



# Course: Regulation and control

**Lecture 6:** *Stability Analysis I*

**Lecturer:** Chalachew Werku

# Stability

- Stability

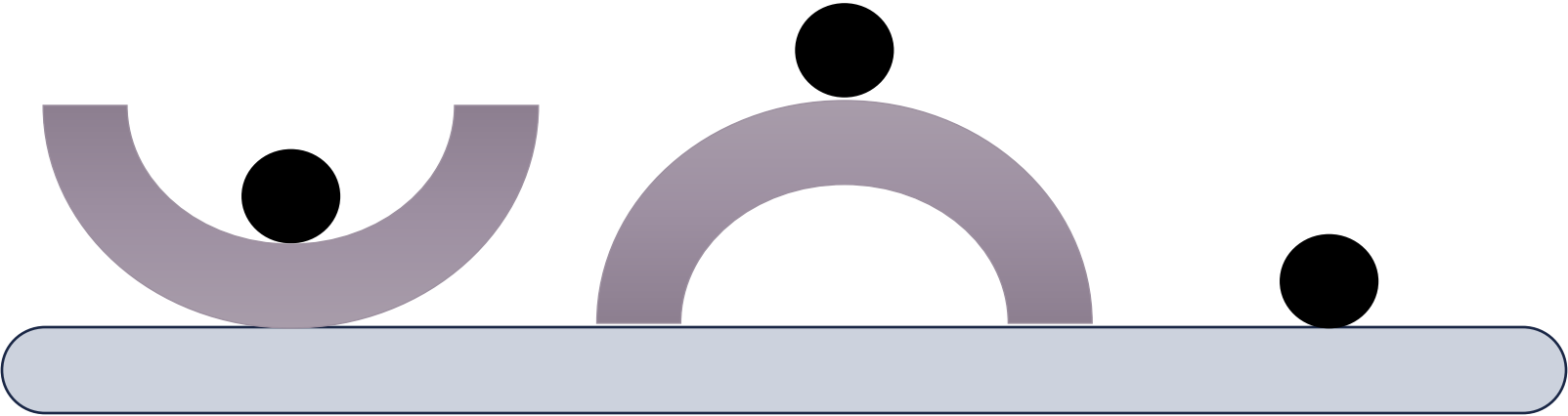


Fig 1: Stability :-Marble and bowl [1]

# Stability

Stability for linear, time-invariant systems. Using the natural response a system is

- **Stable** if the response approaches zero as time approaches infinity.
- **Unstable** if the response approaches infinity as time approaches infinity.
- **Marginally stable** if the response neither decays nor grows but remains constant or oscillates.

# Stability *BIBO*

Using the total response (BIBO):

- A system is **stable** if every bounded input yields a bounded output.
- A system is **unstable** if any bounded input yields an unbounded output
  - **Question:** "Will my system's output remain bounded for a bounded input?"
  - **Answer:** Determined entirely by the location of the system's poles.

# The Pole-Location Stability Criterion

- **Stable:** All poles have negative real parts (Left-Half Plane, LHP).
- **Marginally Stable:** Non-repeated poles on the imaginary axis.
- **Unstable:** At least one pole has a positive real part (Right-Half Plane, RHP), or repeated poles on the imaginary axis.

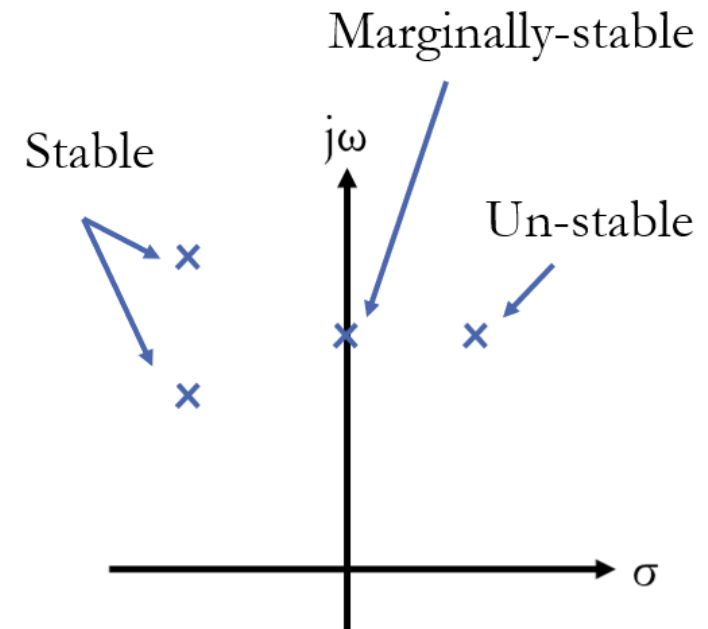


Fig 2: Stability and pole location [2]

# The Pole-Location Stability Criterion

Cont....

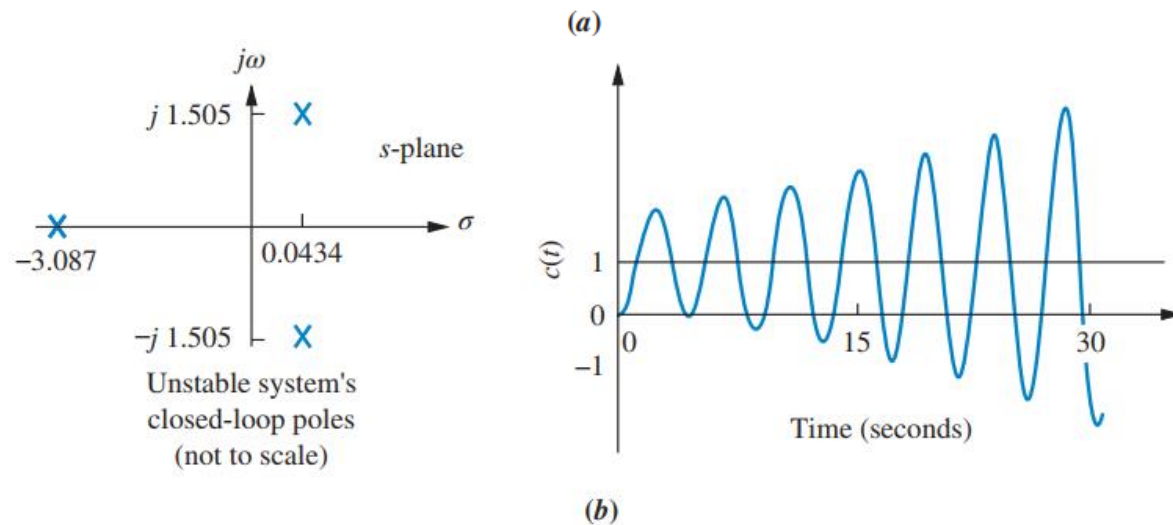
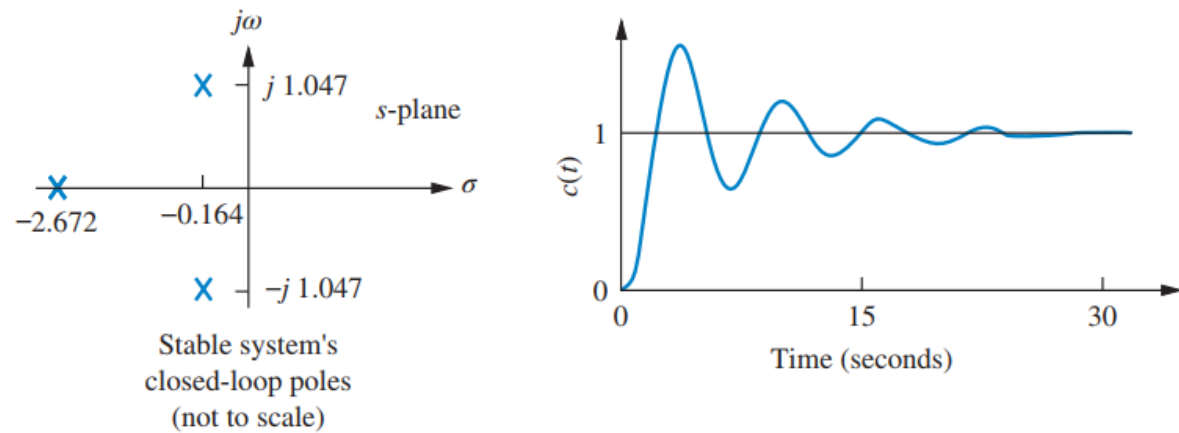


Fig 3: Closed-loop poles and response, a. stable system b. unstable system [3]

# Why Do We Need Routh-Hurwitz?

- Simple systems (1st/2nd order): Easy to find poles analytically
- Complex systems ( $\geq 3$ rd order): Factoring becomes difficult or impossible
- Routh-Hurwitz advantage: Determines stability without solving for roots
- Design applications: Find parameter ranges (like gain  $K$ ) for stability

# The Routh-Hurwitz Criterion

- A tabular method to determine the number of roots of a polynomial that are in the Right-Half Plane.
- Start with the characteristic polynomial:

$$a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0$$

- Write the first two rows.
- Fill in the remaining rows using the determinant formula.

# Constructing the Routh Array - Step 1

- Start with the characteristic polynomial:

$$a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0$$

- Let's say we have  $a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$
- Write the first two rows.

$s^4$	$a_4$		$a_2$		$a_0$
$s^3$	$a_3$		$a_1$		0
$s^2$					
$s^1$					
$s^0$					

Fig 4: First two rows of RH table [4]

# Constructing the Routh Array - Step 2

- Fill in the remaining rows using the determinant formula.

$s^4$	$a_4$	$a_2$	$a_0$
$s^3$	$a_3$	$a_1$	0
$s^2$	$b_1 = \frac{a_3 a_2 - a_4 a_1}{a_3}$	$b_2 = \frac{a_3 a_0 - a_4 \cdot 0}{a_3}$	$\frac{a_3 \cdot 0 - a_4 \cdot 0}{a_3} = 0$
$s^1$	$c_1 = \frac{b_1 a_1 - a_3 b_2}{b_1}$	$\frac{b_2 \cdot 0 - a_1 \cdot 0}{b_1} = 0$	0
$s^0$	$d_1 = \frac{c_1 b_2 + b_1 \cdot 0}{c_1}$	0	0

Fig 5: Completing RH table [5]

# Interpreting the Routh Array

- The necessary and sufficient condition the system to be **stable** is there **SHOULD NOT BE** any sign change in the first column.
- If there is any sign change the system is unstable.
- The number of sign change shows the number of poles in the RHP.
- If there is zero the system is marginally stable.

$s^4$	$a_4$	$a_2$	$a_0$
$s^3$	$a_3$	$a_1$	$0$
$s^2$	$b_1$	$b_2$	$0$
$s^1$	$c_1$	$0$	$0$
$s^0$	$d_1$	$0$	$0$

Fig 6: First column of RH array [6]

# Example 1

- Check stability of a system given by

a.  $G(s) = \frac{12}{s^2 + 4s + 2}$  and  $H(s) = 0.3$ ;

b.  $\frac{C(s)}{R(s)} = \frac{12}{s^3 + 5s^2 + 10s + 2}$

c.  $\frac{C(s)}{R(s)} = \frac{s^3 + 5s^2 + 10s + 2}{s^5 + 3s^4 + 3s^3 + 6s^2 + 5s + 3}$

d.  $\frac{C(s)}{R(s)} = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$

# Example 1

## *Solution*

a.  $G(s) = \frac{12}{s^2 + 4s + 2}$  and  $H(s) = 0.3$ ;

Find the overall tf

$$\frac{G(s)}{1 + G(s)H(s)} = \frac{\frac{12}{s^2 + 4s + 2}}{1 + \frac{12}{s^2 + 4s + 2} \cdot 0.3} = \frac{12}{s^2 + 4s + 5.6}$$

# Example 1

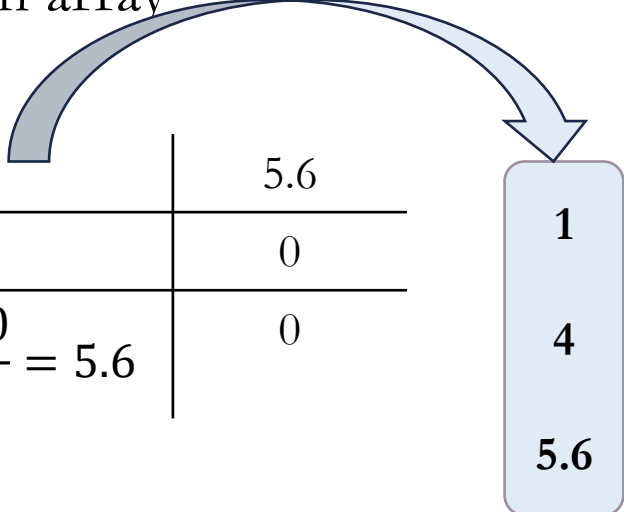
## *Solution*

Cont....

We have  $s^2 + 4s + 5.6$

Writing it in Routh array

$s^2$	1	5.6
$s^1$	4	0
$s^0$	$\frac{4 * 5.6 - 1 * 0}{4} = 5.6$	0



The diagram shows a curved arrow pointing from the '1' in the  $s^2$  row to a vertical box containing the numbers 1, 4, and 5.6. This box represents the first column of the Routh array.

In the first column all the elements are same sign so the system is stable.

# Example 1

## *Solution*

$$b. \frac{C(s)}{R(s)} = \frac{12}{s^3 + 5s^2 + 10s + 2}$$

Taking the denominator and constructing Ruth's array

$s^3$	1	10
$s^2$	5	2
$s^1$	$\frac{5 * 10 - 1 * 2}{5} = 9.6$	$\frac{5 * 0 - 1 * 0}{5} = 0$
$s^0$	$\frac{9.6 * 2 - 5 * 0}{9.6} = 2$	0

## Cont....

- All the elements of the first column are positive this system is also stable.

# Example 1

## Solution

$$C. \quad \frac{C(s)}{R(s)} = \frac{s^3 + 5s^2 + 10s + 2}{s^5 + 3s^4 + 3s^3 + 6s^2 + 5s + 3}$$

- We use only the denominator
- We have a sign change in the first column, so the system is unstable and
- we have two poles lie on the RHP of s-plane.

+1   +3   +1   -6   +4.5   3



Cont....

$s^5$	1	3	5
$s^4$	3	6	3
$s^3$	$\frac{3 * 3 - 1 * 6}{3} = 1$	$\frac{3 * 5 - 1 * 3}{3} = 4$	0
$s^2$	$\frac{1 * 6 - 3 * 4}{1} = -6$	$\frac{1 * 3 - 3 * 0}{1} = 3$	0
$s^1$	$\frac{-6 * 4 - 1 * 3}{-6} = 4.5$	0	0
$s^0$	$\frac{4.5 * 3 - (-6 * 0)}{4.5} = 3$	0	0

# Example 1

## *Solution*

Cont....

$$d. \frac{C(s)}{R(s)} = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

$s^5$	1	3	5
$s^4$	2	6	3
$s^3$	$\frac{2 * 3 - 1 * 6}{2} = 0$	$\frac{2 * 5 - 1 * 3}{1} = 7$	0
$s^2$	$\frac{0 * 6 - 2 * 7}{0} = \cancel{E}$		
$s^1$			
$s^0$			

# Routh-Hurwitz: Special Case 1 - Zero in the First Column

- **Problem:** Division by zero when calculating next row
- **Solution 1:** Replace zero with small positive number  $\varepsilon$ 
  - Complete the array with  $\varepsilon$
  - Take limit as  $\varepsilon \rightarrow 0^+$
  - Determine sign changes
- **Solution 2:** Reverse coefficients: reverse coefficients of the polynomial.

# Example 1

Cont....

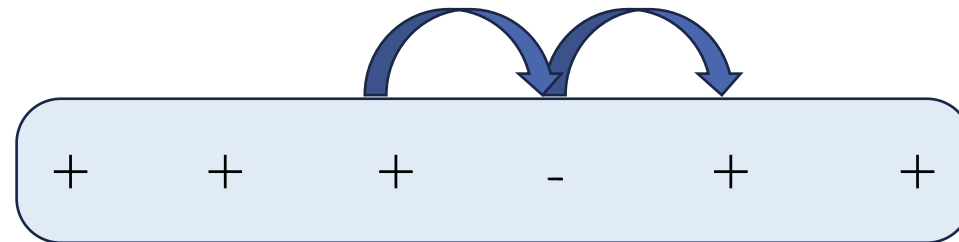
*Solution: Special Case 1: using  $\varepsilon$*

$$d. \frac{C(s)}{R(s)} = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

$s^5$	1	3	5
$s^4$	2	6	3
$s^3$	$\frac{2 * 3 - 1 * 6}{2} = \varepsilon$	$\frac{2 * 5 - 1 * 3}{1} = 7$	0
$s^2$	$\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon * 6 - 2 * 7}{\varepsilon} = a$	$\frac{3\varepsilon - 2 * 0}{\varepsilon} = 3$	0
$s^1$	$\lim_{\varepsilon \rightarrow 0} \frac{7a - 3\varepsilon}{a} = b$	0	
$s^0$	3		

- $a = \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon * 6 - 2 * 7}{\varepsilon} = -ve$

- $b = \lim_{\varepsilon \rightarrow 0} \frac{7a - 3\varepsilon}{a} = +ve$



- We have two sign changes so the system is unstable.

# Example 1

Cont....

*Solution: Special Case 1: Reverse coefficients*

$$d. \frac{C(s)}{R(s)} = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

The original polynomial

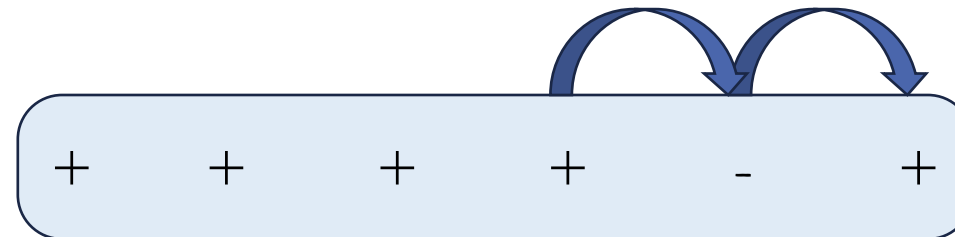
$$s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3$$

*So if we inverse the coefficients*

$$3s^5 + 5s^4 + 6s^3 + 3s^2 + 2s + 1$$

*Then we build Ruth array*

$s^5$	3	6	2
$s^4$	5	3	1
$s^3$	$\frac{5 * 6 - 3 * 3}{5} = 4.2$	$\frac{5 * 2 - 3 * 1}{5} = 1.4$	0
$s^2$	$\frac{4.2 * 3 - 5 * 1.4}{4.2} = 1.33$	$\frac{4.2 * 1 - 0 * 5}{4.2} = 1$	0
$s^1$	$\frac{1.33 * 1.4 - 4.2 * 1}{1.33} = -1.75$	0	
$s^0$	1		



The system is unstable

## Example 2

$$b. \frac{C(s)}{R(s)} = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56}$$

- We have row of zero

$s^5$	<b>1</b>	<b>6</b>	<b>8</b>
$s^4$	7	42	56
$s^3$	$\frac{7 * 6 - 1 * 42}{7} = 0$	$\frac{7 * 8 - 1 * 56}{7} = 0$	$\frac{7 * 0 - 1 * 0}{7} = 0$
$s^2$			
$s^1$			
$s^0$			

## Routh-Hurwitz: Special Case 2 - Entire Row is Zero

- Indicates: Symmetrical root locations about origin
- Solution:
  1. Form auxiliary equation from previous row
  2. Differentiate auxiliary equation
  3. Replace zero row with coefficients from derivative
  4. Continue the array

## Example 2

*Solution: Special Case 2: using auxiliary equation*

$$b. \frac{C(s)}{R(s)} = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56}$$

$s^5$	1	6	8
$s^4$	7	42	56
$s^3$	$\frac{7 * 6 - 1 * 42}{7} = 0$	$\frac{7 * 8 - 1 * 56}{7} = 0$	$\frac{7 * 0 - 1 * 0}{7} = 0$
$s^2$			
$s^1$			
$s^0$			

- Form auxiliary equation from previous row

$$A(x) = 7s^4 + 42s^2 + 56$$

- Differentiate auxiliary equation

$$A'(x) = 28s^3 + 84s$$

## Example 2

Cont....

*Solution: Special Case 2: using auxiliary equation*

- Replace zero row with coefficients from derivative and continue the array

$s^5$	1	6	8
$s^4$	7	42	56
$s^3$	28	84	0
$s^2$	$\frac{28 * 42 - 7 * 84}{28} = 21$	56	0
$s^1$	$\frac{21 * 84 - 28 * 56}{21} \approx 9.33$	0	0
$s^0$	56		

Even if the first column is positive since it was initially unstable the system is Marginally stable.

# Application: Finding the Range of K for Stability

**Common control problem:** Determine gain  $K$  that maintains stability

**Procedure:**

1. Form closed-loop characteristic equation with parameter  $K$
2. Construct Routh array
3. Set conditions for all first-column elements  $\neq 0$
4. Solve inequalities for  $K$  range

## Example 3: Finding Stable Gain K

$$G(s) = \frac{k}{s(1+0.6s)(1+0.4s)} \text{ and } H(s) = 0.3$$

- i. To be stable
- ii. To be marginally stable,
- iii. How much is its' frequency of sustained oscillation

## Example 3: Finding Stable Gain K

Cont....

Form closed-loop characteristic equation with parameter K

$$\frac{G(s)}{1 + G(s)H(s)} = \frac{\frac{k}{s(1+0.6s)(1+0.4s)}}{1 + \frac{k}{s(1+0.6s)(1+0.4s)}} \quad (0.3)$$

$$\frac{k}{0.24s^3 + s^2 + s + 0.3k}$$

## Example 3: Finding Stable Gain K

Cont....

Construct Routh array

$s^3$	0.24	1
$s^2$	1	$0.3k$
$s^1$	$\frac{1 * 1 - 0.24 * 0.3k}{1} = 1 - 0.072k$	0
$s^0$	$0.3k$	0

- i. To be stable all column elements should be positive

so,  $1 - 0.072k > 0$  and  $0.3k > 0$

$0.072k < 1$  and  $k > 0$

$k < 13.9$  and  $k > 0$

$$0 < k < 13.9$$

## Example 3: Finding Stable Gain K

Cont....

ii. To be marginally stable,

$$1 - 0.072k = 0 \text{ and } 0.3k = 0$$

$$k = 13.9 \text{ and } k = 0$$

iii. How much is its' frequency  
of sustained oscillation

@  $k = 13.9$  3<sup>rd</sup> row becomes row  
of zero

• Construct auxiliary equation

$$A(x) = s^2 + 0.3k = s^2 + 0.3(13.9)$$

$$A(x) = s^2 + 4.17$$

• Equating with zero and solving  
for s,

$$s^2 = -4.17$$

$$s = j2.04 \rightarrow \omega = 2.04 \text{ rad/sec}$$

# Introduction to Pole-Zero Plots

- **Purpose:** Visual representation of system dynamics
- **Poles ( $\times$ ):** Roots of denominator  $\rightarrow$  Natural response modes
- **Zeros ( $\circ$ ):** Roots of numerator  $\rightarrow$  Affect response amplitudes

# Interpreting Pole Locations

- **Real pole on real axis:** Pure exponential response (over-damped or critically-damped)
- **Complex conjugate poles:** Oscillatory response (under-damped)
- **Distance from origin:** the natural frequency  $\omega_n$
- **Real part ( $\sigma$ ):** Decay rate  $\rightarrow$  Settling time  $\approx 4/|\sigma|$
- **Angle ( $\theta$ ):** with the real axis determines the damping ratio ( $\zeta = \cos \theta$ ).

# Interpreting Pole Locations

Cont....

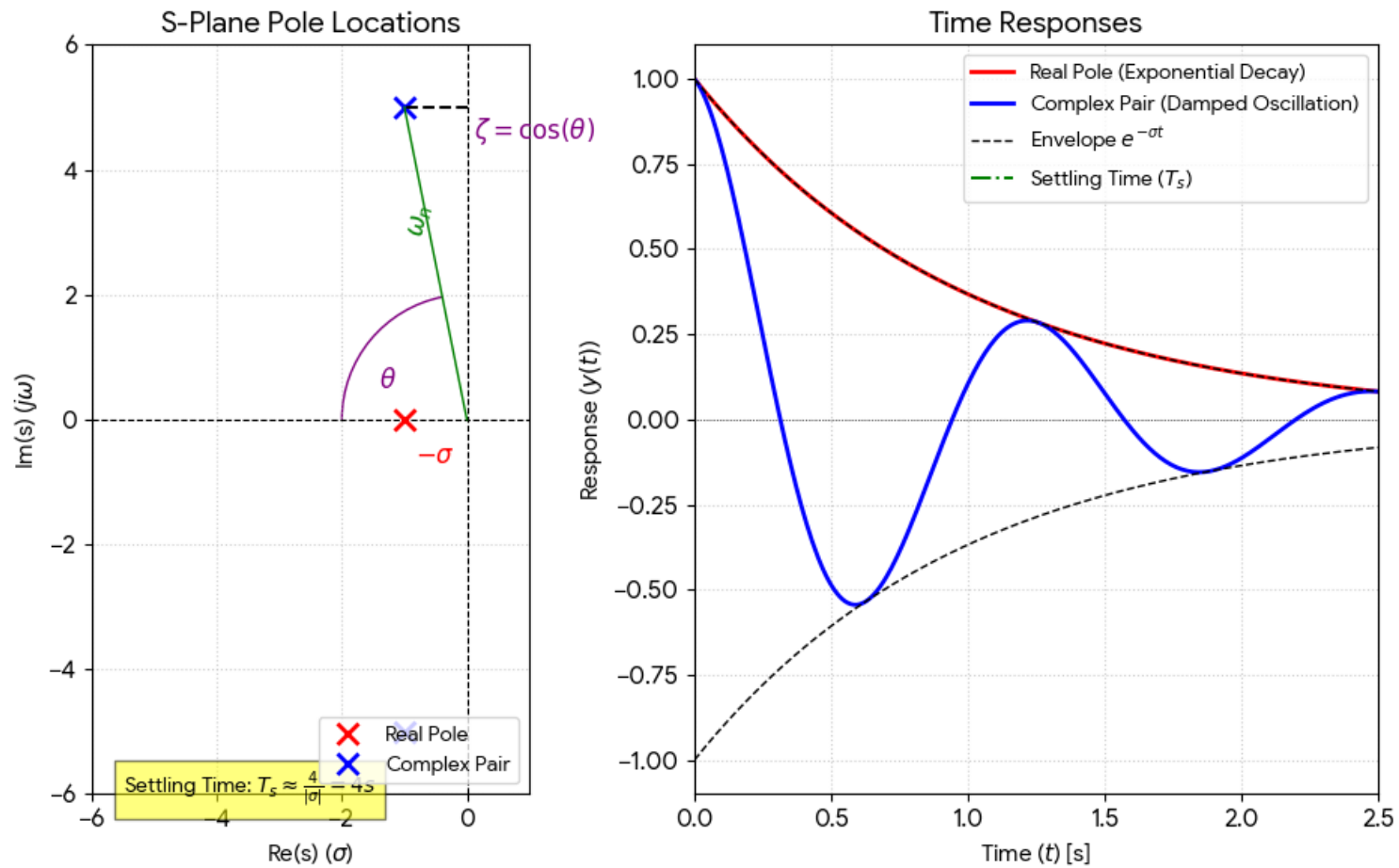


Fig 7: Interpreting pole locations [7]

# Relating Pole Location to Time Response

- A series of small plots showing a pole location and its corresponding impulse response.

1	Real, left ( $s=-a$ )	Exponential decay
2	Complex, left	Damped oscillations
3	Imaginary axis	Sustained oscillations
4	Real, right	Exponential growth
5	Complex, right	Growing oscillations

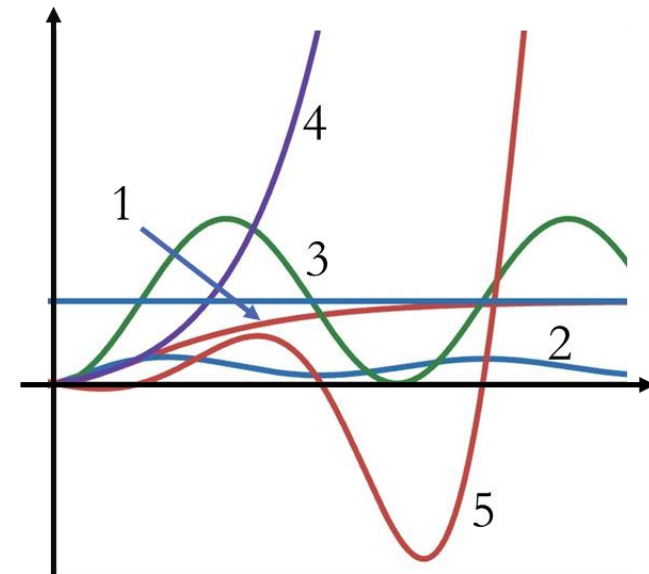


Fig 8:-Relating Pole Location to Time Response [8]

# The Effect of a Zero

- **LHP Zero:** Increases overshoot, speeds up response
- **RHP Zero:** Causes initial inverse response ("wrong-way" behavior)
- **Zero near origin:** Strong effect on transient response
- **Zero far from origin:** Minimal effect

# The Concept of Dominant Poles

- **Definition:** Poles closest to imaginary axis dominate transient response
- **Reason:** Their natural modes decay slowest
- **Rule of thumb:** Poles with real parts  $\sigma > 5\times$  farther left can often be neglected

## Example 4: Identifying Dominant Poles

- Given a 4th order system, identify which poles are dominant and approximate the response.

- $$G(s) = \frac{1512}{s^4 + 38s^3 + 423s^2 + 1458s + 1512}$$

- If we factor it

- $$G(s) = \frac{1512}{(s^2 + 5s + 6)(s^2 + 33s + 252)} =$$
$$\frac{1512}{(s+2)(s+3)(s+12)(s+21)}$$

- Real part poles  $s=-2$  and  $s=-3$  are closer to imaginary axis and they are more than 5x near so they are dominant
- So we can approximate to
- $$G(s) = \frac{1512}{(s^2 + 5s + 6)}$$
 which is 2<sup>nd</sup> order

# Synthesis: From Specs to Pole Locations

- *Performance Specifications* → *Desired Pole Regions*:
- **Settling time:**

$$T_s = \frac{4}{\zeta\omega_n} \rightarrow \sigma \leq -\frac{4}{T_s}$$

- **Overshoot:**

$$M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \rightarrow \text{Fixed } \zeta \text{ lines}$$

- **Design region:** Wedge-shaped area in LHP

# Visualizing the Design Region

- Show a region of acceptable pole location for the specification given below.
- $T_s=0.5\text{sec}$  and  $M_p=9.5\%$

# Visualizing the Design Region

## *Solution*

- A vertical line for maximum  $\sigma$  (from  $T_s$ ).
  - Radial lines for the required  $\zeta$  (from  $M_p$ ).
- $\zeta = 0.6 = \cos(\theta)$
  - $\Rightarrow \cos^{-1}(0.6) = 53^\circ$

- $\sigma = -\zeta\omega_n \rightarrow = -\frac{4}{T_s} = -8$

- $M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \rightarrow \zeta = 0.6$

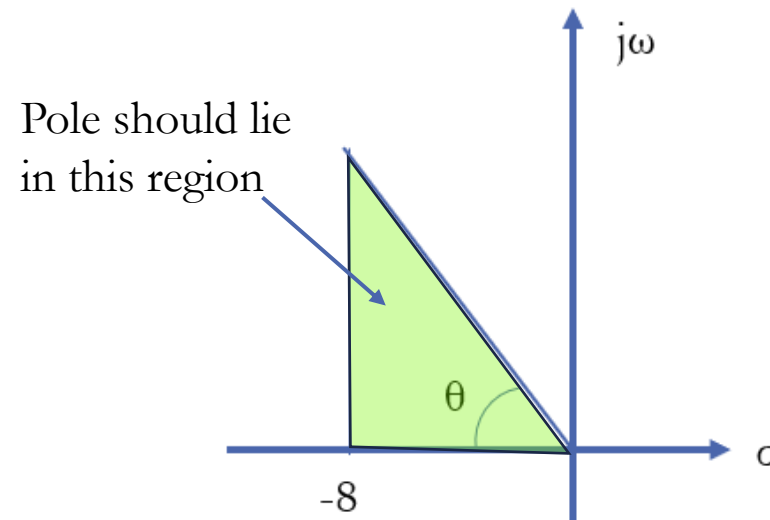


Fig 9:- Visualizing design region [9]

# Summary

- **Stability Definition & Pole Location:** System stability depends entirely on pole locations: left-half plane poles ensure stability, right-half plane poles cause instability, and imaginary axis poles create marginal stability.
- **Routh-Hurwitz Method:** This technique determines stability without polynomial factoring by analyzing sign changes in a computed array, also identifying stable parameter ranges for controllers.
- **Pole-Zero Analysis:** Pole locations determine response nature while zeros affect response shape, with dominant poles closest to the imaginary axis governing transient behavior.
- **Design Applications:** Stability analysis enables translating performance specifications into desired pole regions and ensures systems remain stable under all operating conditions.

# References

- [1] Chalachew Werku, 2025, Stability: -Marble and bowl, Figure, Self-created
  - [2] Chalachew Werku, 2025, Stability and pole location, Figure, Self-created
  - [3] Closed-loop poles and response, a. stable system b. unstable system, Norman S. Nise - Control Systems Engineering (2015, Wiley)
  - [4] Chalachew Werku, 2025, First two rows of RH table, Figure, Self-created
  - [5] Chalachew Werku, 2025, Completing RH table, Figure, Self-created
  - [6] Chalachew Werku, 2025, First column of RH array, Figure, Self-created
  - [7] Interpreting pole locations, “Plot s-plane with poles and their time responses: a real pole (exponential decay) and complex pair (damped oscillation). Label  $\sigma$  for decay rate and  $\omega_n$  for natural frequency. Annotate settling time  $\approx 4/|\sigma|$  and  $\theta$  to show damping ratio.”, Gemini, Google, 6 Oct. 2025
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