Course: Regulation and control

Lecture 7: Stability Analysis II

Lecturer: Chalachew Werku

Learning Objectives

By the end of this lecture, you will be able to:

- Differentiate between Absolute and Relative Stability.
- Define and interpret Gain Margin (GM).
- Define and interpret Phase Margin (PM).
- Explain the relationship between Margins and a system's robustness.
- Prepare for the graphical calculation techniques introduced on the next lecture.

Recap: Absolute Stability - The Cliff Edge

- **Definition**: All poles of the closed-loop transfer function lie strictly in the LHP.
- **Tool**: Routh-Hurwitz Criterion.
- **Limitation**: Binary (Yes/No) result. A stable system could still be highly oscillatory or barely stable.

- Analogy: Standing on a cliff edge.

Fig 1: Car on cliff edge. [1]

The Problem: Robustness and Performance

A Stable System is Not Always a Good System.

- **Issue 1: Robustness**: Real-world parameters change (friction, temperature, component aging). Small changes can push a barely stable pole into the RHP.
- **Issue 2: Performance**: Poles near the axis lead to a low damping ratio (ξ), causing excessive overshoot and long settling times.
- **Goal of Relative Stability:** Guarantee stability and acceptable transient performance.

Introducing the Safety Buffers

Relative Stability is Measured by Safety Buffers:

- **Buffer 1: Gain Margin (GM):** How much the system's strength (gain) can be increased before instability occurs.
- **Buffer 2: Phase Margin (PM):** How much delay or lag (phase shift) can be introduced before instability occurs.
- **Goal:** Maintain both margins above a critical minimum value(e.g., GM>6dB, $PM>30^0$)

The Stability Condition Revisited

- **The Core Instability Principle**: Instability occurs when the loop gain is 1 and the phase shift is -180^0.
- **Loop Transfer Function** $G(s)H(s)$: This is the function we analyze in the frequency domain.
- **Instability Point**: When $|G(j\omega)H(j\omega)|=1$ and $\angle G(j\omega)H(j\omega)=-180^0$
- **Physical meaning:** The signal returns to the summer the exact size it started at, but exactly inverted. This causes positive feedback and exponential growth.

Analogy: The Cornering Car (Setup)

- **The System**: A car (closed-loop system) taking a corner.
- **Input/Command**: The driver turning the steering wheel (Reference Input).
- **Output**: The car's actual direction and position (Controlled Output).
- **Feedback**: The driver's eyes correcting the steering (Feedback Loop).
- **Instability (Crash):** The car skids out or spins uncontrollably.
- **Margins**: The safety features preventing the crash.

The Car Analogy: Gain Margin (GM)

- **GM in the Car**: How much harder can you press the gas pedal (increase speed/gain) while maintaining the current steering input before the car skids out?
- **Core Concept**: Measuring the safety buffer against over-amplification.
- **Physical Impact**: Low means the car is sensitive to speed changes. It means a small increase in power will cause excessive wobble or spin.

The Car Analogy: Phase Margin (PM)

- **PM in the Car:** How much extra reaction time or delay can be introduced in the driver's response (phase lag) before the car begins to oscillate uncontrollably.
- **Core Concept:** Measuring the safety buffer against excessive time delay.
- **Physical Impact:** Low means the system is very sensitive to reaction time. It means a small lag will cause the driver to overcorrect and weave back and forth.

Defining the Critical Frequencies

- Instability Point: $|G(j\omega)H(j\omega)|=1$ and $\angle G(j\omega)H(j\omega)=-180\circ$ simultaneously.

1. **Gain Crossover Frequency (ωgc):** The frequency at which the magnitude is unity: $|G(j\omega gc)H(j\omega gc)|=1$.
2. **Phase Crossover Frequency (ωpc):** The frequency at which the phase shift is $-180\circ$:

Phase Margin (PM) Calculation

- **Definition**: Phase Margin is the additional phase lag required to make the system unstable.
- **Measured at ωgc**: $PM = 180^{\circ} + \angle G(j\omega_{gc})H(j\omega_{gc})$
  - Note: Since $\angle G(j\omega_{gc})H(j\omega_{gc})$ is typically a negative number (a lag), we add 180^{0} to find the positive buffer.
- **Stable Condition:** $PM > 0^{0}$.;

Phase Margin and Damping Ratio (ζ)

- **Direct Link:** is directly correlated with the Damping Ratio (ζ) of the dominant closed-loop poles.
- **Approximation** (for $30^{\circ} < PM < 60^{\circ}$): $\zeta \approx PM/100$ (where PM is in degrees).
- **Impact:** Higher PM means higher ζ, which means less overshoot and oscillation.
- **Design Tool:** We use PM to meet time-domain overshoot specifications.

Gain Margin (GM) Calculation

- **Definition:** Gain Margin is the factor by which the open-loop gain must be increased to drive the system to marginal stability.

- Measured at ωpc: $GM=\frac{1}{|G(j\omega_{pc})H(j\omega_{pc})|}.$

- **GM in Decibels (dB):** $GM_{dB}=-20\log_{10}|G(j\omega_{pc})H(j\omega_{pc})|$

- **Stable Condition**: $GM>1$ (or $GM_{dB}>0$ dB).;

Gain Margin and System Robustness

- **Interpretation**: indicates the system's tolerance to parameter variations that affect the gain.
- **Physical Example**: Amplifier circuits, motor torque constants, and sensor calibration all introduce gain uncertainty.
- **Impact of Low** : The system is "too sensitive" to minor changes. If the manufacturer of a component provides a slightly higher gain, the system may oscillate.
- **Design Goal**: Maintain a large to accommodate component tolerances and aging effects.

Summary of Stability Margins

- **Stability is Guaranteed if:** $PM > 0^0$ and $GM_{dB} > 0dB$ $(GM > 1)$
- **Performance/Robustness is Good if:** $PM \approx 45^0 - 60^0$ **and** $GM_{dB} \approx 6dB - 12dB$.
- **Low Margins Signal**: Excessive ringing, large overshoot, and vulnerability to noise and parameter changes.

Visualizing Poles and Relative Stability

- **The s-Plane Connection**: The location of the closed-loop poles directly reflects relative stability.
- **Poles near the jω axis**: Low ζ, low PM, high overshoot, low relative stability.
- **Poles far into the LHP**: High ζ, high PM, low overshoot, high relative stability.
- **The Line of Constant Damping Ratio**: Poles along a ray from the origin maintain the same ζ and, therefore, the same PM.

Marginal Stability: $PM=0^0, GM=1$

- **Definition:** A system that is neither stable nor unstable, It exhibits sustained, bounded oscillation.
- **In the s-Plane**: Closed-loop poles are located exactly on the jω axis.
- **In the Frequency Domain**: $|G(j\omega)H(j\omega)|=1$ and $\angle G(j\omega)H(j\omega)=-180\ ^0$ simultaneously at one frequency ω.
- **Margins:** $PM=0^0$ and $GM=1$ (0dB).
- **Physical Analogy**: A perfect pendulum swing with no friction—it never stops, but it never gets bigger.

Understanding Why Phase Margin is Crucial

- Phase Margin → Transient Response: PM largely dictates the shape of the system's step response.
- **0^0 PM:** Sustained, undamped oscillation..
- **Low PM ($<30^0$):** Highly oscillatory, high overshoot, long settling time.
- **High PM ($>45^0$):** Fast, smooth, low overshoot response.
- **Real-life Example:** Tuning a thermostat. A low PM leads to "hunting"—the temperature constantly overshoots the setpoint and swings wildly back and forth.

Understanding Why Gain Margin is Crucial

- **Gain Margin System Stability Reserve**: GM dictates how much power or amplification the system can withstand.
  - *Example*: Robotic Arm: A motor's actual torque constant is 10% higher than specified. If the GM is only 5%, the arm will start shaking uncontrollably.
- **Importance for Safety**: In systems like aircraft or nuclear reactors, a high GM ensures stability reserve even in degraded or fault conditions where sensor gains might unexpectedly increase.
- **Design Tool**: GM provides confidence that the final, physical system will match the simulated stability.

The Dual Challenge: Trading Off Margins

- The Trade-off: Often, increasing PM (better damping) means reducing GM (less gain buffer), and vice versa.
- Increasing Gain (K): Generally increases ωgc (good) but decreases PM (bad) and increases GM (good). The overall effect is complex.
- Engineering Challenge: The goal is to find a controller that simultaneously meets the minimum requirements for PM (performance) and GM (robustness).

Practical Example: Satellite Attitude Control

- **System:** A satellite constantly firing small thrusters to maintain a specific orientation (attitude).
- **Actuator:** The thrusters (introduce thrust *gain* and firing *delay*).
- **Control Loop:** Gyroscopes measure orientation error, send command to thrusters.
- **Low PM Result:** The satellite keeps oscillating around the target angle, wasting fuel and blurring images. The controller reacts too late, causing overshoot.
- **Low GM Result**: A slight pressure variance in a fuel line causes a thruster to output more force than expected, leading to wild, unstable spinning.

Interpreting the 0 dB and -180^0 Lines

- **The 0 dB Line**: Represents a magnitude of 1 (unity gain). Crossing this line means the signal is neither amplified nor attenuated.
- **The −180∘ Line**: Represents the perfect phase inversion required for positive feedback.
- **The Intersection of Danger**: The region near the intersection of these two lines is the domain of low relative stability.
- **Analogy**: The Dead Zone**.** If you are in the dead zone, any small perturbation will cause the system to grow unstable.

Relationship: PM, Overshoot, and ζ

- **High** PM **(e.g.,** 60^0**):** Corresponds to a high ζ (e.g., ζ≈0.6). This yields low overshoot, typically <10%.
- **Low** PM **(e.g.,** 10^0**):** Corresponds to a low ζ (e.g., ζ≈0.1). This yields high overshoot, often >70%, and prolonged oscillation.
- **The Rule of Thumb:** ζ≈PM/100 (degrees).
- **Design Focus:** The desired PM is usually selected first based on the required transient response, specifically maximum overshoot.

Relationship: GM, Resonant Peak, and System Gain

- **Gain Margin → Frequency Domain Peaking:** GM is linked to the magnitude of the resonant peak (Mr) in the closed-loop frequency response.
- **Low** GM**:** High resonant peak (Mr), meaning the system strongly amplifies signals at a specific frequency ωr; leads to high noise sensitivity.
- **High** GM**:** Low resonant peak, indicating smooth, consistent frequency response.
- **Physical Impact:** Low GM systems sound "tinny" or "bloomy" if they are audio systems, or they vibrate violently at one specific frequency if they are mechanical.

Example Calculation: Finding ω_{gc}

- **Definition:** The Gain Crossover Frequency (ω_{gc}) is the frequency where the loop gain magnitude is exactly unity (1, or 0 dB).
- **Formula:** Set the magnitude $|G(j\omega)H(j\omega)|=1$ and solve the resulting algebraic equation for ω.
- **Significance:** This frequency is the anchor point for the Phase Margin calculation.

Example Calculation: Finding ω_{pc}

- **Definition:** The Phase Crossover Frequency (ω_{pc}) is the frequency where the loop phase shift is exactly −180°.
- **Formula:** Set the phase angle $\angle G(j\omega)H(j\omega) = -180°$ and solve for ω.
- **Significance:** This frequency is the anchor point for the Gain Margin calculation.

Step-by-Step: Calculating Phase Margin (PM)

1. Find ω_{gc} : Set $|G(j\omega)H(j\omega)|=1$ and solve for ω.
2. Calculate Phase at ω_{gc} : Substitute ω_{gc} into $\angle G(j\omega_{gc})H(j\omega_{gc})$.
3. Compute PM: $PM=180^0+\angle G(j\omega_{gc})H(j\omega_{gc})$.

Step-by-Step: Calculating Gain Margin (GM)

1. Find ωpc : Set $\angle G(j\omega)H(j\omega)=-180^0$ and solve for ω.

2. Calculate Magnitude at ω_{pc} : Substitute ω pc into $|G(j\omega\ pc\ )H(j\omega\ pc\ )|$.

3. Compute GM: $GM=\frac{1}{|G(j\omega\ pc\ )H(j\omega\ pc\ )|}$.

(In dB: $GM_{dB} =-20\log_{10} |G(j\omega\ pc\ )H(j\omega\ pc\ )|$).

The Role of the Loop Gain K

- **Impact on PM:** Increasing the proportional gain K generally reduces PM because the unity gain frequency (ω_{gc}) shifts to a higher, more lagged frequency.
- **Impact on GM:** Increasing K generally increases GM_{dB} because the magnitude at the phase crossover frequency (ω_{pc}) is now further below 0dB.
- **Trade-off:** Adjusting K simultaneously affects both margins, often in opposing directions.

The Significance of ω_{gc} (Bandwidth)

- **Definition of Bandwidth:** The range of frequencies the system can track effectively.
- **Impact:** A higher ω_{gc} means a faster system (better rise time, better responsiveness).
- **The Trade-off:** Increasing ω_{gc} for speed usually costs you PM (oscillation) and risks exciting high-frequency noise.

Introduction to Time Delay (e^{-sT})

- **Time Delay (Transport Lag):** Physical phenomenon where the output response is delayed by a fixed time T after the input is applied.
- **Mathematical Model:** $G(s)=e^{-sT}$.
- **Phase Impact:** The transfer function $e^{-j\omega T}$ has a constant magnitude of 1 but introduces a phase lag of $\angle e^{-j\omega T}=-\omega T$ radians (or $-57.3\omega T$ degrees).
- **Effect on PM:** Time delay introduces infinite, non-attenuated phase lag, which rapidly and severely reduces the Phase Margin.

Design Implications of Stability Margins

- **Scenario 1: Low** PM**, High** GM**:** System is very oscillatory (poor performance) but safe from gain variations (robust).
  - **Solution:** Must increase phase lead to raise PM.
- **Scenario 2: Low** GM**, High** PM**:** System has low overshoot (good performance) but is fragile to gain variations (poor robustness).
  - **Solution:** Must adjust the gain curve at ω_{pc} to raise GM.
- **Engineering Goal:** Simultaneously achieve the target PM and GM with minimum cost and complexity.

The Concept of Conditional Stability

- **Definition**: A system where stability depends on the loop gain K falling within a specific range (e.g., Kmin<K<Kmax).
- **Frequency Domain View:** The magnitude plot crosses the 0 dB line at multiple frequencies where the phase is -180°.
- **Physical Danger**: If the gain is too low (below Kmin) or too high (above Kmax), the system becomes unstable.
- **Analogy**: Driving a car—you can't go too slow (stall/loss of control) or too fast (spin out).

The Nyquist Stability Criterion (Preview)

- Tool for Conditional Stability: Routh-Hurwitz fails to clearly identify conditional stability.
- The Nyquist Path: A contour in the s-plane that encircles the entire RHP.
- Nyquist Plot: The mapping of the Nyquist path onto the $G(s)H(s)$ plane.
- Criterion (Concept Only): Stability is determined by the number of times the Nyquist plot encircles the critical point $(-1, j0)$.
- Lecture 8/9 Setup: This criterion is the formal, graphical method we use to determine both absolute and relative stability accurately.

Design Workflow: Using Margins

- **Step 1: Define Specs**: The client specifies maximum overshoot (Mp) and maximum steady-state error (ess).
- **Step 2: Translate to Margins**: Convert Mp$\rightarrow$ required $\zeta\rightarrow$ required PMmin. Set a required GMmin (e.g., 6 dB) for robustness.
- **Step 3: Analyze Current System**: Calculate the actual PM and GM of the current system (using the Bode/Nyquist tools we learn next).
- **Step 4: Design Compensator**: If actual PM<PMmin (too much overshoot), use a Lead compensator. If actual GM<GMmin (too fragile), use a Lag compensator.

The Importance of Phase Margin $\longrightarrow 45^0$

- **The Magic Number:** A Phase Margin of 45^0 is a good starting point for design.
    - Why 45^0?: It yields $\zeta \approx 0.45$, which corresponds to an overshoot of approximately 20%.
- **Balance:** 45^0 is often a good compromise between speed (low ω_{gc}) and performance (acceptable overshoot).
- **Best Practice**: Aim for 45^0 as a minimum, but ideally design for 50^0 to 60^0 for high-quality transient performance.

The Importance of Gain Margin $\longrightarrow$ 6dB

- **The Engineering Minimum**: A Gain Margin of 2 (or 6 dB) is often considered the bare minimum.
  - Why 6 dB?: It provides a factor of 2 safety buffer against gain variations. The system can handle a doubling of the loop gain before instability.
- **Robustness:** 6 dB accounts for component tolerances ($\pm 10\% - 20\%$ variation) and ensures stability over the system's operational temperature range.
- **Best Practice:** For safety-critical systems, or those with highly variable components, GM should be 10 dB or more.

Troubleshooting: When Margins are Too Low

- **Problem**: Low PM (<30^0).; Symptom: High overshoot, ringing, oscillatory step response.
  - **Diagnosis**: The system has insufficient damping (ζ).
- **Action**: Introduce phase lead (Lead Compensator) to boost PM at ωgc.
  - **Problem**: Low GM (<6 dB).
- **Symptom**: Sensitive to gain changes, strong amplification of noise at ωpc, prone to oscillation.
- **Diagnosis**: The system is too close to the −180° phase line.; Action: Introduce a compensator to shift the magnitude curve away from 0 dB at ωpc.

Case Study: The Boeing 777 Fly-by-Wire

- **Context**: Modern aircraft use "fly-by-wire," where pilot inputs are processed by a computer before reaching the control surfaces.
- **Requirement:** High PM: Critical to ensure that when the pilot commands a change (e.g., pulling back on the stick), the plane settles to the new altitude smoothly without dangerous, sustained pitch oscillation (pilot-induced oscillation).
- **Requirement:** High GM: Essential to handle variations in aerodynamic gain (due to speed, altitude, and weather) and actuator performance (due to fluid temperature changes).
- **Conclusion:** Aircraft design uses these margins as absolute safety requirements to guarantee predictable and safe flight under all operating conditions

Conclusion and Next Steps

- **Key Takeaways**: Relative Stability quantifies the degree of stability using PM and GM.
- **Phase Margin (PM):** Buffer against phase lag/delay. Dictates damping (ζ) and overshoot (Mp). Target: $45^0 - 60^0$.
- **Gain Margin (GM):** Buffer against magnitude/gain change. Dictates robustness. Target: 6 dB−12 dB.
- On the next lecture: We begin the graphical methods of analysis with **Bode Plots** and the formal **Nyquist Criterion**—the tools required to calculate these margins easily.

References

[1] Car on cliff edge, https://media.istockphoto.com/id/157286211/photo/over-the-edge-2.jpg?s=612x612&w=0&k=20&c=5IyXB4m-Qf1kMDPz51rbs8NokpF7a_KUl6bSIruEvfk= , accessed on October 12, 2025

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