



Course: Regulation and control

Lecture 4: Time Domain Analysis I

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Revision

- Mathematical model
- Transfer function
- LTI systems
- State-space

Introduction to the Time Domain

- What is the Time Domain?
- Analyzing system behavior and signals as a function of time.
- Key Question: "How does the output of a system change over time in response to an input?"

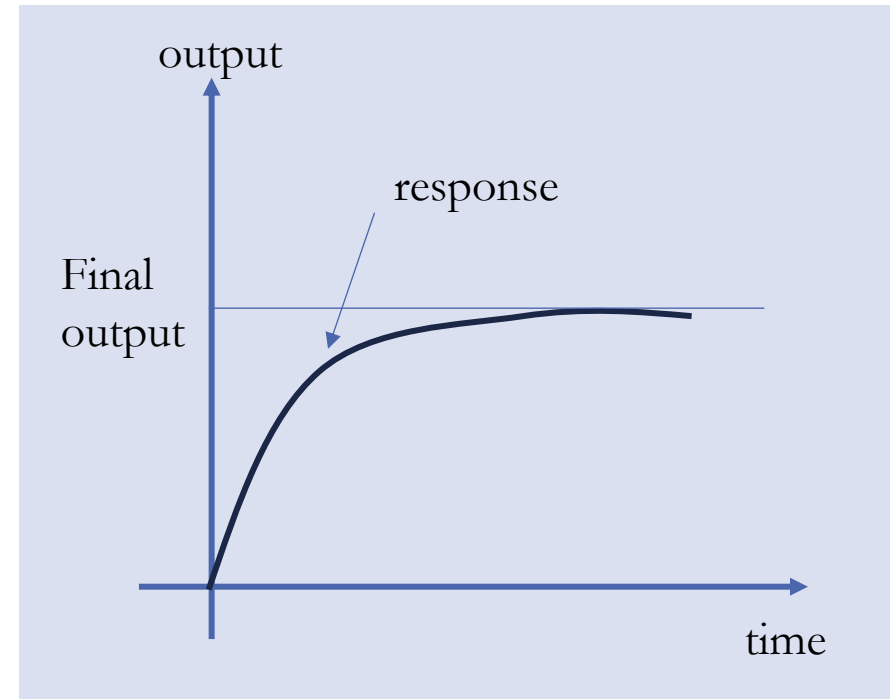


Figure 1: Simple step response. [1]

Why Time Domain Analysis?

- To evaluate performance metrics:
 - Speed of response
 - Stability of the system
 - Accuracy in the long run
- To design controllers to meet desired specifications.

Systems and Signals

- **System:** A physical or mathematical entity that processes an input to produce an output.

Examples: Electrical circuit, motor, chemical plant.

- **Input, $r(t)$:** The command or excitation.
- **Output, $c(t)$:** The resulting response.



Standard Test Inputs

- Why use standard inputs?
 - Predict performance.
 - Compare different systems.
 - Simplify the math.

The Impulse Input, $\delta(t)$

- **Mathematical Concept:** Infinite amplitude, zero width, unit area.
- **Physical Interpretation:** A sudden, sharp shock or disturbance.
- **Significance:** The impulse response completely characterizes a system.

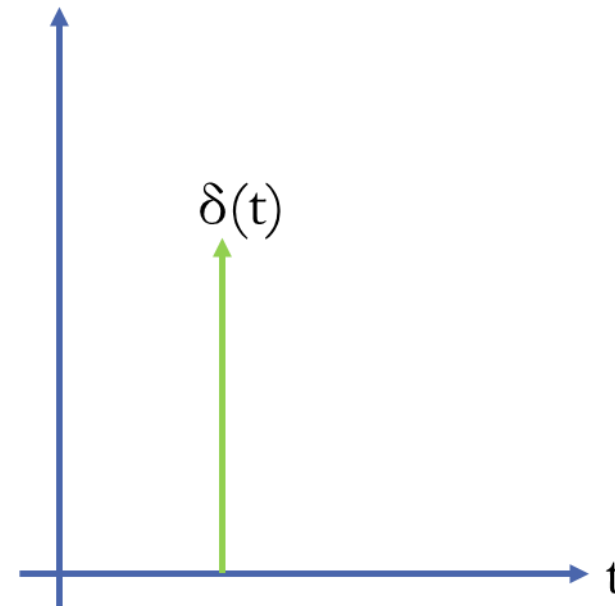


Figure 2:- impulse input [2]

The Step Input, $u(t)$

- **Mathematical Definition:** 0 for $t < 0$, 1 for $t \geq 0$.
- **Physical Interpretation:** A sudden, sustained change in setpoint.
Example: Switching a thermostat to a new temperature.
- **Focus:** This is our primary test signal.

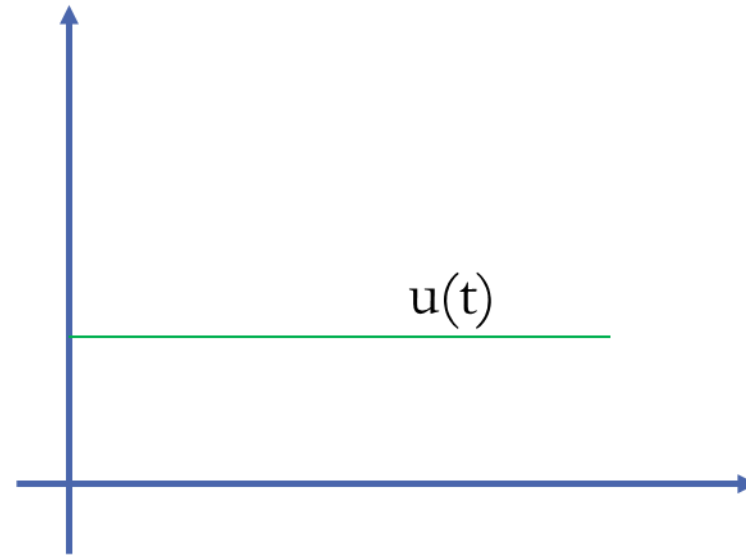


Figure 3:- step input [3]

The Ramp Input, $t \cdot u(t)$

- **Mathematical Definition:** A linearly increasing signal.

- **Physical Interpretation:** Tracking a moving target.

Example: A radar dish tracking a satellite.

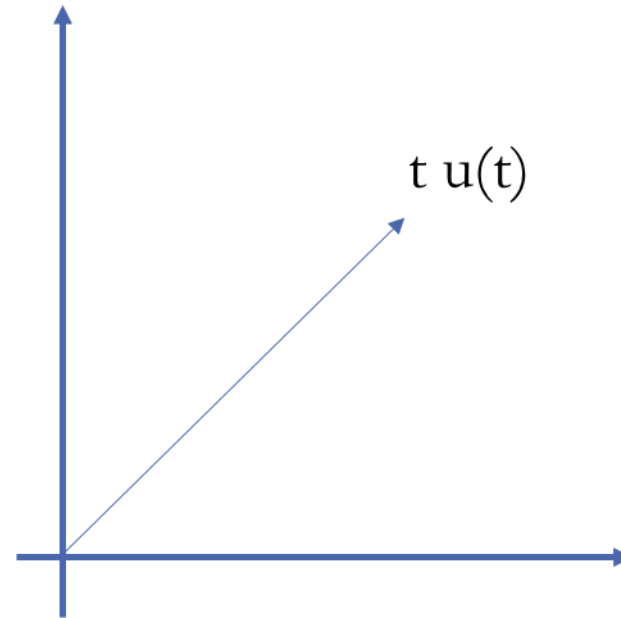


Figure 4:- ramp input [4]

The Parabolic Input, $(t^2/2) \cdot u(t)$

- Mathematical Definition: A quadratically increasing signal.
- Physical Interpretation: Tracking an accelerating target.
Example: missile guidance control system

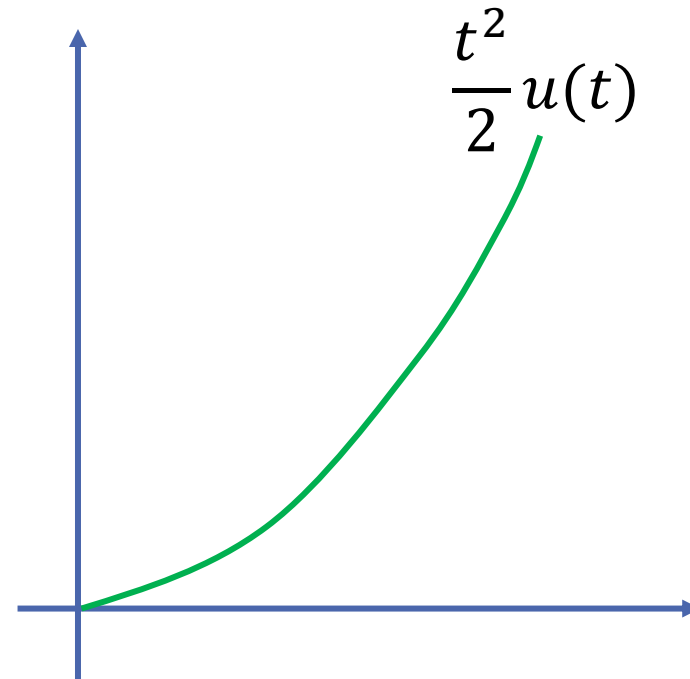


Figure 5:- parabolic input [5]

Decomposing the Total Response

- Total Response, $c(t) = \text{Transient Response} + \text{Steady-State Response}$
- **Transient/ Natural** $[C_{tr}(t)]$: response of the system as the input is applied and die out as time goes. $\lim_{t \rightarrow \infty} C_{tr}(t) = 0$
- **Steady-state/ forced** $[C_{ss}(t)]$: response of a system which remain after transient response die out.

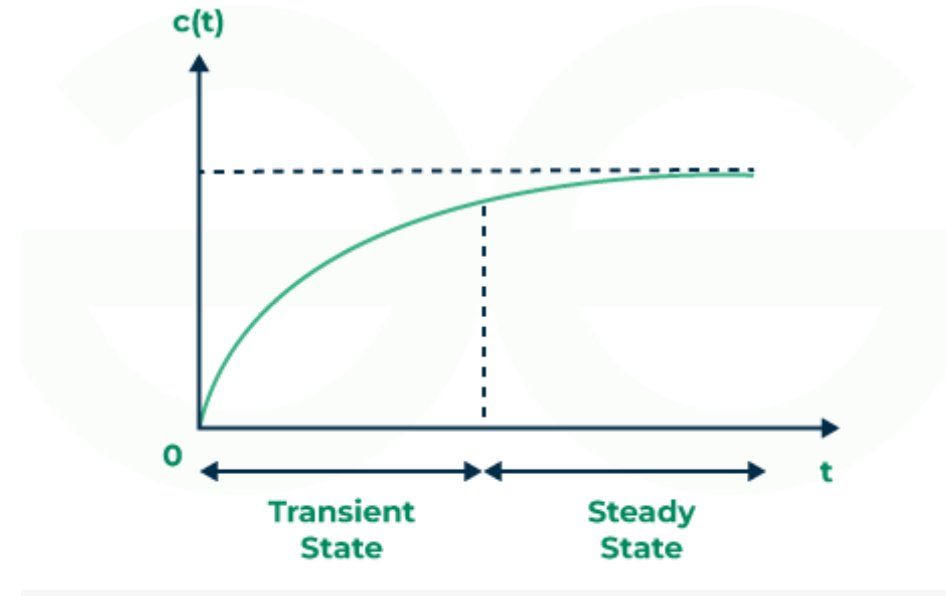


Figure 6: Total Response of a control system [6]

Transient vs steady-state response

Transient Response

- The part of the response that decays to zero as time becomes large.
- Determined by the system's natural modes (poles).
- Dictates the initial speed and damping of the response.

Steady-State Response

- The part of the response that remains after the transients have died out.
- Determines the system's final accuracy.
- Steady-State Error, e_{ss} : The difference between the input and output as $t \rightarrow \infty$.

Type and order of transfer function

- **Type:** The number of free integrators (poles at origin, s in denominator) in the open-loop transfer function $G(s)$.
- **Order:** The highest power of “ s ” in the characteristic equation (denominator of closed-loop TF, $1 \pm G(s)H(s)$).
- **Poles:** zeros of a transfer function denominator.
- **Zeros:** zeros of a transfer function numerator.

First-Order Systems: Introduction

- **First order system:** A system whose dynamics are described by a first-order differential equation. A system having only one energy storage element (like capacitor or mass)

- **Standard Transfer Function:**

$$G(s) = \frac{K}{\tau s + 1}$$

- **K:** System Gain (affects the final value).
- **τ (tau):** Time Constant (affects the speed).

First-Order System Step

Performance specifications

- **Time constant τ** : Time to reach 63.2% of the final steady-state value.

The fundamental measure of speed for a first-order system.

- **Rise time (T_r)**: Time for the response to go from 10% to 90% of its final value.

$$T_r \approx 2.2\tau$$

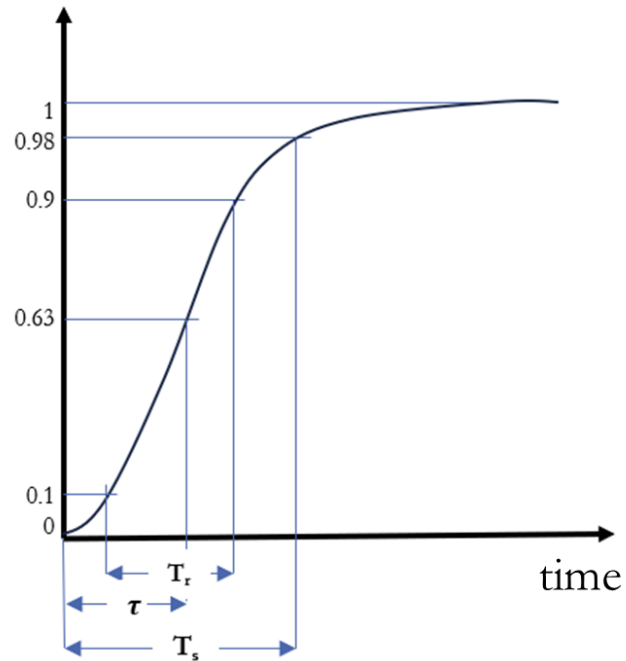
- **Settling time (T_s)**: Time for the response to reach and stay within $\pm 2\%$ of the final value.

$$T_s = 4\tau$$

First-Order System Step

Performance specifications summary

$$\text{Output } c(t) = K(1 - e^{-\frac{t}{\tau}})$$



- **Time constant: τ :** to reach 63% of the final value.
- **Rise time: T_r :** to reach from 10% to 90% of its final output.
- **Settling time: T_s :** to reach 98% of final output.

Figure 7:- First-order system specifications [7]

Second-Order Systems: Why They Matter

- Very common in engineering (spring-mass-damper, RLC circuit, motor position).

Spring-mass-damper	RLC circuit	Motor speed
$\frac{x(s)}{F(s)} = \frac{1}{s^2m + sB + k}$	$\frac{v_c}{v_i} = \frac{1}{s^2LC + sRC + 1}$	$\frac{\omega_m}{v_a} = \frac{k_t}{s^2J_aL_a + s(J_aR_a + B_aL_a) + k_bk_t}$

- Exhibit a wider range of behaviors (oscillation, overshoot).
- Concepts extend to higher-order systems.

Second-Order Systems: Standard Form

- **Standard Transfer Function:**

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- **ω_n (*omega n*), Natural Frequency:** How fast the system oscillates naturally. Higher ω_n faster response.
- **ζ (*zeta*), Damping Ratio:** how oscillatory vs how sluggish the system is. Determines system behavior.

The Damping Ratio (ζ) - The Behavior Dictator

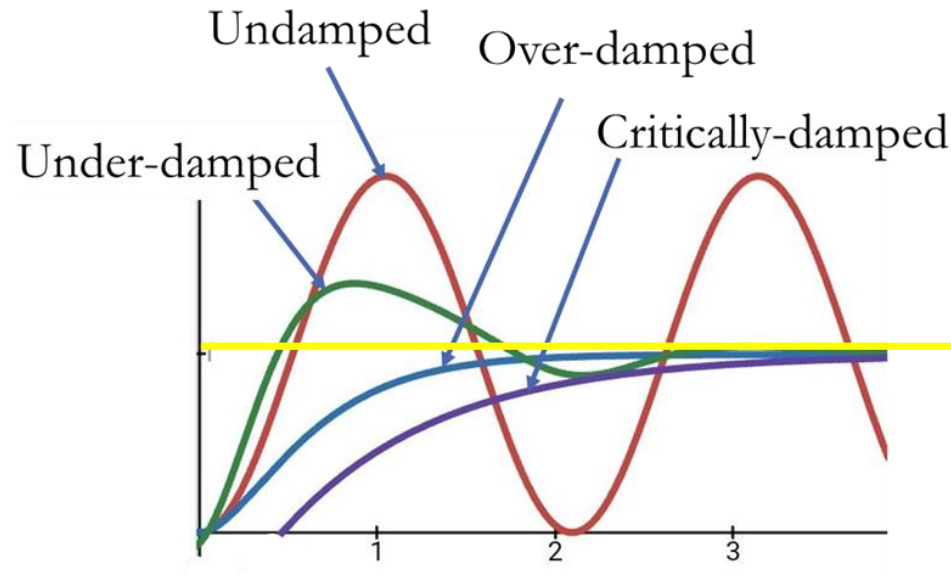


Figure 8:-Response of second-order system for different value of damping ratio [8]

- $\zeta = 0$: Undamped (Sustained Oscillation)
- $0 < \zeta < 1$: Underdamped (Oscillatory, most common case)
- $\zeta = 1$: Critically Damped (Fastest response without overshoot)
- $\zeta > 1$: Overdamped (Slow, sluggish, no overshoot)

Pole Locations for Different Damping

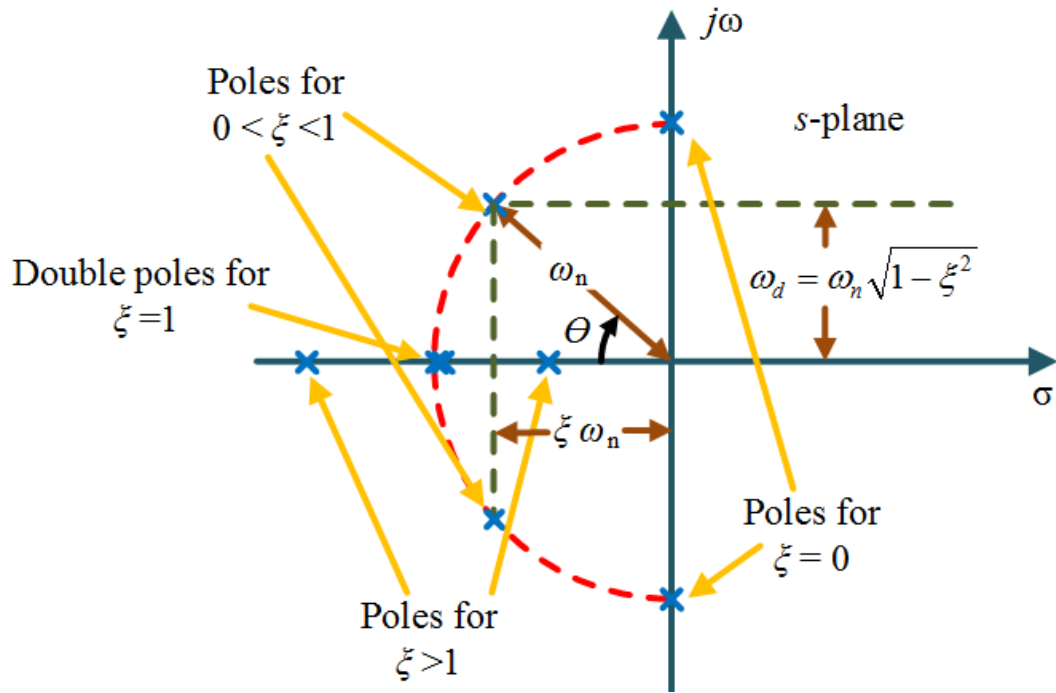


Figure9:- Pole zero map second-order system [9]

- **Undamped:** Purely imaginary poles.
- **Underdamped:** Complex conjugates in LHP.
- **Critically Damped:** Two identical, real poles.
- **Overdamped:** Two distinct, real poles.

Underdamped Step Response ($0 < \zeta < 1$)

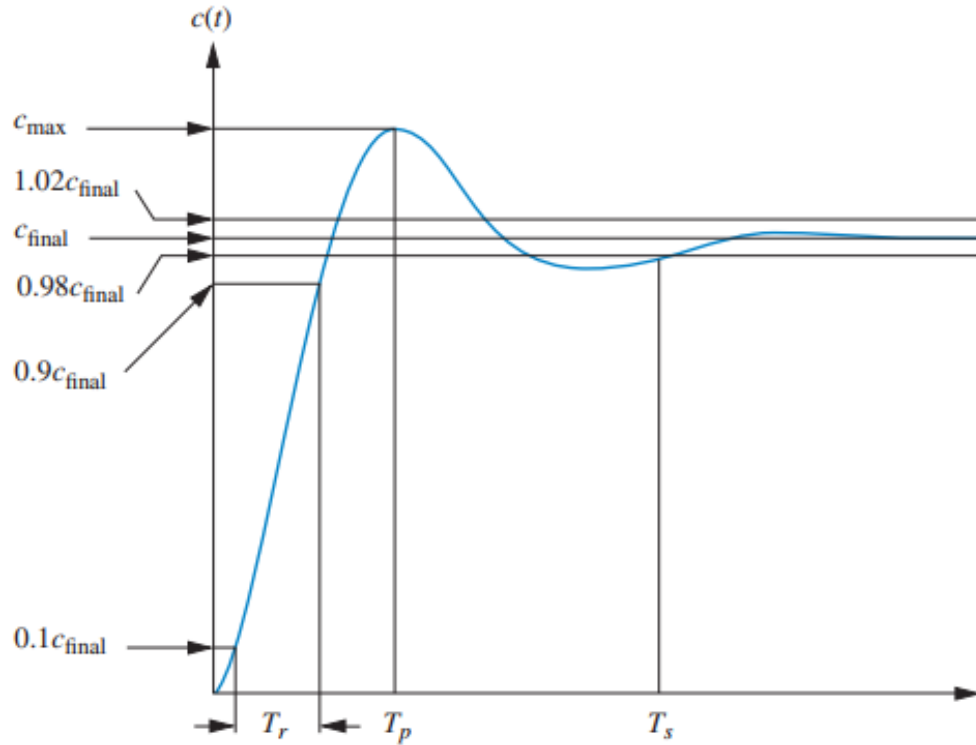


Figure 10:- Second-order underdamped response specification Norman [10]

- Characterized by:

Rise Time (T_r): $\frac{\pi - \alpha}{\omega_d}$

where $\omega_d = \omega_n \sqrt{1 - \zeta^2}$

and $\tan \alpha = \frac{\sqrt{1 - \zeta^2}}{\zeta}$

Peak Time (T_p): $\frac{\pi}{\omega_d}$

Settling Time (T_s): $\frac{4}{\zeta \omega_n}$

Overshoot (%OS):

$\%OS = e^{-\frac{\zeta \pi}{\sqrt{1 - \zeta^2}}} \times 100\%$

Steady-State Error (e_{ss})

Steady state error: the difference between the desired output (reference input) and the actual system output as time approaches infinity, when the response has reached its final, constant value.

$$e_{ss} = e(\infty) = \lim_{t \rightarrow \infty} \mathbf{e}(t) = \lim_{t \rightarrow \infty} [\mathbf{r}(t) - \mathbf{c}(t)]$$

- Using the Final Value Theorem (i.e. $\lim_{t \rightarrow \infty} f(t) = f(\infty) = \lim_{s \rightarrow 0} sF(s)$)

$$e_{ss} = \lim_{s \rightarrow 0} sE(s)$$

Static Error Constant

- **Static error constants:** numerical values that quantify a control system's steady-state accuracy in tracking reference inputs.

- $e_{ss} = \lim_{s \rightarrow 0} sE(s)$

- $e_{ss} = \lim_{s \rightarrow 0} \frac{sR(s)}{1 + G(s)H(s)}$

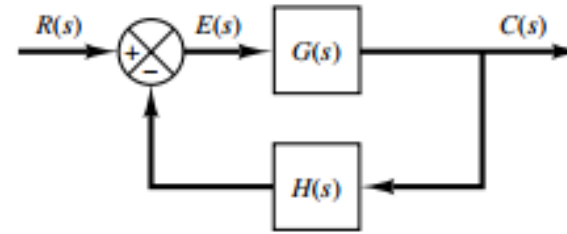


Figure 11:- Closed loop system [11]

$$E(s) = R(s) - H(s)C(s)$$

$$C(s) = E(s)G(s)$$

$$E(s) = R(s) - H(s)E(s)G(s)$$

$$E(s)(1 + G(s)H(s)) = R(s)$$

$$E(s) = \frac{R(s)}{1 + G(s)H(s)}$$

Static Error Constant

Purpose: Measure a system's ability to track different input types with zero steady-state error.

Constant	Input Type	Measures Accuracy for
K_p - Position Constant	Step Input	Constant position
K_v - Velocity Constant	Ramp Input	Constant velocity
K_a - Acceleration Constant	Parabolic Input	Constant acceleration

Relationship between input, system type, static error constants and steady state errors

Input	Steady-state error formula	Type 0		Type 1		Type 2	
		Static error constant	Error	Static error constant	Error	Static error constant	Error
Step, $u(t)$	$\frac{1}{1 + K_p}$	$K_p = \text{Constant}$	$\frac{1}{1 + K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v = \text{Constant}$	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a = \text{Constant}$	$\frac{1}{K_a}$

Where

$$K_p = \lim_{s \rightarrow 0} G(s)H(s)$$

$$K_v = \lim_{s \rightarrow 0} sG(s)H(s)$$

$$K_a = \lim_{s \rightarrow 0} s^2G(s)H(s)$$

Figure 12: Relationship between input, system type, static error constants and steady state errors. [12]

Worked Examples & Problem Solving

Example 1: Identifying System Order & Parameters

Classify these systems (type, order) and identify key parameters:

1. $G(s) = \frac{5}{2s+2}$

2. $G(s) = \frac{100}{4s^2+20s+100}$

3. $G(s) = \frac{4(s+2)}{(s+1)(s+3)(s+5)}$

4. $G(s) = \frac{s^2+1}{s(s+2)(s+3)}$

1. *First order, $K=5/2$, $\tau=1$ sec, type 0.*

2. *Second order, $\omega_n=5$, $\zeta=0.5$ under damped, type 0.*

3. *Third order, type 0.*

4. *Third order, type 1.*

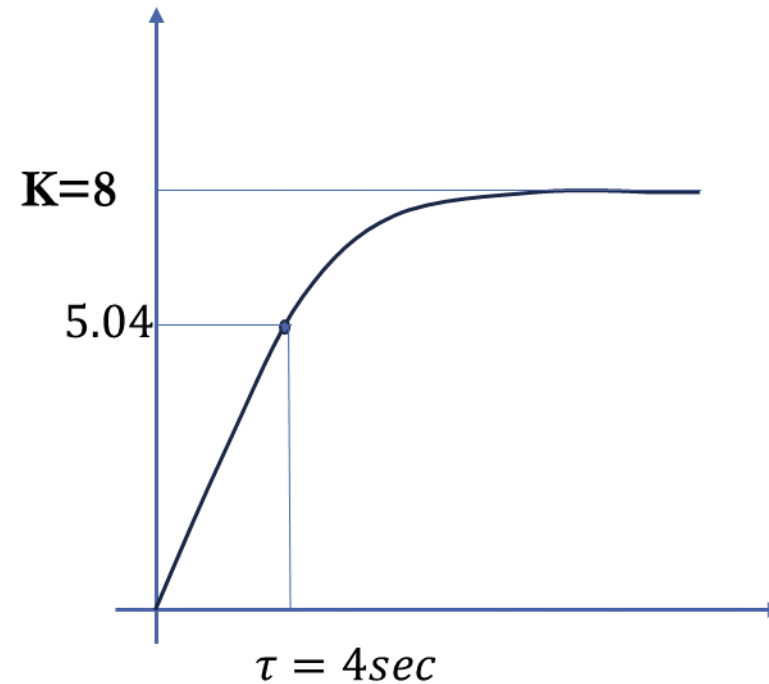
Worked Examples & Problem Solving

Example 2: First-Order Step Response Sketching

Problem: A thermal system has

transfer function $G(s) = \frac{8}{4s+1}$

Sketch its step response.



Worked Examples & Problem Solving

Example 3: First-Order Parameter Calculation

- **Problem:** A liquid level system reaches 63% of final height in 12 seconds after a valve opening. The final level is 2 meters.

- Find τ , K
- Calculate T_r and T_s
- Write the transfer function

- **Solution:**

$$\tau = 12 \text{ seconds (given)}$$

$$K = 2 \text{ (final value for unit step)}$$

$$T_r \approx 2.2 \tau = 2.2 \times 12 = 26.4\text{s}$$

$$T_s \approx 4 \tau = 4 \times 12 = 48\text{s}$$

$$G(s) = \frac{2}{12s + 1}$$

Worked Examples & Problem Solving

Example 4: Second-Order Parameter Extraction

Problem: A mass-spring-damper system has transfer function:

$$G(s) = \frac{64}{s^2 + 4s + 64}$$

- Find:
 - Natural frequency (ω_n)
 - Damping ratio (ζ)
 - Damped natural frequency (ω_d)
 - Predict the response type

Worked Examples & Problem Solving

Example 4: Second-Order Parameter Extraction

$$G(s) = \frac{64}{s^2 + 4s + 64} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- $\omega_n^2 = 64 \Rightarrow \omega_n = 8 \text{ rad/s}$
- $2\zeta\omega_n = 4 \Rightarrow \zeta = 4/(2 \times 8) = 0.25$
- $\omega_d = \omega_n \sqrt{1 - \zeta^2} = 8\sqrt{(1 - 0.25^2)} = 7.75 \frac{\text{rad}}{\text{s}}$
- Underdamped ($0 < \zeta < 1$)

Worked Examples & Problem Solving

Example 5: Response Prediction from ζ

Problem: Predict the step response

characteristics for

1. $\zeta = 0.2$ (Suspension system)
2. $\zeta = 1.0$ (Door closer)
3. $\zeta = 2.0$ (Overdamped mechanical system)

Analysis:

1. $\zeta = 0.2$: Large overshoot oscillatory, slow settling
2. $\zeta = 1.0$: No overshoot, fastest non-oscillatory response
3. $\zeta = 2.0$: Very slow, sluggish, no overshoot

Worked Examples & Problem Solving

Example 6: Steady-State Error Calculation

- **Problem:** For system with:

$$G(s) = \frac{10}{(s + 1)(s + 5)}$$

1. Step input $r(t) = 5 \cdot u(t)$
2. Ramp input $r(t) = 2t$

- Calculate steady-state error for both cases.

- **Solution:**

1. $r(t) = 5 \cdot u(t) \rightarrow R(s) = \frac{5}{s}$

$$e_{ss} = \lim_{s \rightarrow 0} \frac{sR(s)}{1 + G(s)H(s)}$$

$$= \lim_{s \rightarrow 0} \frac{s * \frac{5}{s}}{1 + \frac{10}{(s + 1)(s + 5)}} (1) = \frac{5}{3}$$

Worked Examples & Problem Solving

Cont....

Example 6: Steady-State Error Calculation

2. Ramp input $r(t) = 2t$

$$R(s) = \frac{2}{s^2}$$

$$e_{ss} = \lim_{s \rightarrow 0} \frac{sR(s)}{1+G(s)H(s)} = \lim_{s \rightarrow 0} \frac{s * \frac{2}{s^2}}{1 + \frac{10}{(s+1)(s+5)}}$$

$$e_{ss} = \lim_{s \rightarrow 0} \frac{\frac{2}{s}}{1 + \frac{10}{(s+1)(s+5)}} = \infty$$

Worked Examples & Problem Solving

Example 7: Comprehensive Design Problem

- **Problem:** A DC motor position control system has:

$$G(s) = \frac{K}{s(s + 4)}$$

- **Design requirements:**

- Zero steady-state error for step input
- Settling time < 2 seconds
- Overshoot $< 10\%$

Find:

- What type system is this originally?
- What K value gives $\zeta = 0.6$?
- Does it meet settling time requirement?
- What's the predicted overshoot?

Worked Examples & Problem Solving

Example 7: Comprehensive Design Problem

$$\bullet G(s) = \frac{K}{s(s+4)}$$

$$\bullet \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$\bullet \frac{C(s)}{R(s)} = \frac{\frac{k}{s(s+4)}}{1+\frac{k}{s(s+4)}}(1)$$

$$\bullet \frac{C(s)}{R(s)} = \frac{k}{s^2+4s+k}$$

- Originally Type 1

$$\rightarrow s^2 + 2\omega_n\zeta s + \omega_n^2$$

$$4 = 2\omega_n\zeta, \quad K = \omega_n^2$$

- $\omega_n = 4/(2 \times 0.6) = 3.33 \text{ rad/s}$
- $\omega_n^2 = K \Rightarrow K = 11.11$
- $T_s \approx 4/(\zeta\omega_n) = 4/(0.6 \times 3.33) = 2 \text{ seconds YES}$
- $M_p = e^{(-\pi \times 0.6 / \sqrt{1-0.36})} = 0.095 = 9.5\% \text{ YES}$

We know that

$$\text{Settling Time (T}_s\text{): } \frac{4}{\zeta\omega_n}$$

$$\text{Overshoot (\%OS): } \%OS = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\%$$

Worked Examples & Problem Solving

Example 8: Design trade-off

A system has $\zeta = 0.3$
with $T_s = 8s$.

If we want $T_s = 4s$ while
keeping same ζ , what
should we change?

We know that

$$T_s = \frac{4}{\zeta \omega_n}$$

$$\Rightarrow \omega_{n1} = \frac{4}{8 * 0.3} = 1.67, \quad \text{for } T_s = 8 \text{ sec}$$

$$\Rightarrow \omega_{n1} = \frac{4}{4 * 0.3} = 3.33, \quad \text{for } T_s = 4 \text{ sec}$$

The natural frequency has to be doubled.

Summary

- Time Domain Analysis evaluates control systems using **standard test inputs**: impulse, step, ramp, and parabola.
- **Total system response** consists of transient (decaying) and steady-state (persistent) components.
- **First-order systems** are defined by time constant (τ), while **second-order systems** are governed by damping ratio (ζ), determining key specs like overshoot and settling time.
- System Type and static error constants predict **steady-state error**. These principles form the core of classical control analysis.

References

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