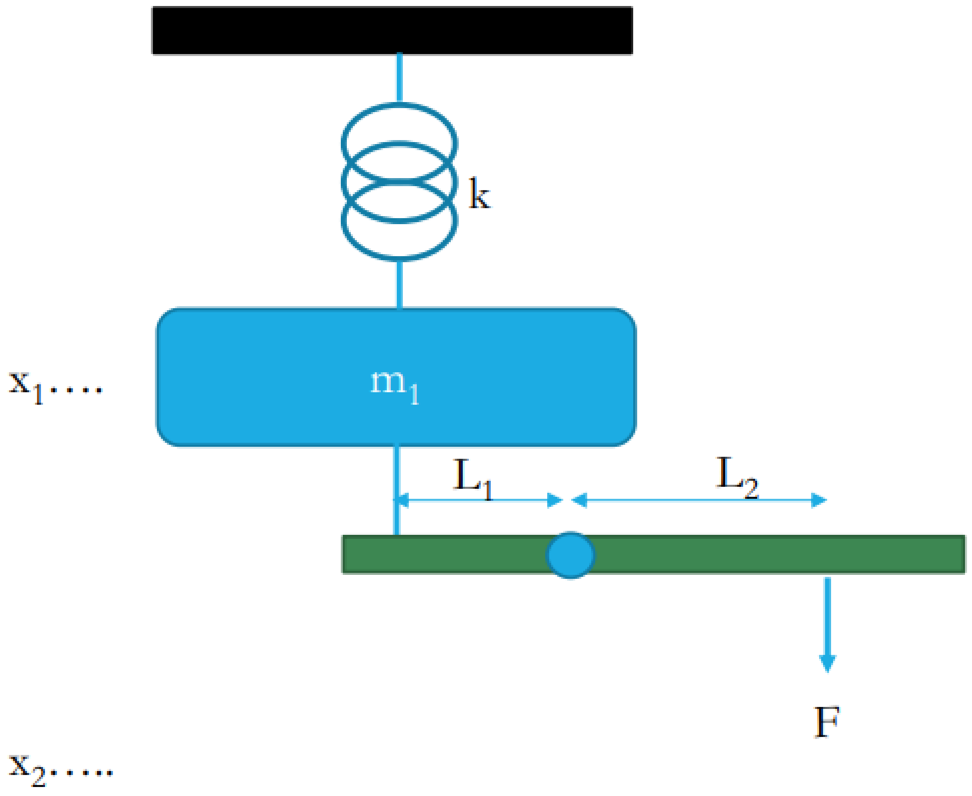
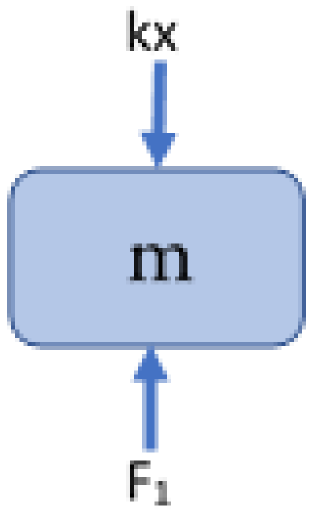


Fig 6: Mathematical modelling of a lever system [6]

Example derive mathematical equation of lever

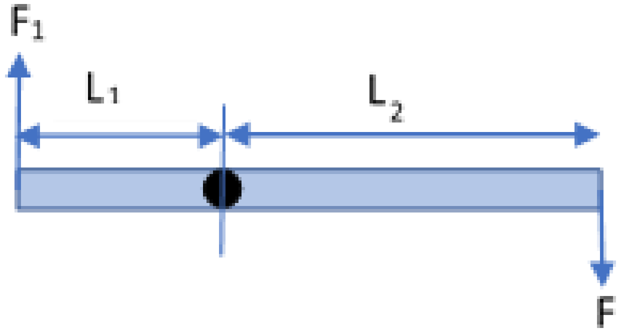
Free body diagrams



From newtons second law

$$F_1 - kx_1 = ma_1$$

$$F_1 = ma_1 + kx_1$$



from mechanical advantage we know that

$$F_1 x L_1 = F x L_2$$

$$F_1 = -\frac{L_2}{L_1} F$$

Re arranging the equations gives us

$$F = -\frac{L_1}{L_2}(ma_1 + kx_1)$$

$$\frac{L_1}{L_2} = \frac{a_1}{a} \Rightarrow a_1 = a\frac{L_1}{L_2}$$

$$F = \left(\frac{L_1}{L_2}\right)^2 (ma + kx)$$

Torque-speed characteristics

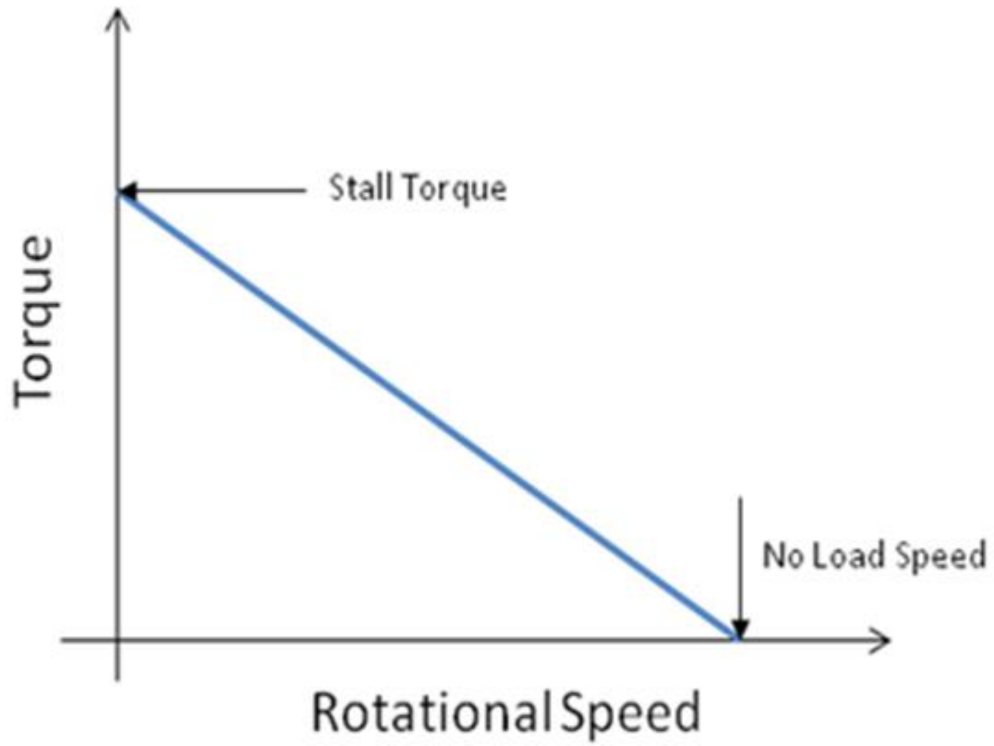


Fig 7: Torque-speed characteristics of a DC motor. [7]

- The graph shows the relationship between torque and angular speed on DC motors.
- We can use this graph to determine motor constants (k_b and k_t)

Torque-speed characteristics

The two golden points

- We can use motor ***no load speed*** and ***stall torque*** to reproduce Torque-speed graph and determine k_b and k_t from DC motor equations.
- **No-load speed (ω_{nl}):** This is the speed the motor spins at when it's doing no work, when the load torque is zero.
- **Stall torque (T_{stall}):** This is the torque the motor produces when its shaft is prevented from rotating.

Torque-speed characteristics

The Mathematical Model & The Link to Constants

- We have

For electrical part : $$v = R_a I_a + L_a \frac{di_a}{dt} + e_b$$

For mechanical part: $$T = k_t I_a = J\frac{d\omega}{dt} + B\omega$$

- At steady state (i.e. when there is no change with respect to time) and substituting e_b with $k_b\omega$ gives us

For electrical part : $$v = R_a I_a + k_b\omega$$

For mechanical part: $$T = k_t I_a$$

Torque-speed characteristics cont….

The Mathematical Model & The Link to Constants

- *Solving for Torque*

$$T = -\frac{k_b k_t}{R_a}\omega + v\frac{k_t}{R_a}$$

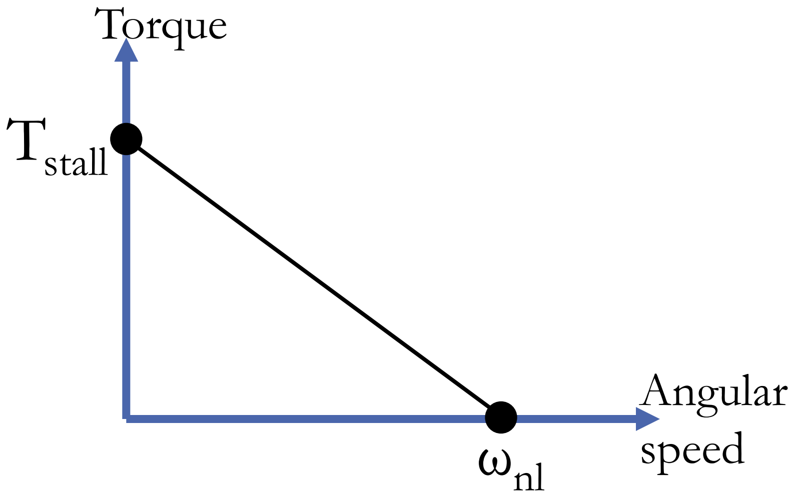


Fig 8: The relationship between T_{stall} and ω_{nl} [8]

- *This is equation of line with negative slope*

$$T = -m\omega + b$$

- *When speed =0, T=b=* $v\frac{k_t}{R_a}$*=*T_{stall}
- *When torque =0,* $\omega = \frac{v}{k_b} = \omega_{nl}$

➔ $k_t = \frac{T_{stall} R_a}{v}$ *and* $k_b = \frac{v}{\omega_{nl}}$

Analogous

- Analogous means comparable in certain respects, typically in a way that makes similarities clear. It describes a relationship of similarity between two things that are otherwise different.
- Example: "The wings of a bird are analogous to the wings of an airplane." (Both are used for flight, but their structure and origin are completely different).

Analogous cont....

- Analogous in Engineering & Control Systems: a specific, formal correspondence between two different physical systems where they are governed by the same mathematical equations.

In short, analogous means different in physics, but identical in mathematics.

Analogous circuits

- An analogous circuit is an **electrical circuit** specifically constructed to model the **mathematical behavior** of a **non-electrical** system (e.g., mechanical, hydraulic, thermal, acoustic).
- The voltages and currents in the electrical circuit directly analogous to the forces and flows in the original system.
- The primary goal is to allow the analysis of a physical system using well-established electrical circuit theory and simulation tools.

Mechanical-Electrical Analogy

- **If force in a mechanical is analogous to voltage in an electrical.**
  - Force- a voltage (cause of flow or change.)
  - Velocity – a current (both are flows).
  - Mass - an Inductor (both resist changes in flow: velocity/current).
  - Spring -a Capacitor (both store energy).
  - Damper -a Resistor (both dissipate energy).

Why we need electrical analogous circuits

- **Unified Analysis:** We can use a single, powerful set of tools (e.g., Kirchhoff's laws, transfer functions, circuit simulation software like SPICE) to analyze systems from many different physical domains.
- **Simplification of Complex Systems:** Electrical analogs create a single, unified model for electro-mechanical systems, allowing seamless analysis of interactions between physical parts and control circuits using one consistent set of laws.

Why we need electrical analogous circuits cont....

- **Simulation Efficiency:** It is vastly cheaper, faster, and safer to simulate a circuit on a computer than to build and test a complex mechanical prototype. Changing a resistor value in a simulation is instant; machining a new part is not.
- **Design Insight**: The analogy provides an intuitive way for electrical engineers to understand mechanical dynamics (like resonance and damping) by relating them to the familiar behavior of RLC circuits.

Types of analogies

- **Force-Voltage (F-V) Analogy (Impedance Analogy):**This is the most widely used analogy. It is called the "Impedance Analogy" because mechanical impedance, defined as Z_m = Force / Velocity, is analogous to electrical impedance, Z_e = Voltage / Current.
- **Force-Current (F-I) Analogy (Mobility Analogy):** This analogy is often considered more intuitive for certain systems because it preserves the physical layout, or topology.

Types of analogies cont….

Force-voltage (Series) analogy

- We have derived mathematical equation of simple mass-damper-spring system as

$$F = m\frac{dv}{dt} + Bv + k\int v\,dt$$

- For series **Resistor-Inductor-Capacitor** circuit with voltage source

$$v = L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt$$

- If we compare these equations **F-v, m-L, B-R** and **k-1/C**

Types of analogies cont….

Force-current (parallel) analogy

- We have derived mathematical equation of simple mass-damper-spring system as

$$F = m\frac{dv}{dt} + Bv + k\int v\,dt$$

- For parallel **Resistor-Inductor-Capacitor** circuit with current source

$$i = c\frac{dv}{dt} + \frac{1}{R}v + \frac{1}{L}\int v\,dt$$

- If we compare these equations **F-i, m-C, B-1/R** and **k-1/L**

Table 1: Analogies summary

Mechanical		Electrical	
Translational	**Rotational**	**F-V**	**F-I**
Force F	Torque T	Voltage V	Current I
Mass M	Inertia J	Inductance L	Capacitance C
Damper B	Damper B	Resistance R	Conductance 1/R
Lever L_1/L_2	Gear N_1/N_2	Transformer N_1/N_2	Transformer N_1/N_2
Spring k	Spring k	Elastance 1/C	Reluctance 1/L
Velocity v	Angular velocity ω	Current I	Voltage V
Position x	Angular position θ	Charge Q	Flux linkage ψ

Derive F-V and F-I analogous equation and draw electrical circuit

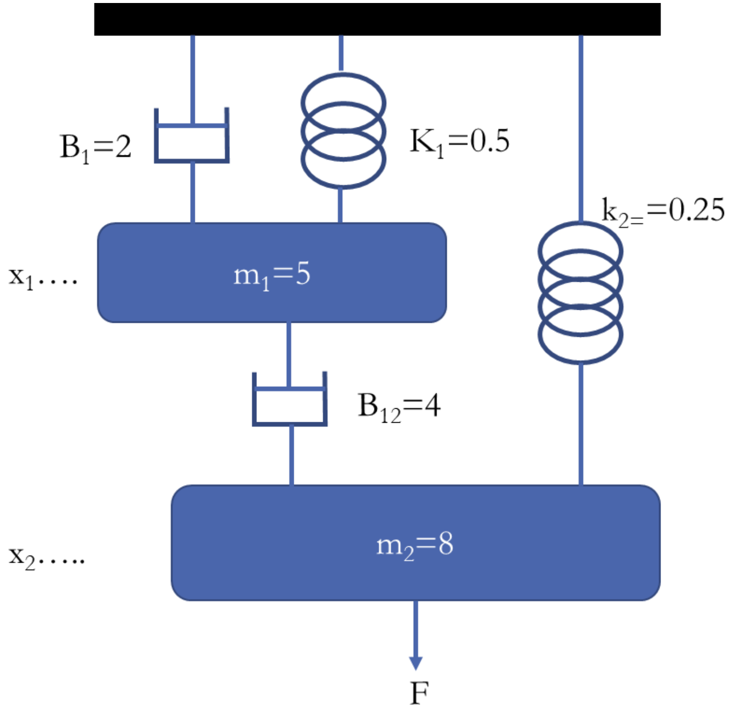


Fig 9: Two mass-spring-damper system [9]

The mathematical equation becomes

For mass 1

$$-B_1 v_1 - k_1 x_1 - B_{12} v_{12} = m_1 a_1$$

For mass 2

$$\mathrm{F} = m_2 a_2 + B_{12} v_{12} + k_2 x_2$$

F-V analogy

Electrical counter part equations

$$L_1 \frac{di_1}{dt} + R_1 i_1 + \frac{1}{C_1}\int i_1 dt + R_{12} i_{12} = 0$$

And

$$v = L_2 \frac{di_2}{dt} + R_{12} i_{12} + \frac{1}{C_2}\int i_2 dt$$

- We have two currents so we should have two loops the components are placed according to their number, if the component has two digits it should be placed between the respective loops

F-V analogy circuit

Electrical counter part equations

$$L_1 \frac{di_1}{dt} + R_1 i_1 + \frac{1}{C_1}\int i_1 dt + R_{12} i_{12} = 0$$

And

$$v = L_2 \frac{di_2}{dt} + R_{12} i_{12} + \frac{1}{C_2}\int i_2 dt$$

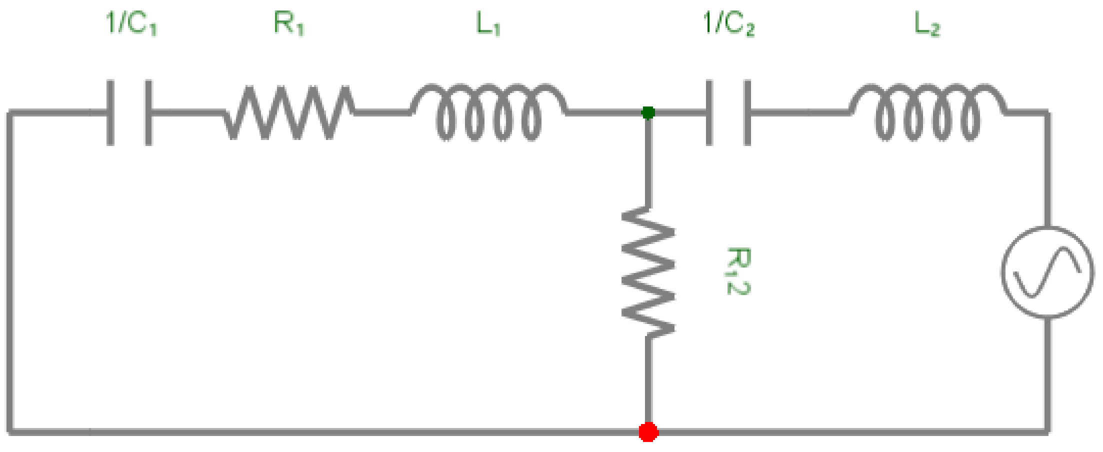


Fig 10: F-V analogous circuit of the mechanical system shown on fig 9 [10]

$1/C_1$	R_1	L_1	R_{12}	$1/C_2$	L_2
0.5	2	5	4	0.25	8

Fig 11: Numerical values of the F-V analogous circuit [11]

F-I analogy

Electrical counter part equations

$$C_1 \frac{dv_1}{dt} + \frac{1}{R_1} v_1 + \frac{1}{L_1} \int vdt + \frac{1}{R_{12}} \; v_{12} = 0$$

And

$$i = C_2 \frac{dv_2}{dt} + \frac{1}{R_{12}} \; v_{12} + \frac{1}{L_2} \int v_2 dt$$

- We have two voltages so we should have two nodes the components are placed between the node and ground parallelly according to their number, if the component has two digits it should be placed between the respective nodes.

F-I analogy circuit

Electrical counter part equations

$$C_1 \frac{dv_1}{dt} + \frac{1}{R_1} v_1 + \frac{1}{L_1} \int vdt + \frac{1}{R_{12}} \ v_{12} = 0$$

And

$$i = C_2 \frac{dv_2}{dt} + \frac{1}{R_{12}} \ v_{12} + \frac{1}{L_2} \int v_2 dt$$

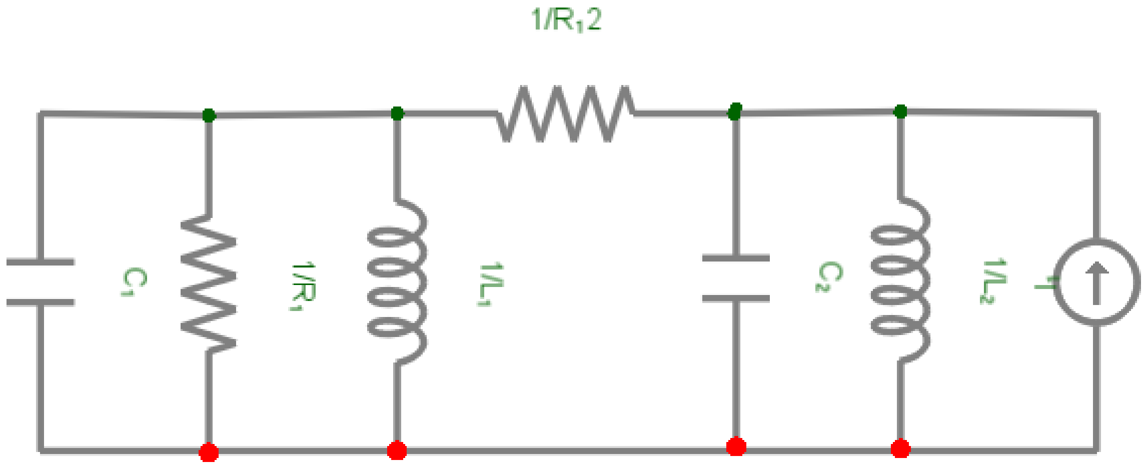


Fig 12: F-I analogous circuit of the mechanical system shown on fig 9 [12]

$1/L_2$	C_2	$1/R_{12}$	$1/L_1$	$1/R_1$	C_1
0.25	8	4	0.5	2	5

Fig 13: Numerical values of the F-I analogous circuit [13]

Derive T-V analogous equation and draw electrical circuit

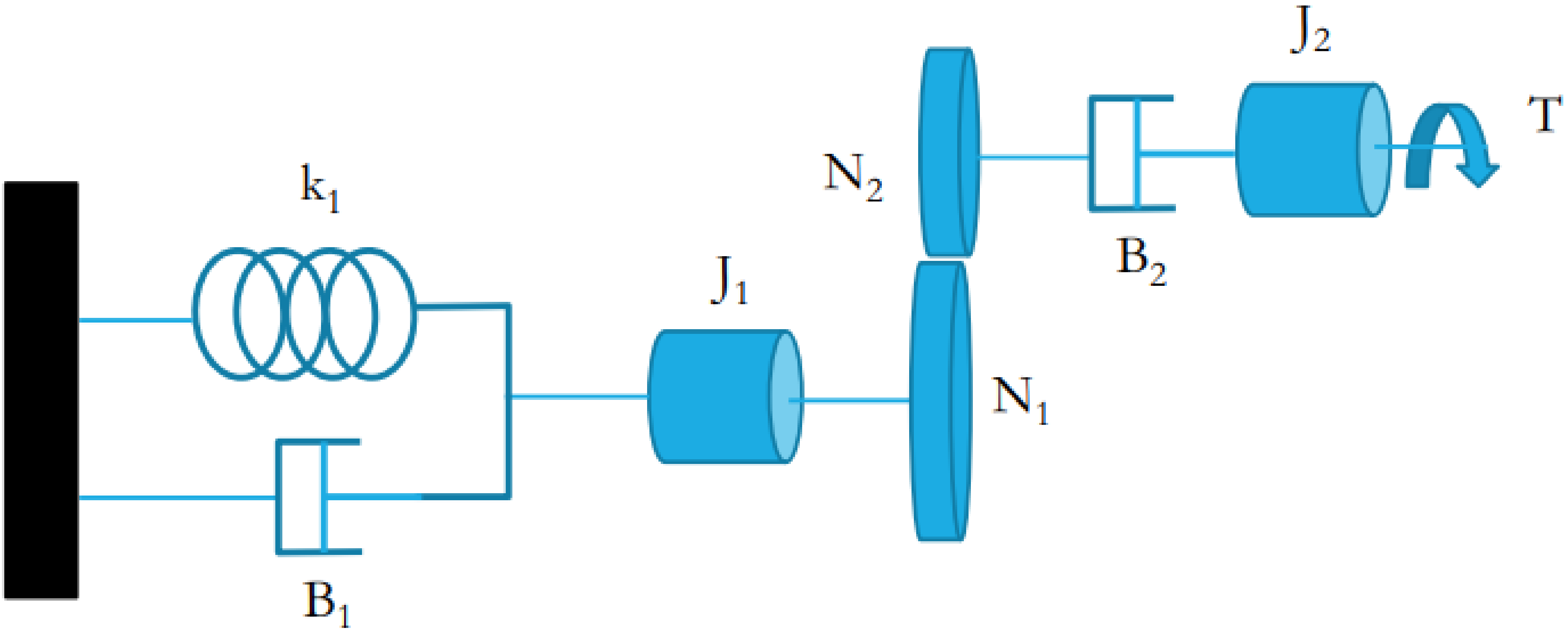


Fig 14: Rotational mechanical system linked by gear train [14]

Derive F-V…equation

- We already derived it's mathematical equation

$$T = \left(J_2 + J_1\left(\frac{N_2}{N_1}\right)^2\right)\alpha_2 + \left(B_2 + B_1\left(\frac{N_2}{N_1}\right)^2\right)\omega_2 + \left(\frac{N_2}{N_1}\right)^2 \mathrm{k}_1\theta_2$$

- By referring table 1

- $T = \left(L_2 + L_1\left(\frac{N_2}{N_1}\right)^2\right)\left(\frac{di_2}{dt}\right) + \left(R_2 + R_1\left(\frac{N_2}{N_1}\right)^2\right) i_2 + \left(\frac{N_2}{N_1}\right)^2 \frac{1}{\mathrm{C}_1}\int i_1 dt$

Derive F-V…circuit

- We have derived these equations before

$$T = T_2 + B_2\omega_2 + J_2\alpha_2$$

$$T_1 = k_1\theta_1 + B_1\omega_1 + J_1\alpha_1$$

$$T_2 = \frac{N_2}{N_1}T_1$$

- Their electrical counter parts are

$$v = v_2 + R_2 i_2 + \frac{L_2 di_2}{dt}$$

$$v_1 = \frac{1}{C_1}\int i_1 dt + R_1 i_1 + \frac{L_1 di_1}{dt}$$

$$v_2 = \frac{N_2}{N_1}v_1$$

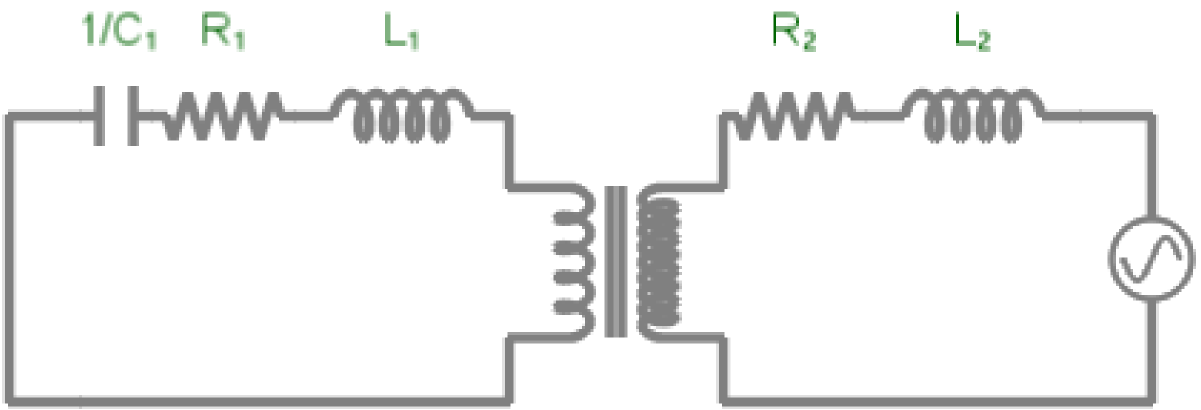


Fig 15: F-V analogous circuit of rotational mechanical system shown on fig 14. [15]

Summary

- Gears act as transformers, trading speed for torque via a gear ratio, with load inertia and damping reflected to the motor shaft reduced by N^2.
- DC motor constants are derived from torque-speed curve characteristics like no-load speed and stall torque.
- Finally, electro-mechanical analogies were introduced to translate physical systems into electrical circuits, enabling unified analysis and simulation using circuit theory.

Next class

- The modeling techniques you learned today (DC motor, gear equations, analogies) result in differential equations. The next class introduces the powerful tools (Laplace transforms, transfer functions) used to solve and analyze those equations, and the visual languages (block diagrams, SFGs, state-space) used to represent complex systems built from those components.
- This is the crucial next step from modeling physical systems to analyzing and designing control systems for them.

References

[1] Chalachew Werku, 2025, DC motor layout ,Figure, Self-created

[2] Chalachew Werku, 2025, DC motor schematics, Schematic , Self-created

[3] Chalachew Werku, 2025, Gear train, Figure, Self-created

[4] Chalachew Werku, 2025, Layout of motor and load connected via gear, Figure, Self-created

[5] Chalachew Werku, 2025, Rotational mechanical system, Figure, Self-created

[6] Chalachew Werku, 2025, Mathematical modelling of a lever system, Figure, Self-created

[7] Torque-speed characteristics of a DC motor, https://www.motioncontroltips.com/wp-content/uploads/2017/01/Torque-Speed-Curve.jpg, accessed on September 25, 2025

[8] Chalachew Werku, 2025, The relationship between Tstall and ωnl, Graph, Self-created

[9] Chalachew Werku, 2025, Two mass-spring-damper system, Figure, Self-created

[10] Chalachew Werku, 2025, F-V analogous circuit of the mechanical system shown on fig 9, Figure, Self-created

References Cont. …

[11] Chalachew Werku, 2025, Numerical values of the F-V analogous circuit, Table, Self-created

[12] Chalachew Werku, 2025, F-I analogous circuit of the mechanical system shown on fig 9, Figure, Self-created

[13] Chalachew Werku, 2025, Numerical values of the F-I analogous circuit, Table, Self-created

[14] Chalachew Werku, 2025, Rotational mechanical system linked by gear train, Figure, Self-created

[15] Chalachew Werku, 2025, F-V analogous circuit of rotational mechanical system shown on fig 14., Figure, Self-created

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