Course: Regulation and control

Lecture 3: System representations

Lecturer: Chalachew Werku

Revision on the last lecture

On the last lecture we have seen about

- Reflecting load damping and inertia to the motor shaft through gearbox.
- How we can use torque-speed curve characteristics like no-load speed and stall torque to derive motor constants.
- Electro-mechanical analogies to translate physical systems into electrical circuits, enabling unified analysis and simulation using circuit theory.

Lecture objective

- Utilize the **Laplace Transform** to convert differential equations from the time domain into algebraic equations in the frequency domain, simplifying their analysis.
- Define and derive **Transfer Functions** to create a compact, input-output model of a system's dynamic behavior.

Lecture objective Cont….

- Construct and interpret **Block Diagrams** to visualize the structure, subsystems, and signal flow of complex control systems.
- Apply **Signal Flow Graphs** and Mason's Gain Formula to efficiently determine input-output relationships in intricate networks.
- Understand **State-Space Representation** as a modern method for modeling a system's internal state, enabling the analysis of complex, multi-variable systems.

Laplace Transform L{f(t)}

- The Laplace Transform is an integral transform that converts a function of time, f(t)—like a voltage or a displacement—into a function of a complex variable “s”, which we call F(s).
- Laplace transform converts **differential equations calculus—derivatives and integrals—into simple algebra**.
- Problems that were dark and winding in the time domain become well-lit and straightforward in this new **s-domain**.

Laplace Transform Cont….

S-domain

- The variable $s = \sigma + j\omega$ is a complex frequency.
- The real part σ tells us about exponential growth or decay (stability),
- while the imaginary part $j\omega$ relates to oscillation frequency.

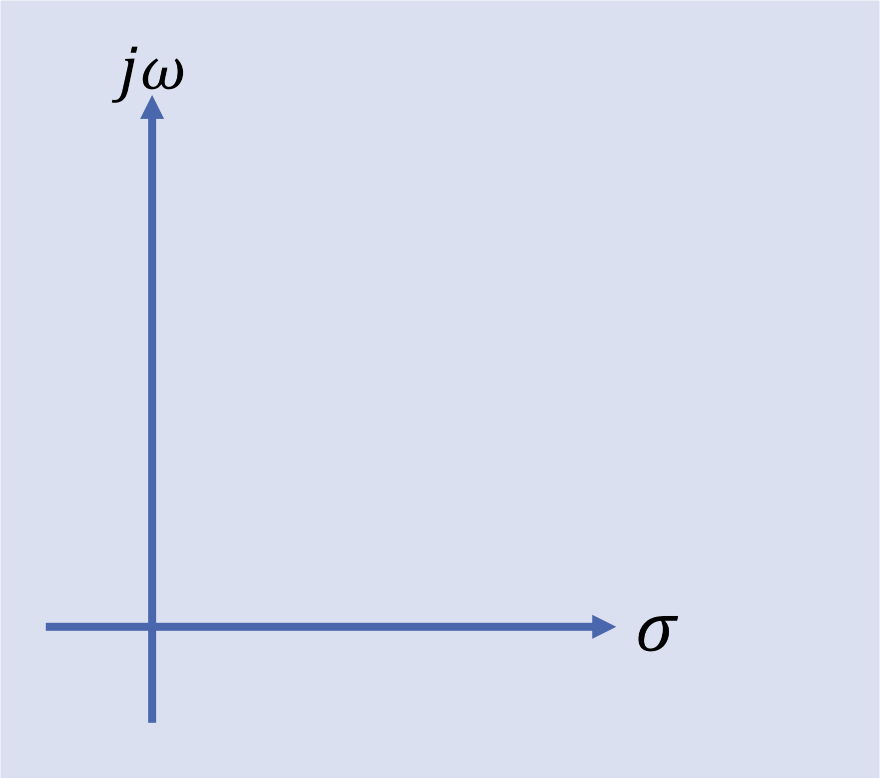


Fig 1: S-plane [1]

Laplace Transform Cont….

Properties of Laplace transform

- **Linearity**: The transform of a sum is the sum of the transforms.

$$L\{f(t) \pm g(t)\} = F(s) \pm G(s)$$

- **Differentiation Property**: Laplace transform of a derivative,

$$L\left\{\frac{df}{dt}\right\} = sF(s) - f(0)$$

  - Under zero initial conditions, sF(s). A second derivative becomes $s^2F(s)$.

- **Integration Property**: Similarly, the transform of an integral becomes $\frac{F(s)}{s}$.

Laplace Transform Cont….

Inverse-Laplace transform $L^{-1}\{f(t)\}$

- **Inverse Laplace Transform:** used to transform s-domain answers into time domain.
- **Partial Fraction Expansion** is the crucial technique that makes this possible. It allows us to break down a complex rational function in "s" into a sum of simpler terms, each of which has a known inverse transform.

Laplace Transform Cont….

Basic Laplace transforms of functions

Time domain	Laplace domain
u(t) step input	$\frac{1}{s}$
tu(t)	$\frac{1}{s^2}$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$
$e^{-at}u(t)$	$\frac{1}{s+a}$
$\int f(t)\,dt$	$\frac{F(s)}{s}$
$\frac{d^n f(t)}{dt^n}$	$s^n F(s)$
$f(t) \pm g(t)$	$F(s) \pm G(s)$

Fig 2: Laplace transforms of basic time domain functions [2]

Transfer Function

- A mathematical representation of the relationship between the input and output of a Linear Time-Invariant (LTI) system in the Laplace domain. The Primary Purpose is to characterize the system's dynamics (behavior) without needing to solve differential equations in the time domain.

- $G(s) = \frac{Y(s)}{X(s)}$ where

  G(s) = Transfer Function

  Y(s) = Laplace transform of the output signal

  X(s) = Laplace transform of the input signal

Transfer Function cont….

Fundamental Assumptions

- **Linear**: Superposition applies.
- **Time-Invariant (LTI):** System parameters do not change over time.
- **Zero Initial Conditions**: All initial conditions are assumed to be zero.

Transfer Function cont….

Derivation from a Differential Equation

1. Start with an ordinary differential equation (ODE) describing the system.

   Example: $f(t) = ma + Bv + kx$

   Find transfer function velocity (output) to the force (input)

2. Convert/ derive mathematical equation in terms of the required output variable.

$$f(t) = m\frac{dv(t)}{dt} + Bv(t) + k\int v(t)dt$$

Transfer Function cont....

3. Take the Laplace transform of both sides, assuming zero initial conditions.

$$L\{f(t)\} = L\left\{m\frac{dv(t)}{dt} + Bv(t) + k\int v(t)dt\right\}$$

$$F(s) = m\big(sv(s) - v(0)\big) + Bv(s) + k\frac{v(s)}{s}$$

equating initial conditions to zero gives us

$$F(s) = msv(s) + Bv(s) + k\frac{v(s)}{s}$$

4. Solve for the ratio output to input

$$F(s) = v(s)\left(ms + B + \frac{k}{s}\right)$$

$$\frac{v(s)}{F(s)} = \frac{1}{ms + B + \frac{k}{s}} \quad or$$

$$\frac{v(s)}{F(s)} = \frac{s}{ms^2 + Bs + k}$$

Transfer Function cont….

example

Find the transfer function x_1/F

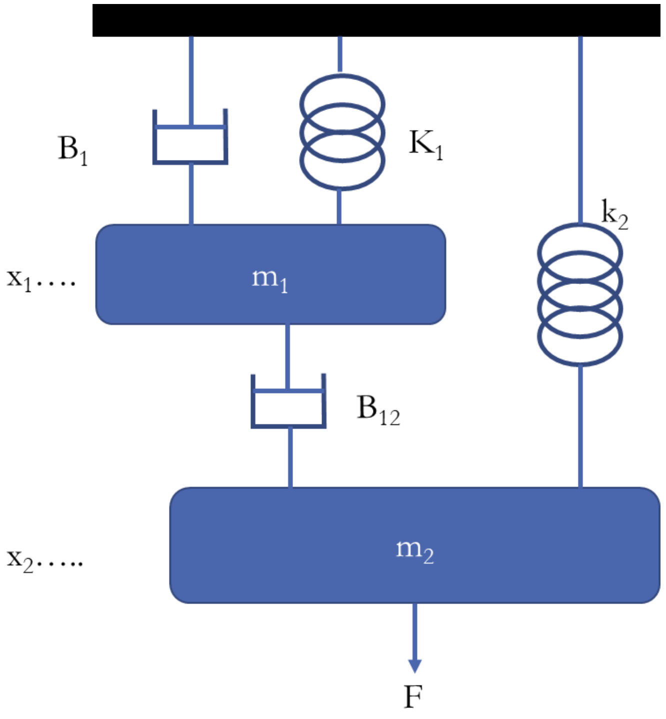


Fig 3: Two mass-spring-damper system [3]

The mathematical equations become

$$-B_1v_1 - k_1x_1 - B_{12}v_{12} = m_1a_1$$

$$\text{F} = m_2a_2 + B_{12}v_{12} + k_2x_2$$

Since we are asked for position

$$m_1\frac{d^2x_1}{dt^2} + B_1\frac{dx_1}{dt} + B_{12}\frac{dx_{12}}{dt} + k_1x_1 = 0$$

$$\text{F} = m_2\frac{d^2x_2}{dt^2} + B_{12}\frac{dx_{12}}{dt} + k_2x_2$$

Transfer Function cont….

example

- Taking Laplace transfer

$$m_1 s^2 x_1 + B_1 s x_1 + B_{12}\, s x_{12} + k_1 x_1 = \mathbf{0}$$

$$\mathrm{F} = m_2 s^2 x_2 + B_{12} s x_{12} + k_2 x_2$$

$$x_{12} = x_1 - x_2 \ @\ mass\ 1\ and\ x_{12} = x_2 - x_1\ @\ mass2$$

- Substitute value of x_{12}

$$m_1 s^2 x_1 + B_1 s x_1 + B_{12}\, s(x_1 - x_2) + k_1 x_1 = \mathbf{0}$$

$$\mathrm{F} = m_2 s^2 x_2 + B_{12} s(x_2 - x_1) + k_2 x_2$$

Transfer Function cont….

example

- Expand and collect similar terms

$$m_1s^2x_1 + B_1sx_1 + B_{12}\,sx_1 - B_{12}\,sx_2 + k_1x_1 = \mathbf{0}$$

$$(m_1s^2 + B_1s + B_{12}\,s + k_1)x_1 = B_{12}\,sx_2$$

- Write x_2 in terms of x_1

$$x_2 = \left(\frac{m_1s^2 + B_1s + B_{12}s + k_1}{B_{12}s}\right)x_1$$

$$\mathrm{F} = (m_2s^2 + B_{12}s + k_2)x_2 - B_{12}sx_1$$

Transfer Function   cont….

example

Substituting the values of x_2

$$F = (m_2s^2 + B_{12}s + k_2)\left(\frac{m_1s^2 + B_1s + B_{12}s + k_1}{B_{12}s}\right)x_1 - B_{12}sx_1$$

$$F = (m_2s^2 + B_{12}s + k_2)\left(\frac{m_1s^2 + B_1s + B_{12}s + k_1}{B_{12}s} - B_{12}s\right)x_1$$

Transfer Function cont....

example

Re-arranging the equations and solve for x1/F

$$\frac{F}{x_1} = (m_2s^2 + B_{12}s + k_2)\left(\frac{m_1s^2 + B_1s + B_{12}s + k_1}{B_{12}s} - B_{12}s\right)$$

$$\frac{F}{x_1} = (m_2s^2 + B_{12}s + k_2)\left(\frac{m_1s^2 + B_1s + B_{12}s + k_1}{B_{12}s} - B_{12}s\right)$$

$$\frac{x_1}{F} = \frac{B_{12}s}{(m_2s^2 + B_{12}s + k_2)\ ((m_1s^2 + B_1s + B_{12}s + k_1) - (B_{12}s)^2)}$$

Block Diagram

A block diagram is a pictorial representation of a system where each component is represented by a block.

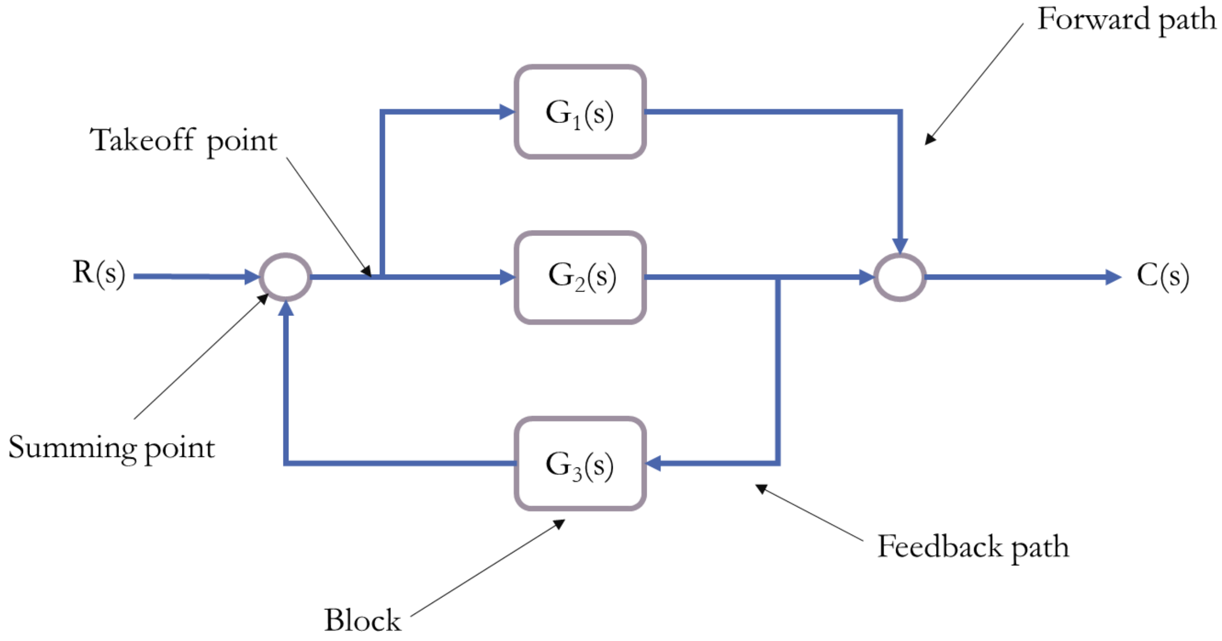


Fig 4: Basic elements of block diagram [4]

Block diagram reduction cont....

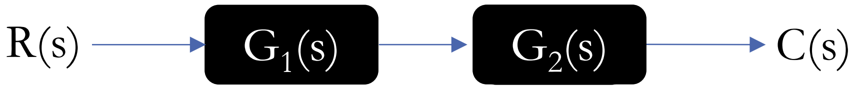
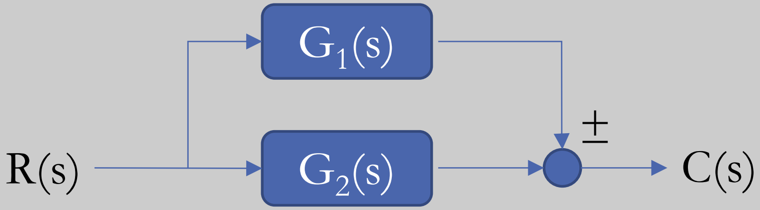
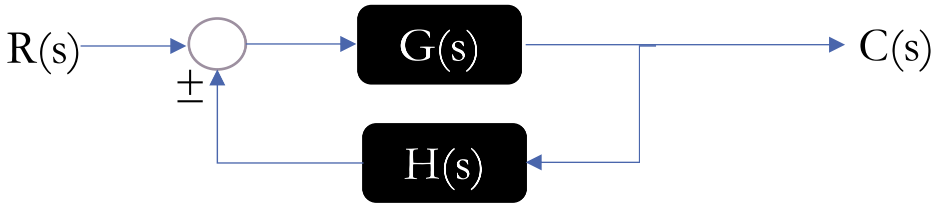
Block diagram	**Transfer function** $\frac{C(s)}{R(s)}$	
	$G(s)$	
	$G_1(s)G_2(s)$	Series blocks
	$G_1(s) \pm G_2(s)$	Parallel blocks
	$\frac{G(s)}{1 \mp H(s)G(s)}$	feedback

Fig 5: Rules for reducing basic blocks [5]

Block diagram reduction MOVING TAKEOFF POINT

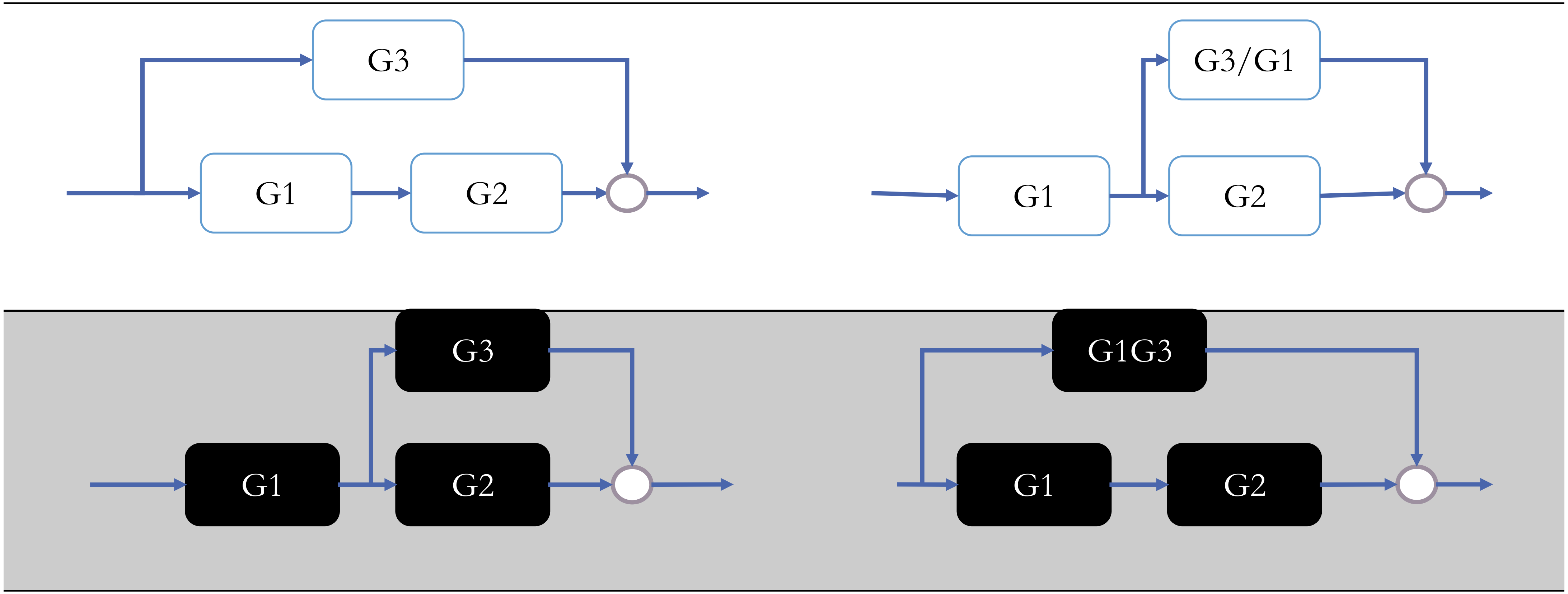


Fig 6: Rules for moving takeoff point [6]

Block diagram reduction MOVING SUMMING POINT

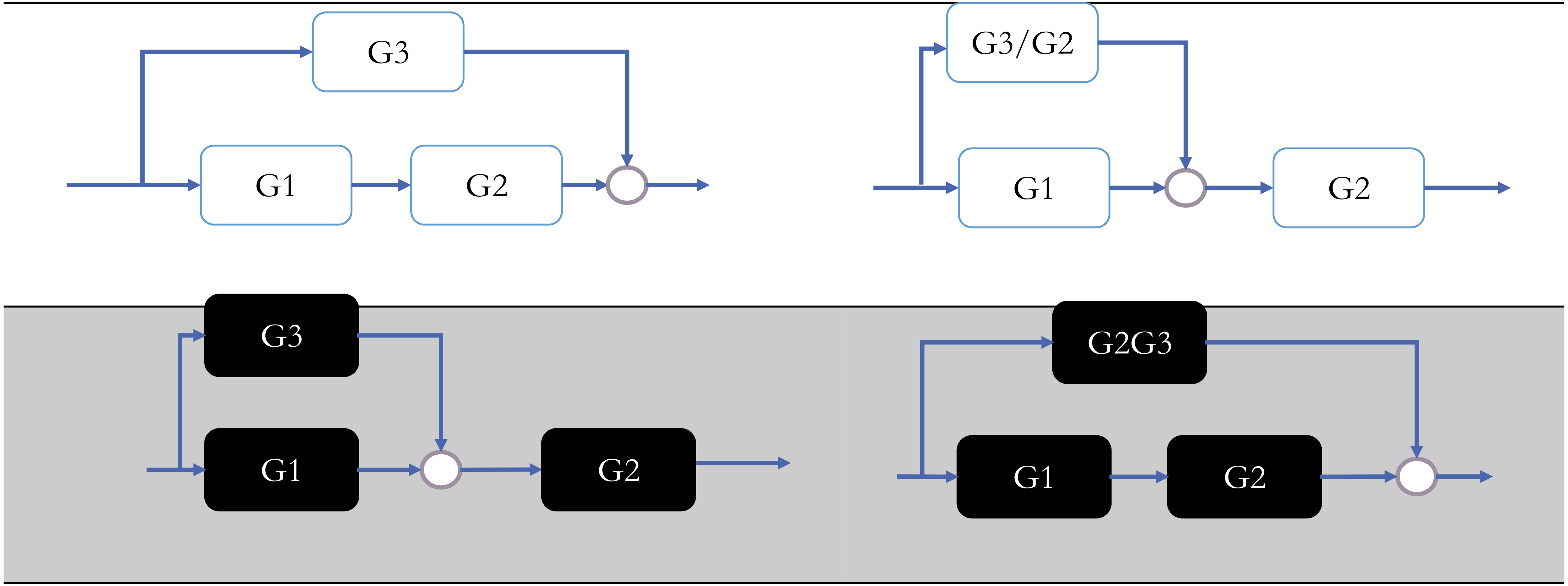


Fig 7: Rules for moving summing point [7]

BLOCK DIAGRAM REDUCTION *example1*

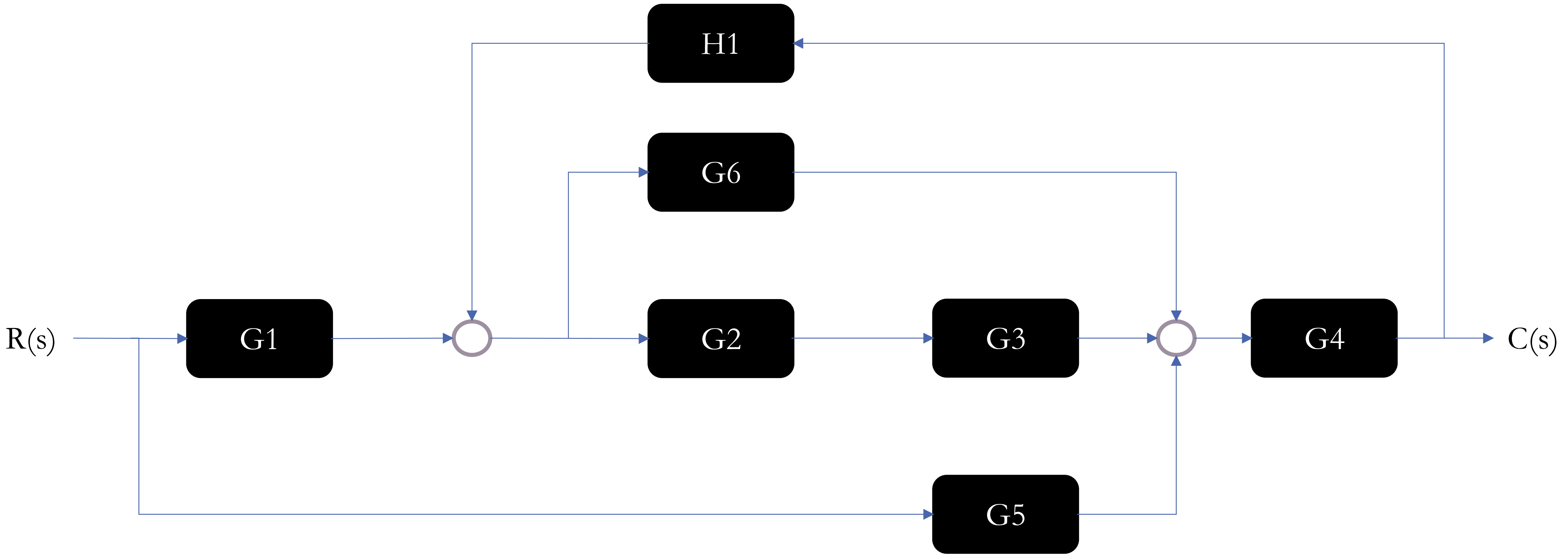
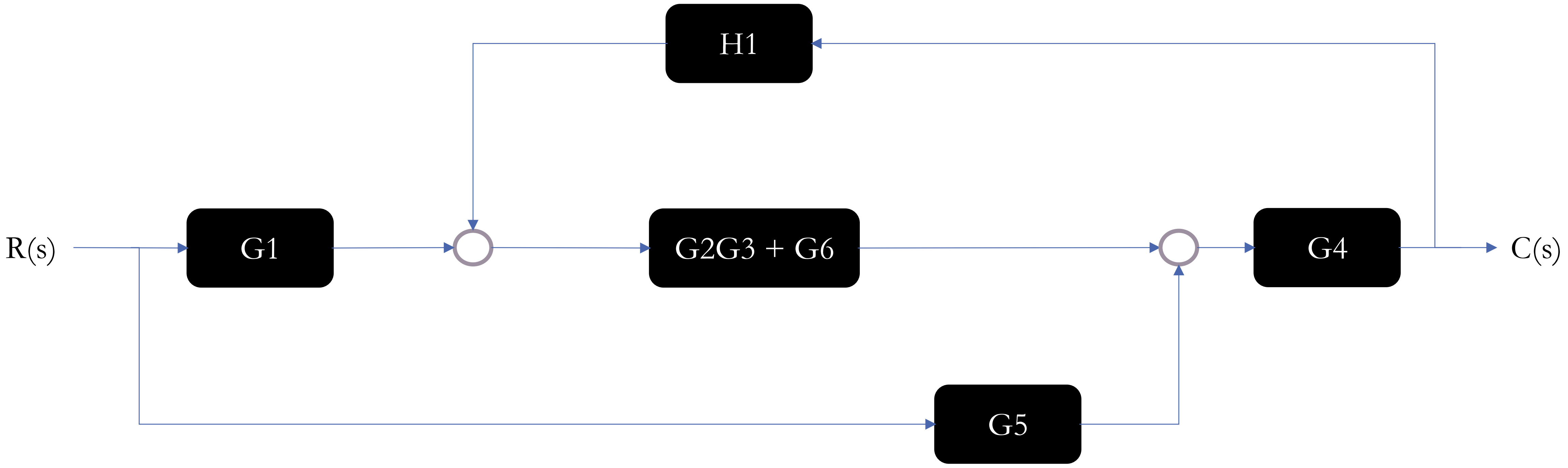


Fig 8: Block diagram for example 1 [8]

BLOCK DIAGRAM REDUCTION *example1* *cont…*

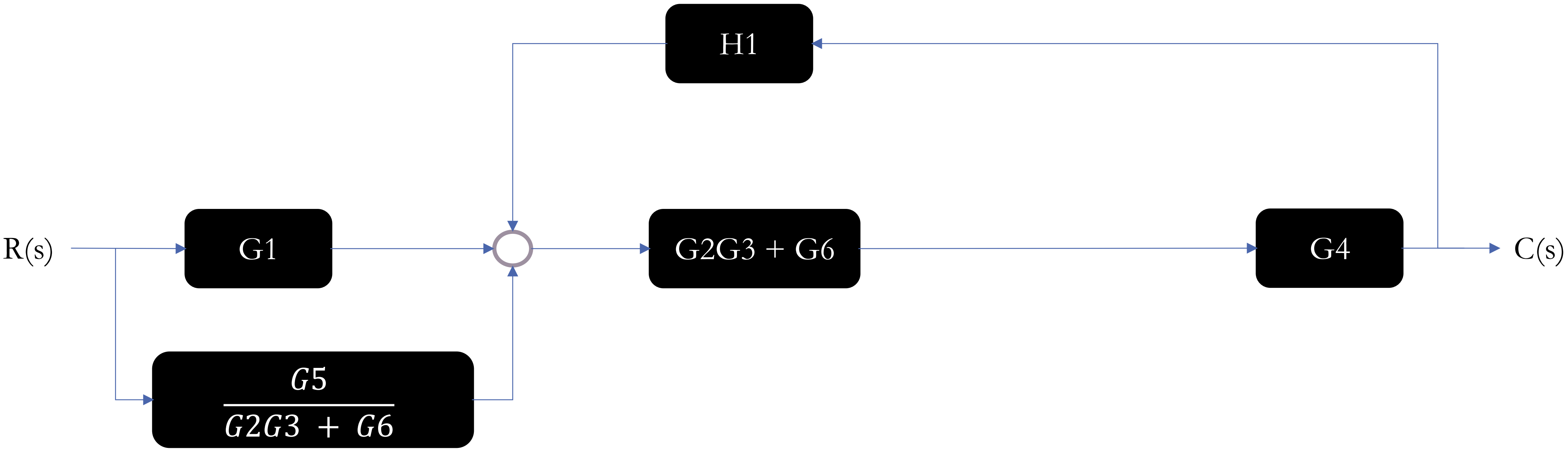
Multiply series blocks G2G3

Summing parallel blocks G2G3 + G6



BLOCK DIAGRAM REDUCTION *example1* *cont…*

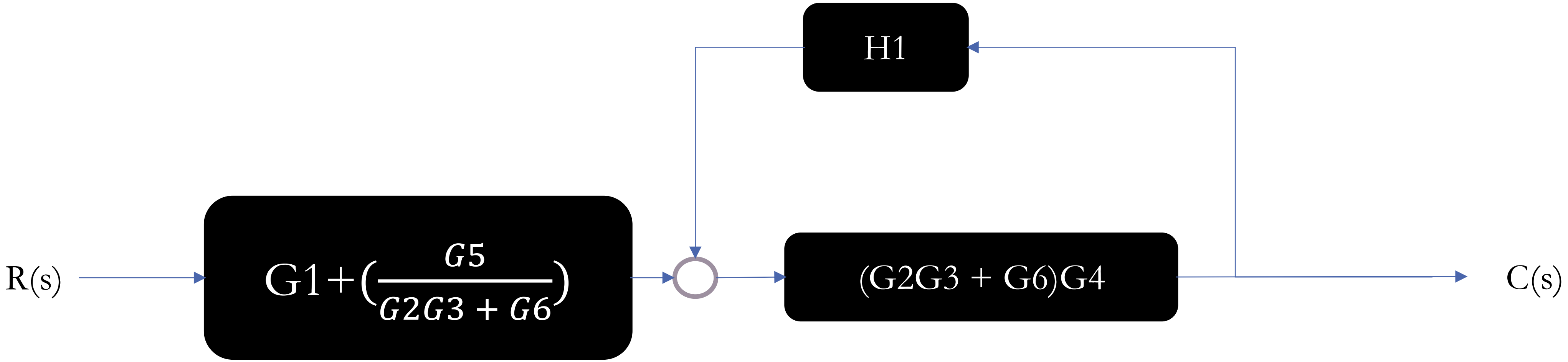
Moving takeoff to the left jumping G2G3 + G6



BLOCK DIAGRAM REDUCTION *example1* *cont…*

Summing parallel blocks G1+G5/(G2G3 + G6)

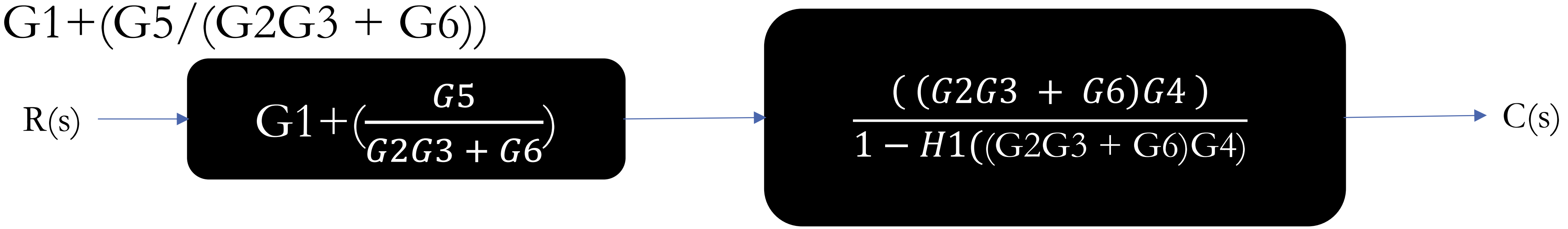
Multiply series blocks (G2G3 + G6)G4



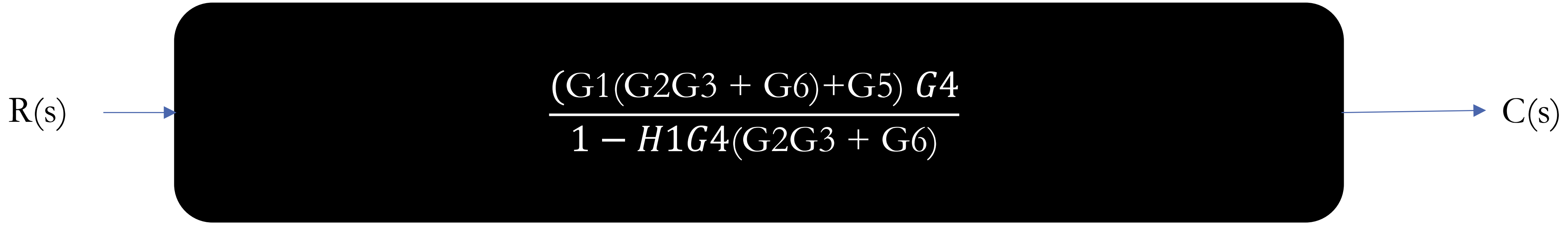
BLOCK DIAGRAM REDUCTION *example1* *cont…*

Use feedback formula for H1 and (G2G3 + G6)G4 make it series with

G1+(G5/(G2G3 + G6))



Multiply series blocks



BLOCK DIAGRAM REDUCTION *example2*

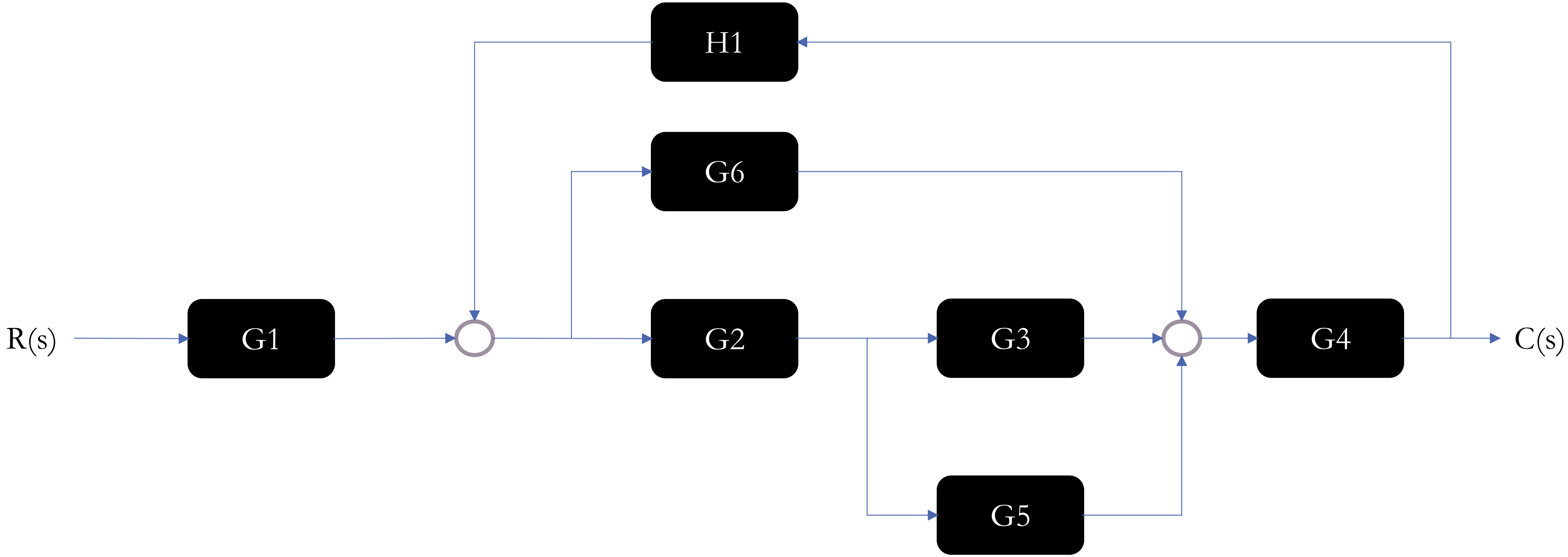
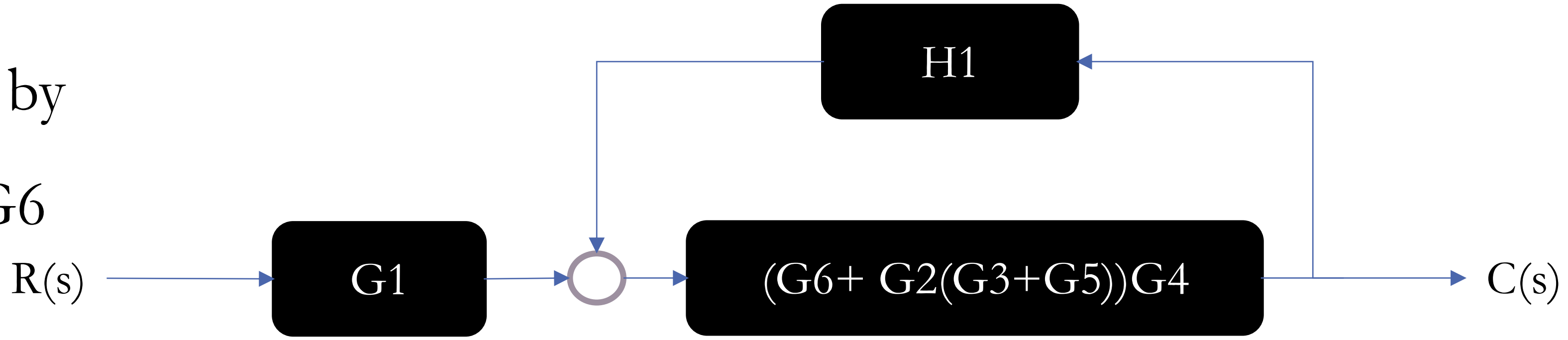


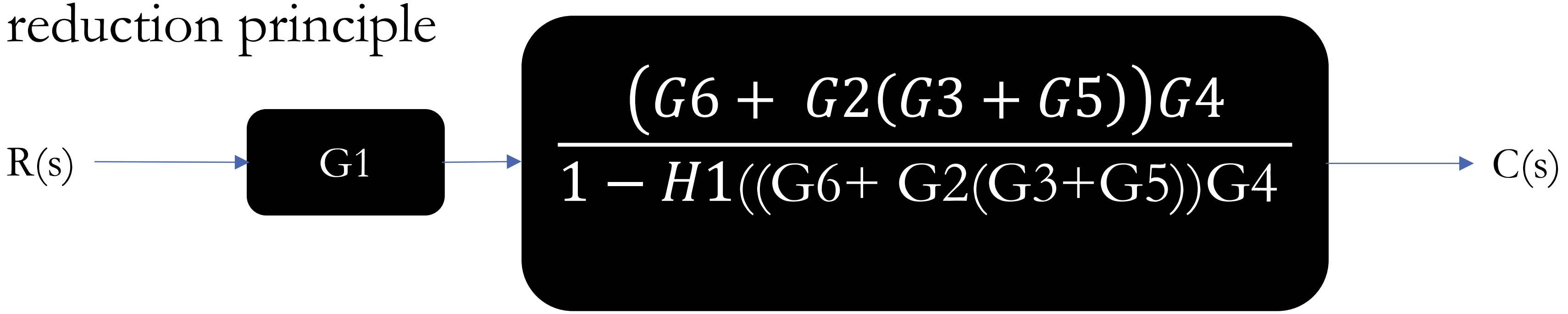
Fig 9: Block diagram for example 2 [9]

BLOCK DIAGRAM REDUCTION *example2* *cont…*

Summing G3 and G5 parallel, multiplying by G2, and sum with G6



Applying feedback block reduction principle



BLOCK DIAGRAM REDUCTION *example2* *cont…*

Multiplying by the series block

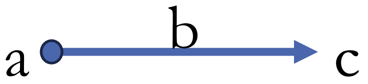
R(s) → $$\frac{G1\big(G6 + G2(G3 + G5)\big)G4}{1 - H1((G6+ G2(G3+G5))G4}$$ → C(s)

Re-arranging and simplifying

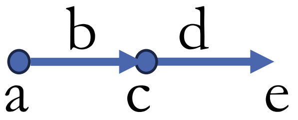
R(s) → $$\frac{G1G4(G6 + G2G3 + G2G5)}{1 - H1G4(G6+ G2G3+G2G5)}$$ → C(s)

Signal Flow Graph

- An SFG is a network of directed branches connecting nodes. It was developed to provide a direct, formulaic method for system analysis.
- *Signal flow graph reduction*

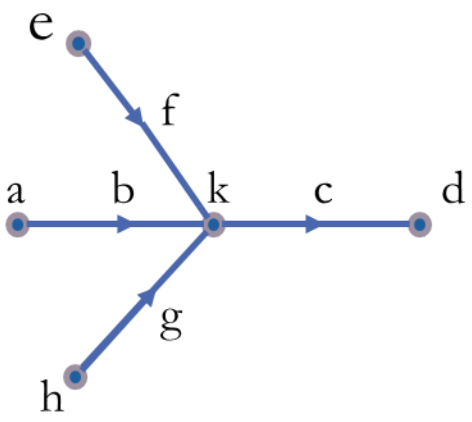


$c=ab$



$c=ab$,

$e=dc=dab$



$d= kc$

$k= ef + ab + hg$

$d= c(ef + ab + hg)$

SIGNAL FLOW GRAPH

- **Node:** a point where a path start or stop (i.e. 1, 2, 3…)
- **Path gain**: value on arrow (i.e. a, b, c…)
- **Loop**: start and sink on same node.
- **Self loop**:- start and sink on same node without intermediate nodes
- **Touching loop**: loops having common node
- **Non-touching loop**: loops without common node.
- **Dummy node**: a path with gain equal to 1.

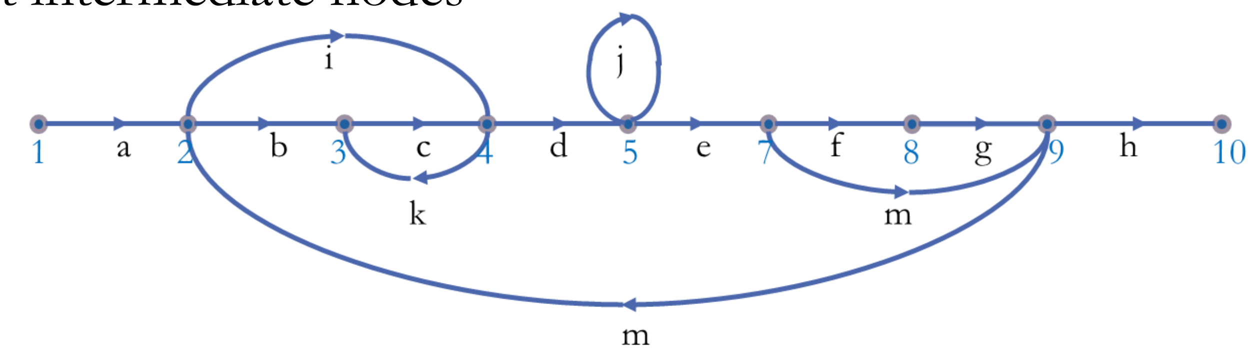


Fig 10: Signal flow graph basic elements [10]

SFG

Masons Gain Formula(MGF)

$$T_f = \frac{C(S)}{R(S)} = \frac{\sum_{i=1}^{n} p_i \Delta_i}{\Delta}$$

Where

P_i-gain of the i^{th} forward path

n- total number of forward paths

Δ=1 - $\sum$ loop gains +

$\sum$ product of non-touching-loop gains taken two at a time -

$\sum$ product of non-touching-loop gains taken three at a time…

Δ_i= value of Δ after eliminating all the loops that touch its forward path.

Block diagram to signal flow graph
example 3

Convert into SFG and find gain using MGF

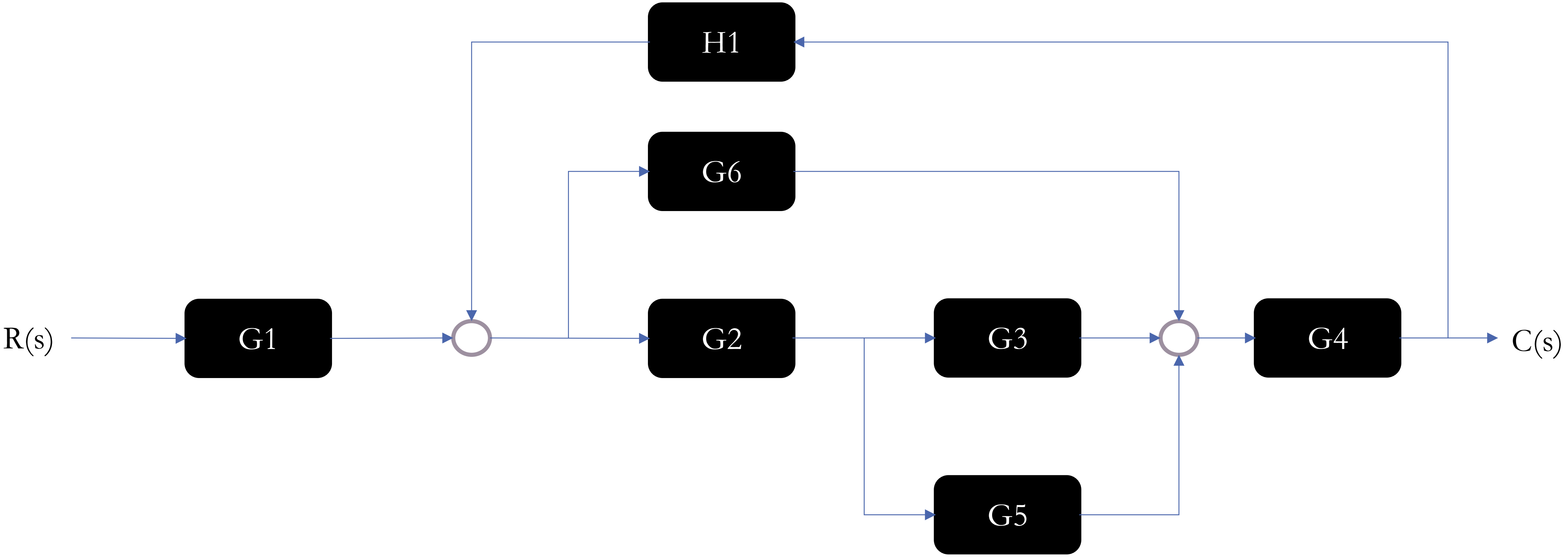
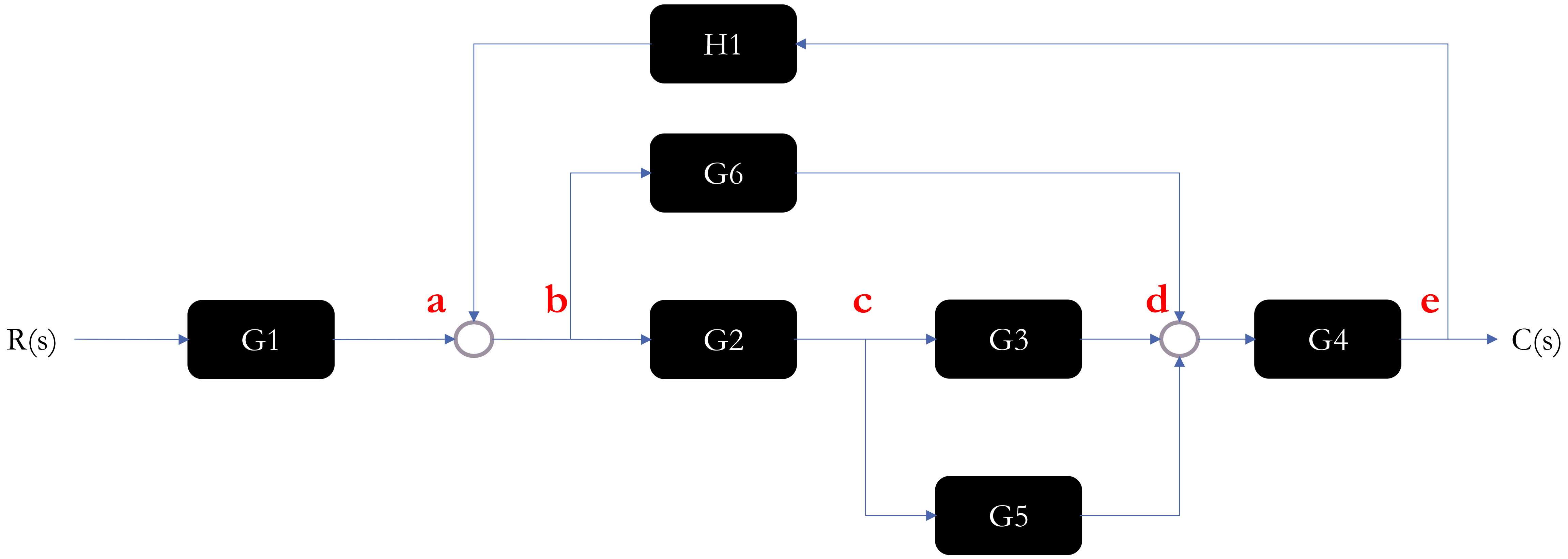


Fig 11: Block diagram for example 3 [11]

Block diagram to signal flow graph cont….

example 3

1. Mark takeoff and summing points



Block diagram to signal flow graph cont….

example 4

- Complete the paths following the marked takeoff and summing points as nodes
- Put path gains for each path and mark the input and output
- Add dummy node to connect input and output if necessary

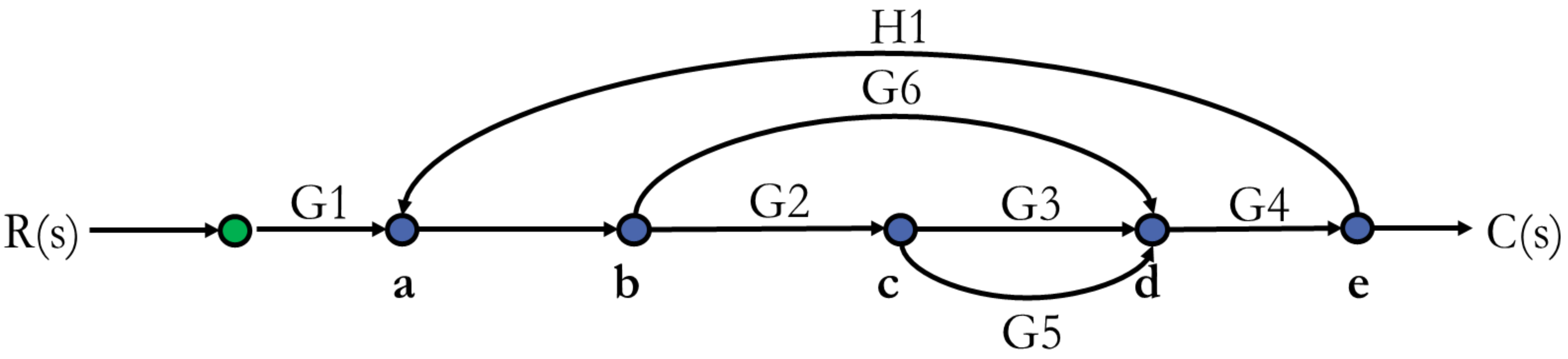


Fig 12: Signal flow graph for example 4 [12]

Block diagram to signal flow graph cont….

example 4

$$from\ MGF, \frac{C(s)}{R(s)} = \frac{\sum P_i \triangle_i}{\triangle}$$

Forward paths

P1: G1G2G3G4

P2: G1G2G5G4

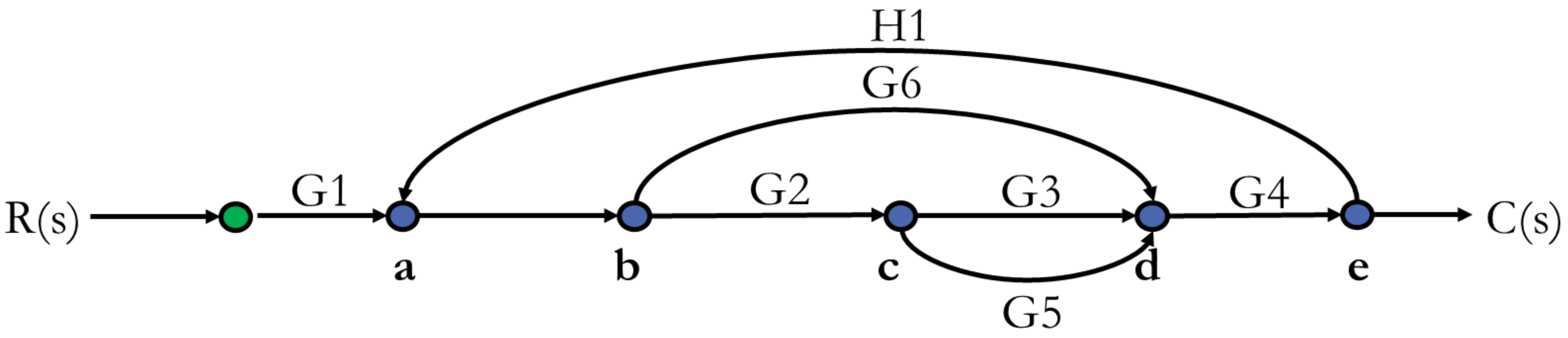
P3: G1G6G4

Loops

L1: G2G3G4H1

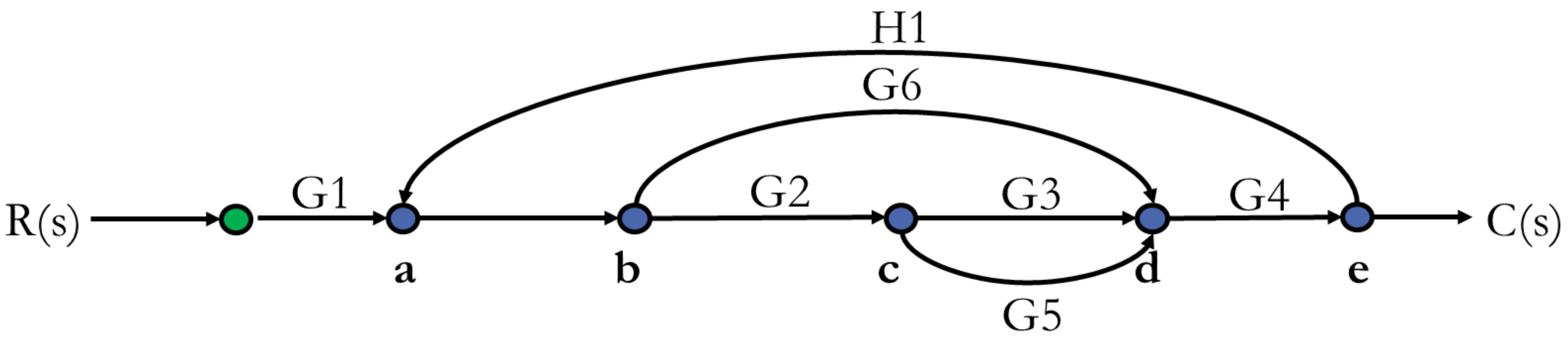
L2: G2GG5G4H1

L3: G6G4H1



Block diagram to signal flow graph cont….

examples



- We do not have non-touching loops
- All the loops touch every forward paths

$\rightarrow \Delta_1=\Delta_2=\Delta_3=1$

$\Delta=1-(L_1+L_2+L_3)$

$$\frac{C(s)}{R(s)}=\frac{G1G2G3G4*1+G1G2G5G4*1+G1G6G4*1}{1-(G2G3G4H1+G2G5G4H1+G6G4H1)}$$

$$\frac{C(s)}{R(s)}=\frac{G1G4(G2G3+G2G5+G6)}{1-G4H1(G2G3+G2G5+G6)}$$

Signal Flow Graph vs Block diagram

Feature	Block Diagram	Signal Flow Graph (SFG)
Primary Use	**Intuitive visualization** of system structure and subsystems.	**Systematic analysis**, especially using **Mason's Gain Formula.**
Basic Elements	Blocks, summing junctions, takeoff points.	Nodes (representing signals) and branches (representing transfers).
Connection Rules	Can become complex with many crossing lines.	**Strict rules**: signals flow along branches in one direction only.
Simplification Method	**Step-by-step reduction** using a set of rules (series, parallel, feedback).	**Mason's Gain Rule** - provides a direct formula for the transfer function.
Best For	Understanding system architecture and control logic.	Quickly finding transfer functions of complex systems with multiple loops and paths.

Fig 13: Comparison between block diagram and signal flow graph representation. [13]

State-Space

- **State-space** representation describes a system using first-order differential equations. Instead of just relating input to output like a transfer function, it models the internal state of the system.
- A state-space model consists of two sets of equations:

State Equation:

$$\mathbf{x}'(t)=\mathbf{A}\mathbf{x}(t)+\mathbf{B}\mathbf{u}(t)$$

Output Equation:

$$\mathbf{y}(t)=\mathbf{C}\mathbf{x}(t)+\mathbf{D}\mathbf{u}(t)$$

State-Space cont....

Here is a breakdown of the vectors and matrices in these equations:

- **x(t) is the state vector**, a column vector containing the minimum number of variables needed to describe the system's state at any time t.
- **u(t) is the input vector**, representing external inputs or forces applied to the system.
- **y(t) is the output vector**, representing the measurable outputs of the system

State-Space cont….

A is the state matrix, defining how the current state variables interact with and influence each other.

B is the input matrix, defining how the external inputs affect the state variables.

C is the output matrix, defining how the state variables contribute to the system's output

D is the feedforward matrix, representing any direct influence of the inputs on the outputs. For many systems, D is a zero matrix

State space cont….

example

Express the system below with state-space equations

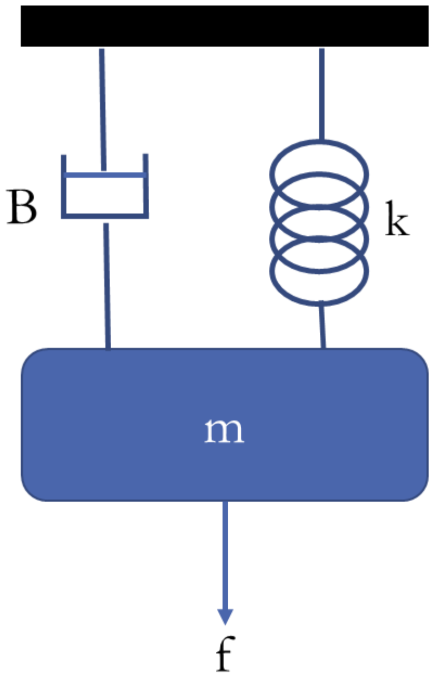


Fig 14: Spring-mass-damper system for example 5 [14]

The mathematical equations become

$$f = ma + Bv + kx$$

$$f = m\ddot{x} + B\dot{x} + kx$$

Define state variables

$x_1 = x$

$x_2 = \dot{x}$

State space cont....

example

Express derivative as first order equations

$\dot{x_1} = x_2$

$\dot{x_2} = \ddot{x} = -\frac{k}{m}x - \frac{b}{m}\dot{x} + \frac{1}{m}f = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}f$

Write the matrix representation

State equation

$$[\dot{x_1} \quad \dot{x_2}] = \left[0 \;\; 1 \;\; -\frac{k}{m} \quad -\frac{b}{m}\right][x_1 \;\; x_2] + \left[0 \;\; \frac{1}{m}\right]f(t)$$

Output equation

$$y = [1 \;\; 0][x_1 \quad x_2]$$

Comparisons of transfer function vs state space

Aspect	Transfer Function	State-Space
System Type	Only for LTI systems	LTI, nonlinear, time-varying
Internal View	No internal dynamics	Shows internal states
MIMO Systems	Difficult to handle	Naturally handles MIMO
Initial Conditions	Cannot incorporate easily	Easily incorporates
Computer Implementation	Difficult	Very suitable

Fig 15: Comparison of transfer function and state-space representation [15]

Summary

- We use Laplace Transform, to convert complex time-domain differential equations into manageable s-domain algebra.
- We define Transfer Functions to describe a system's input-output behavior.
- Then learned to visualize systems using Block Diagrams and Signal Flow Graphs.
- Finally, we advanced to State-Space Representation, which allows us to model not only input and out put but also internal state of a system.

Next class

- .On the next lecture we will begin Time Domain Analysis to assess system performance by analyzing step response, transient behavior, and steady-state error.

References

[1] Chalachew Werku, 2025, S-plane, Graph, Self-created
[2] Chalachew Werku, 2025, Laplace transforms of basic time domain functions, Table, Self-created
[3] Chalachew Werku, 2025, Two mass-spring-damper system, Figure, Self-created
[4] Chalachew Werku, 2025, Basic elements of block diagram, Figure, Self-created
[5] Chalachew Werku, 2025, Rules for reducing basic blocks, Diagram, Self-created
[6] Chalachew Werku, 2025, Rules for moving takeoff point, Diagram, Self-created
[7] Chalachew Werku, 2025, Rules for for moving summing point, Diagram, Self-created
[8] Chalachew Werku, 2025, Block diagram for example 1, Diagram, Self-created
[9] Chalachew Werku, 2025, Block diagram for example 2, Diagram, Self-created
[10] Chalachew Werku, 2025, Signal flow graph basic elements, Diagram, Self-created
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