Course: Regulation and control

Lecture 5: Time Domain Analysis II

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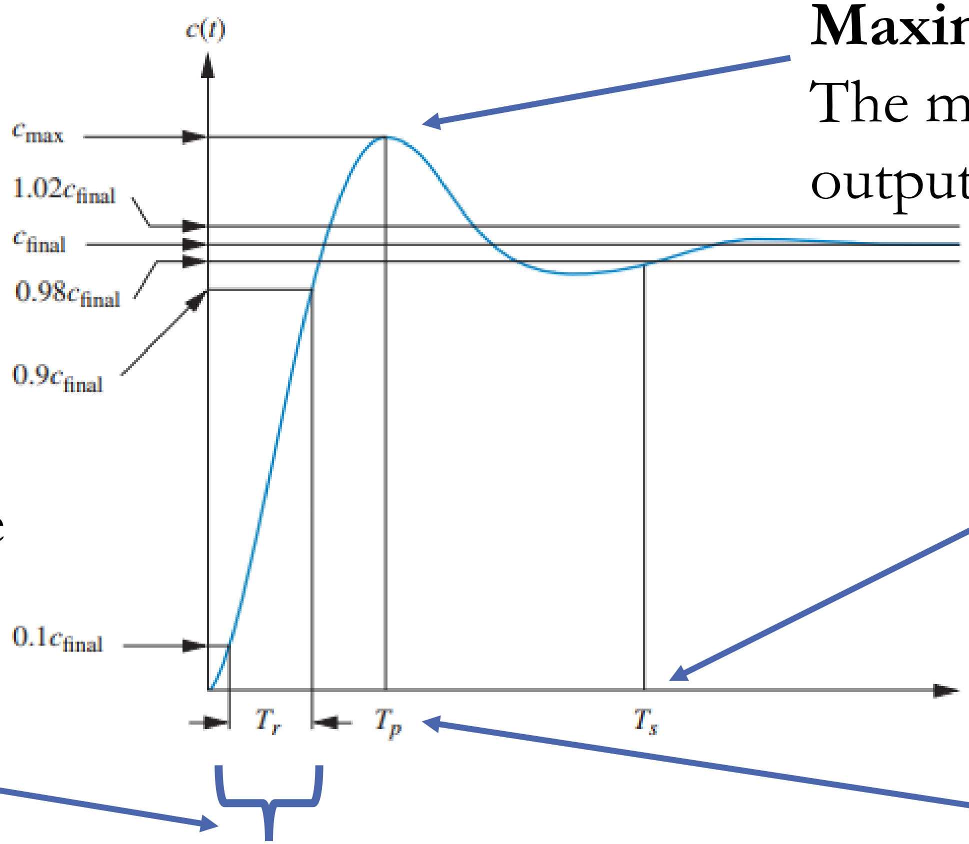
Recap: Underdamped Second-Order Step Response

Maximum percent overshoot: The maximum percentage the output exceeds the final value.



Settling time: Time for the response to enter and stay within a ±2% tolerance band of the final value.

Rise time: Time to go from 10% to 90% of the final value.

Peak time: Time to reach the first peak overshoot.

Fig 1: Response of second-order (under-damped) system. [1]

Performance Specifications: An Overview

- **Rise Time (T_r):** Speed measure: Faster systems have shorter rise times
- **Peak Time (T_p):** Speed measure: Time to reach first overshoot peak, Indicates how quickly system responds to maximum deviation
- **Maximum Overshoot (M_p):** Stability measure: Peak percentage above final value, Lower values indicate better damping and stability.
- **Settling Time (T_s):** Convergence measure: Time to stay within ±2% of final value, Combines speed and damping performance.

Rise Time (Tr)

- The time required for the system's response to go from 10% to 90% of its final steady-state value after a step input.
- It Measures System Speed – How quickly the system initially reacts and approaches its target.
- Critical for applications requiring fast response (e.g., robotics, vehicle control, high-speed manufacturing).
    - **Shorter rise time = faster system.**

Rise Time, Key Relationships

- **Inversely proportional to system bandwidth** – faster systems have higher bandwidth.
- For a **second-order system**, approximated as:

$$T_r = \frac{2.2}{\zeta \omega_n}$$

Design Trade-Off:

- Decreasing(improving) rise time often **increases overshoot** – requiring a balance between speed and stability.

Peak Time (Tp)

- The time required for the system's response to reach its **very first peak overshoot** after a step input.
- **Response Speed to Maximum Deviation** – How quickly the system shows its maximum overshoot.
- Critical for understanding **how fast the system reaches its worst-case deviation** from the desired value.
- Important in systems where **peak stress or maximum temporary error** must be minimized.

Peak Time (Tp), Key Relationships

$$T_p = \frac{\pi}{\omega_d}, \omega_d = \omega_n\sqrt{1-\zeta^2}$$

- **Inversely proportional to damped natural frequency** (ω_d)
- Higher ω_n = **shorter peak time** = faster response
- Lower damping ratio (ζ) = slightly shorter peak time

Design Consideration:

- Reducing peak time often requires **increasing system bandwidth,** which may **increase overshoot** and **reduce stability margins**.

Peak Overshoot (Mp)

- The maximum amount the output exceeds its final value, expressed as a percentage of the final value.
- **System Damping & Stability** – How much the system "rings" or oscillates before settling.
- Critical for applications where **exceeding the target value is dangerous** (e.g., crane operations, robotic arms).
- Indicates how **aggressive** or **conservative** the system response is.

Peak Overshoot (Mp), Key Relationships

$$\%M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} x100\%$$

- **Depends only on damping ratio (ζ)**
- Lower ζ = **higher overshoot** (more oscillatory)
- Higher ζ = **lower overshoot** (more sluggish)

Design Guidelines:

- **ζ=0.7** gives ≈ **4.3% overshoot** (optimal balance for many systems)
- **ζ=1.0** gives **0% overshoot** (critically damped)

Settling Time (Ts)

- Time for the response to reach and stay within a **tolerance band** (typically ±2%) around the final value.
- **Total Convergence Speed** – How long it takes for all transients to effectively die out.
- Critical for applications where the **system must stabilize quickly** to its final value (e.g., data acquisition, positioning systems).
- Determines the **effective operating speed** of the system.

Settling Time (Ts), Key Relationships

$$T_s = \frac{4}{\zeta \omega_n}$$

- **Inversely proportional to $\zeta\omega_n$** (the product of damping ratio and natural frequency)
- To reduce settling time: **increase ζ** or **increase ω_n**

Design Trade-Off:

- Can be improved by increasing system bandwidth, but this may **reduce stability margins** and **increase noise sensitivity**.

The Two Key Parameters: ζ and ω_n

- All four specifications (Tr, Tp, Mp, Ts) are determined by just two parameters.
- **ζ (Damping Ratio):** Primarily controls Mp and the "shape" of the response.
- **ω_n (Natural Frequency):** Primarily controls the speed (Tr, Tp, Ts).

Effect of ζ on response of second order system

- **Small ζ (0.1-0.4):** Fast but oscillatory response with large overshoot and long settling.
- **Medium ζ (0.5-0.8):** Optimal balance - good speed with minimal overshoot and fast settling. ζ=0.7 is ideal.

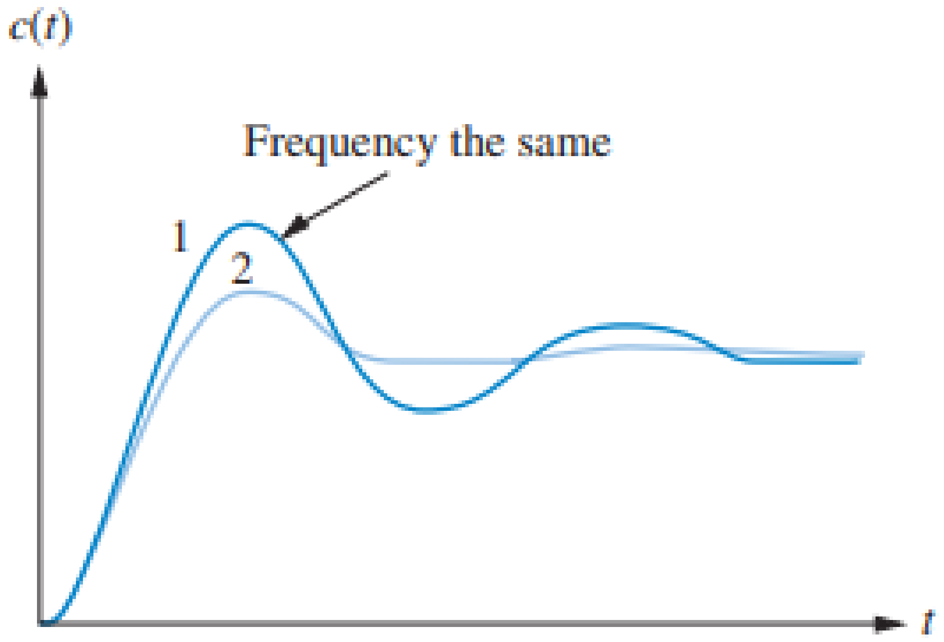


Fig 2: Effect of damping ratio on system response (2nd –order) [2]

Effect of ζ on response of second order system cont…

- **ζ = 1:** Fastest non-oscillatory response - no overshoot but slower than underdamped.
- **Large ζ (>1):** Slow, sluggish response with no overshoot but very long settling time.
- **Trade-off:** Increasing ζ reduces overshoot but increases response time.

Effect of ω_n on response of second order system

- Higher ω_n makes everything faster—rise time, peak time, and settling time all decrease.
- It speeds up the response without affecting overshoot,
- In short, a higher ω_n means a uniformly faster response.

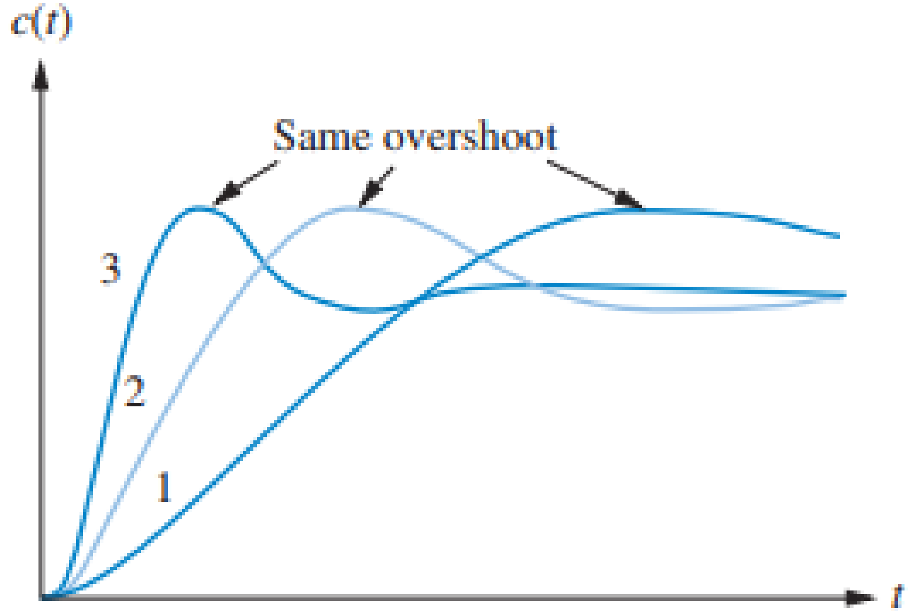


Fig 3: Effect of ω_n on system response (2^{nd} –order) [3]

The Design Trade-Off: Speed vs. Overshoot

- **Problem**: Increasing ζ reduces overshoot (good) but can increase rise time (bad).
- **Solution:** To speed up the system without changing overshoot, you must increase both ζ and ω_n in a specific way.

The Design Process

- From requirements, determine desired Ts, Mp, etc.
- Calculate the required ζ and ω_n.
- Design a controller to place the system's poles at the location that gives this ζ and ω_n.

Example1: Deriving Performance Specs from a Transfer Function

1. For a system with $G(s) = \frac{100}{s^2+6s+100}$

Determine

a. Damping ratio (ζ) and natural frequency (ω_n)

b. Peak time (T_p) and peak overshoot (M_p)

c. Settling time (T_s) and rise time (T_r)

a. Comparing with standard 2nd – order system

$s^2 + 6s + 100$

$= s^2 + 2\zeta\omega_n s + \omega_n^2$

$\omega_n^2 = 100, \omega_n = 10;$

$2\zeta\omega_n = 6,\ \zeta = \frac{6}{2x10} = 0.3$

$\boldsymbol{\omega_n = 10, \zeta = 0.3}$

Example1: Deriving Performance Specs from a Transfer Function cont….

b. Peak time and peak overshoot

$$T_p = \frac{\pi}{\omega_d}, \omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$\omega_d = 10\sqrt{1-0.3^2} = 9.54 rad/sec$$

$$T_p = \frac{\pi}{\omega_d} = \frac{3.142}{9.54} = \mathbf{0.33sec}$$

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} = e^{-\left(\frac{3.142x0.3}{\sqrt{1-0.3^2}}\right)} = 0.37$$

$$\%M_p = 37\%$$

c. Settling time and rise time

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.3x10} = \mathbf{1.33sec}$$

$$T_r = \frac{2.2}{\zeta\omega_n} = \frac{2.2}{0.3x10} = \mathbf{0.73sec}$$

Example 2: Finding ζ and ω_n from Specs

2. Design a second-order system with a percent overshoot of 10% or less and a settling time of 2 seconds or faster.

Step 1: Finding ζ and ω_n

From peak overshoot

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} = 10\% = 0.1$$

$$ln0.1 = -2.3 = -\frac{\pi\zeta}{\sqrt{1-\zeta^2}}$$

$$\Rightarrow \pi\zeta = 2.3\sqrt{1-\zeta^2}$$

$$(\pi^2 + 5.29)\zeta^2 = 5.29$$

$$\zeta \approx 0.6$$

Example 2: Finding ζ and ω_n from Specs Cont....

From settling time

$$T_s = \frac{4}{\zeta\omega_n} = 2sec$$

$$\omega_n = \frac{4}{2\zeta} = \frac{4}{2x0.6} = 3.33\frac{rad}{sec}$$

Checking for %Mp @ ζ =0.6

%Mp=9.47%<10%

it is acceptable

$$G(s)=\frac{3.33^2}{s^2+2(0.6x3.33)s+3.33^2}$$

$$G(s)=\frac{11}{s^2+4s+11}$$

Example 3: The Effect of Changing ω_n

3. We have two second-order systems with same damping ratio of ζ but undamped natural frequency $\omega_{n2} = 2\omega_{n1}$

Compare their

- Percent overshoot
- Rise time
- Settling time and
- Peak time

Example 3: The Effect of Changing ω_n Cont....

Percent overshoot only dependent on the damping ratio so both will have same percent overshoot.

Rise time $T_{r1} = \frac{2.2}{\zeta\omega_{n1}}$, $T_{r2} = \frac{2.2}{\zeta\omega_{n2}}$ $but\ \omega_{n2} = 2\omega_{n1}$

$\Rightarrow T_{r1} = 2\, T_{r2}$

For the same damping ratio if the **undampened natural frequency doubled** the **Rise time, Settling time** and **Peak time** will be two times less (i.e. the response becomes more responsive).

Example 4: Designing Gain K for a Desired Damping Ratio

4. A unity feedback system has G(s)=K/s(s+4) ,find the value of K that gives a damping ratio $\zeta = 0.7$.

- **Solution:**

Step 1: Find the closed-loop transfer function.

$$T(s) = \frac{G(s)}{1+G(s)H(s)} \Rightarrow \frac{\frac{K}{s(s+4)}}{1+\frac{K}{s(s+4)}(1)} = \frac{K}{s^2+4s+K}$$

Example 4: Designing Gain K… Cont….

- **Step 2: Compare with the standard second-order form.**

  Standard form: $s^2+2\zeta\omega_n s+\omega n^2$

  Our system: s^2+4s+K

  By comparison:

  $2\zeta\omega n=4$

  $\omega_n{}^2=K$

Example 4: Designing Gain K… Cont.…

- **Step 3: Substitute the desired ζ and solve for K.**

  Given $\zeta = 0.7$

  $2(0.7)\omega_n=4 \Rightarrow 1.4\omega_n=4 \Rightarrow \boldsymbol{\omega_n=41.4\approx 2.857 rad/s}$

  Now find K:

  $K=\omega_n^2=(2.857)^2\approx 8.16$

- **Final Answer:** K=8.16 gives $\zeta = 0.7$,

Example 5: Interpreting Step Response Plot

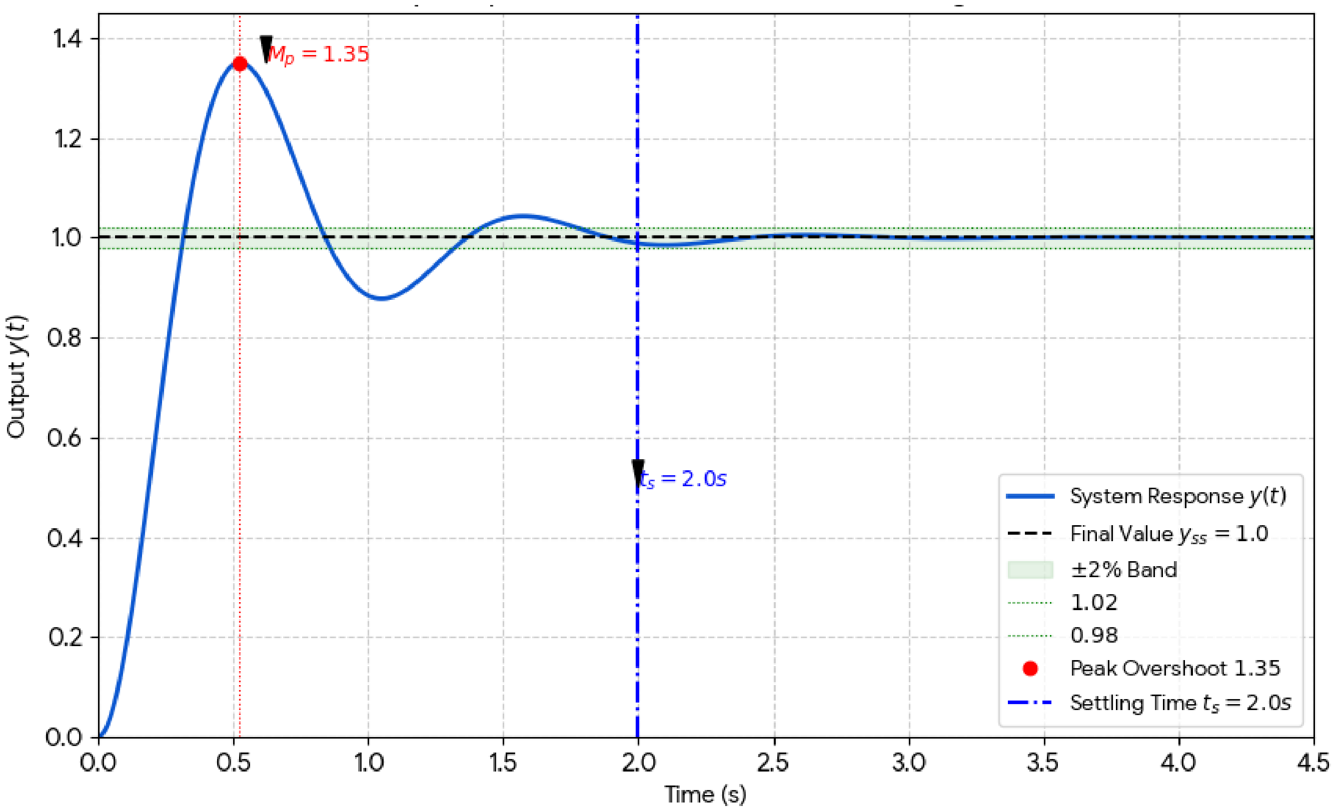


Fig 4: Step response of 2^{nd} –order underdamped system [4]

Determine:

- Settling Time (T_s)
- Peak Time (Tp)
- Percent Overshoot (M_p)
- Estimate ζ and ω_n

Example 5: Interpreting Step Response Plot cont…

- **Step 1: Direct measurements from description.**
  - Settling Time T_s = 2.0s
  - Peak Time T_p = 0.5s
  - Percent Overshoot = (1.35−1.0)×100%=35%
- **Step 2: Estimate ζ from overshoot.**
  - $M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$=0.35

  solving for ζ≈0.32

We have $T_s = \frac{4}{\zeta\omega_n}$

$$\omega_n = \frac{4}{T_s\zeta}$$

$$\Rightarrow \frac{4}{2x0.32} = 6.25 rad/sec$$

Example 5: Interpreting Step Response Plot cont…

- Verify using T_P

$$T_p = \frac{\pi}{\omega_d}, \omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$T_P = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{3.142}{6.25\sqrt{1-0.32^2}} = 0.56\text{sec}$$

Tp≈0.5sec Verified

Practice examples

Q1 Which change will REDUCE settling time T_s?

A) Increase ζ only
B) Increase ω_n only
C) Increase both ζ and ω_n
D) Decrease ω_n

Q2 A system has 60% overshoot. What is the approximate damping ratio?

A) 0.1
B) 0.2
C) 0.4
D) 0.6

Q3 Which pole location gives fastest response without overshoot?

A) $-1 \pm 2j$
B) -3
C) $-2 \pm 1j$
D) $-1, -4$

Q4 A system's step response has T_p=1s and M_p=25%. What is ω_n?

A) 1.0 rad/s
B) 2.5 rad/s
C) 3.4 rad/s
D) 4.0 rad/s

Answers for practice examples

Q1 **Answer: B)** Increase ω_n only (and optionally C)

$T_s \approx 4/(\zeta\omega_n)$

Increasing ζ alone increases denominator

Increasing ω_n alone increases denominator

Increasing both is most effective

Q2 **Answer: B)** 0.2

$M_p = 60\% \Rightarrow \zeta \approx 0.16$ (closest to 0.2)

Q3 **Answer: D)** -1, -4 (critically damped equivalent)

Fastest without overshoot = repeated real poles

Dominant pole at -1 gives $T_s \approx 4s$

Q4 **Answer: C)** 3.4 rad/s

$M_p=25\% \Rightarrow \zeta \approx 0.4$

$T_p=1s \Rightarrow \omega\_d = \pi/1 \approx 3.14$ rad/s

$\omega\_n = \omega\_d/\sqrt{(1-\zeta^2)} = 3.14/\sqrt{(1-0.16)}$

$\approx 3.14/0.916 \approx 3.43$ rad/s

Summary

- The dynamic response of a second-order system is governed by two key parameters:
  - **Damping Ratio (ζ):** Primarily determines the overshoot and oscillation.
  - **Natural Frequency (ω_n):** Primarily controls the speed of the response.
- The core design challenge is a fundamental trade-off between speed versus stability:
  - Increasing ω_n speeds up the response.
  - Increasing ζ reduces overshoot but can make the system more sluggish.
- An optimal balance is often found with a damping ratio of $\boldsymbol{\zeta \approx 0.7}$.

References

[1] Response of second-order (under-damped) system, Norman S. Nise - Control Systems Engineering (2015, Wiley)

[2] Effect of damping ratio on system response (2nd –order), Norman S. Nise - Control Systems Engineering (2015, Wiley)

[3] Effect of ωn on system response (2nd –order), Norman S. Nise - Control Systems Engineering (2015, Wiley)

[4] "Step response of 2nd –order underdamped system, Engineering graph: step response with overshoot to 1.35, settling to 1.0. Fast rise, enters a narrow ±2% band at t=2.0s." prompt, Gemini, Google, 2 Oct. 2025