Course: Regulation and control

Lecture 8: Frequency Domain Analysis I

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Lecture objectives

- Understand frequency response concepts
- Sketch and interpret Bode plots
- Analyze stability using Gain & Phase Margins
- Introduction to Nyquist plots and stability criterion
- Relate frequency and time domain performance

Time Domain vs. Frequency Domain

- **Time Domain Analysis** visualizes how a system's output behaves over time in response to an input (e.g., a step or impulse).
- **Frequency Domain Analysis** reveals the system's steady-state response to sinusoidal inputs of varying frequencies.
- These are not competing views but complementary perspectives, providing a complete picture of system behavior.

Why Frequency Analysis Is Essential

- **Evaluates Robustness**: Determines how a system performs under different operating conditions and frequencies.
- **Predicts Stability**: Directly assesses the inherent stability of a closed-loop system without finding the roots of the characteristic equation.
- **Guides Controller Design**: Provides clear insights for designing compensators (like lead or lag networks) to meet performance specifications.

Concept of Frequency Response

- A sinusoidal input (e.g., $A \sin(\omega t)$) is applied to a linear time-invariant (LTI) system.
- The steady-state output is a sinusoid of the same frequency, but with:
  - **A different amplitude** ($A|G(j\omega)|$).
  - **A phase shift** ($\angle G(j\omega)$).
- The ratio of output to input defines the Frequency Response, characterized by:
  - Magnitude $|G(j\omega)|$: The gain/attenuation.
  - Phase $\angle G(j\omega)$: The time shift/lag.

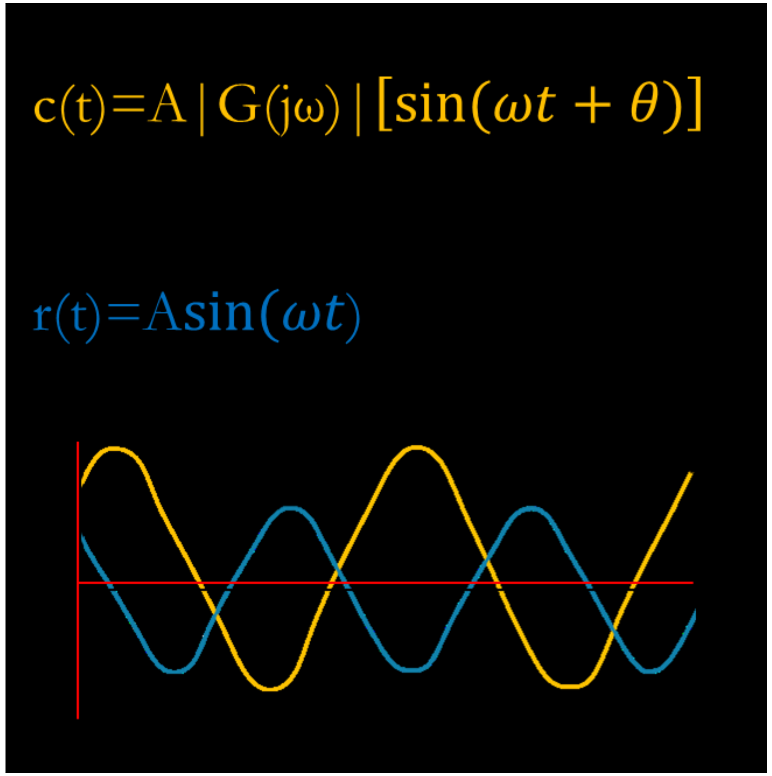

Fig 1:- Frequency input and response [1]

Audio Equalizer Analogy

- An audio equalizer is a practical example of a frequency-domain processor.
- Knobs for **Bass, Mid, and Treble** allow you to selectively amplify or attenuate specific frequency bands (**low, mid, high**).
- This is a direct parallel to how controllers shape a system's frequency response to achieve desired performance.

Fig 2: Audio equalizer [2]

Transfer Function in Frequency Domain

- The frequency response is found by substituting s = jω into the system's Laplace Transfer Function, G(s).
- The resulting complex number G(jω) contains all frequency response information:
- Its **Magnitude** (|G(jω)|) represents the **gain or attenuation** at frequency ω.
- Its **Angle** (∠G(jω)) represents the **phase shift**.
- This method applies directly to Linear Time-Invariant (LTI) Systems.

Plotting Frequency Response

- Frequency response is visualized using two key plots against frequency:
  - **Magnitude Plot:** Shows gain (|G(jω)|) vs. frequency.
  - **Phase Plot**: Shows phase shift (∠G(jω)) vs. frequency.
- Frequency is typically plotted on a **logarithmic scale** to capture a wide range of values effectively.
- Units for the magnitude is magnitude ratio or **decibels** and Phase is degrees
- Decibel = $20 log M$
  - Where M-magnitude ratio or simply |G(jω)|

Bode Plot

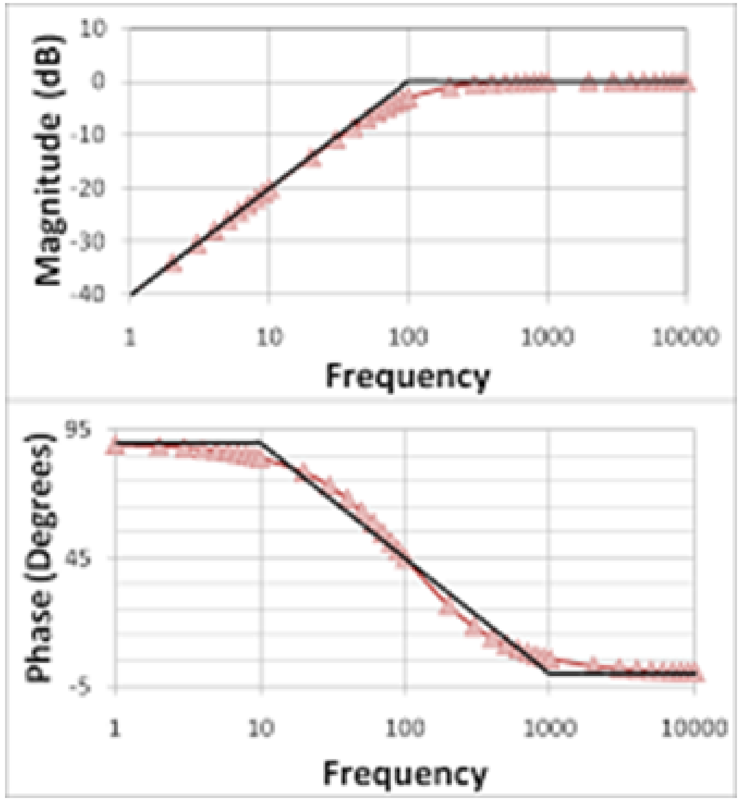

Fig 3: Bode plot [3]

- A Bode Plot is the standard tool for representing frequency response, consisting of two separate graphs:
  - Magnitude (in decibels, dB) vs. log frequency.
  - Phase (in degrees) vs. log frequency.
- Its key feature is the use of straight-line approximations (asymptotes), making it easy to sketch by hand and understand the system's dominant behavior.

Interpreting Bode

Magnitude Plots

- The plot shows regions that are flat, rising, or falling.
- Slopes change at break frequencies (corner frequencies):
  - +20 dB/decade for a zero.
  - -20 dB/decade for a pole.
- The slope indicates the rate of gain change and the system's "order."

Phase Plot

- Each pole contributes up to -90° of phase lag.
- Each zero contributes up to +90° of phase lead.
- These phase shifts occur gradually, centered around their respective break frequencies.

First-Order System Frequency Response

- Characterized by a single pole in its transfer function.
- Low-Frequency Behavior: Constant gain (flat magnitude).
- High-Frequency Behavior: Gain rolls off at -20 dB/decade.
- Phase Shift: Increases from 0° to a maximum of -90° lag.

Second-Order System Frequency Response

- Characterized by two poles, involving natural frequency (ω_n) and damping ratio (ζ).
- Can exhibit a resonant peak in the magnitude plot if the system is underdamped ($\zeta < 0.707$).
- Phase Shift: Increases from 0° to a maximum of -180° lag.

Low-Pass, High-Pass, and Band-Pass Characteristics

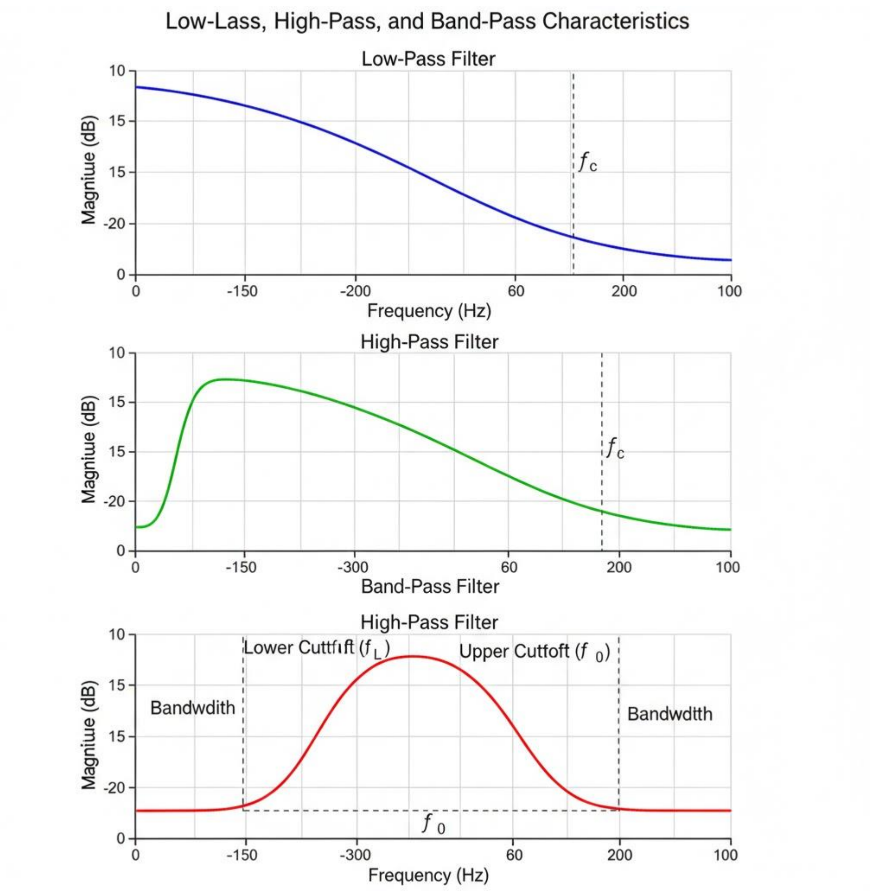

Fig 4: Filter responses [4]

- **Low-Pass Filter**: Passes low frequencies and attenuates high frequencies (e.g., a first-order system).
- **High-Pass Filter**: Passes high frequencies and attenuates low frequencies.
- **Band-Pass Filter**: Passes a specific mid-range band of frequencies, attenuating both lower and higher ones.

Crossover Frequencies

- **Gain Crossover Frequency (ωgc):** The frequency where the magnitude is 0 dB. Indicates the system's bandwidth.
- **Phase Crossover Frequency (ωpc):** The frequency where the phase is -180°. Critical for stability analysis.
- **Relation to stability**: If the system crosses these frequencies at the wrong phase or magnitude, oscillations grow instead of dying.

Gain Margin and Phase Margin

- These are quantitative measures of relative stability—how far the system is from instability.
- **Gain Margin (GM):** The amount of additional gain (in dB) that can be applied before the system becomes unstable.
- **Phase Margin (PM):** The amount of additional phase lag (in degrees) that can be applied at the gain crossover frequency before instability occurs.

Typical Stability Margins

- **Phase Margin:** Typically 30°–60°. Lower PM means more overshoot and oscillations; higher PM means a slower, more sluggish response.
- **Gain Margin:** Typically 6–12 dB. Provides a safety buffer against gain variations.
- Designing these margins is a trade-off between performance (speed) and robustness.

Interpreting Bode Plot for Stability

- A system is stable if both **Gain Margin** and **Phase Margin** are positive.
- As poles move to the right in the s-plane (closer to instability), these stability margins shrink.
  - Adding Poles (e.g., integrators, delays) introduces more phase lag, reducing the Phase Margin and destabilizing the system.
  - Adding Zeros (e.g., differentiators) can introduce phase lead, potentially improving Phase Margin and stability if placed correctly.
  - Proper placement of poles and zeros is crucial for controller design.

Practical Interpretation of Bode Plots

- **Low-Frequency Gain**: Indicates steady-state accuracy (e.g., high gain reduces steady-state error).
- **Bandwidth (ωgc):** Correlates with the speed of response; higher bandwidth means a faster system.
- **Phase Margin**: Correlates with damping and overshoot; higher PM means less overshoot.

Asymptotic Approximations vs. Exact Plot

- **Asymptotic Bode Plot**: A quick, hand-sketched version using straight lines. Excellent for understanding concepts and initial design.
- **Exact Bode Plot**: A computed, smooth curve. Reveals the true system behavior, especially important near break frequencies where the asymptotic plot deviates.

Step-by-Step Sketching Example1: Bode plot

Let's plot: $G(s) = 10 / (s + 10)$

1. Rewrite in Standard Form:
   - $G(s) = 10 / (10 * (s/10 + 1)) = 1 / (0.1s + 1)$
   - Now it's a Constant ($K=1$) and a First-Order Term $[1/(\tau s+1)]$ with time constant $\tau=0.1$.
2. Identify Corner Frequencies:
   - Corner Frequency, $\omega\_c = 1/\tau = 1/0.1 = 10$ rad/s.

Step-by-Step Sketching Example1: Bode plot Cont….

3. Draw the Magnitude Plot:

- **Part A:** Constant Gain (K=1)
  - Magnitude in dB = $20 \cdot \log_{10}(1) = 0$ dB
- This is a horizontal line at 0 dB across all frequencies. It's our starting point.

- **Part B**: First-Order Term [1/(0.1s + 1)]
  - For frequencies ω << ω_c (ω << 10):
  - The term 1/(0.1jω + 1) is approximately 1/1 = 1.
  - Magnitude ≈ $20 \cdot \log_{10}(1) = 0$ dB.
- So, we draw a horizontal line at 0 dB from low frequencies up to ω = 10 rad/s.

Step-by-Step Sketching Example1: Bode plot Cont....

- For frequencies ω >> ω_c (ω >> 10):
  - The term $1/(0.1j\omega + 1)$ is approximately $1/(0.1j\omega)$.
  - Magnitude $\approx 20*\log_{10}(1/(0.1\omega)) = 20*\log_{10}(1) - 20*\log_{10}(0.1\omega)$
    $= 0 - 20*\log_{10}(0.1) - 20*\log_{10}(\omega)$
  - This simplifies to $-20*\log_{10}(\omega) + 20$ (since $20*\log_{10}(0.1) = -20$).
- This is the equation of a straight line on a log-frequency scale with a slope of -20 dB per decade. It decreases by 20 dB for every 10-fold increase in frequency.

Step-by-Step Sketching Example1: Bode plot Cont….

- At the corner frequency (ω = ω_c = 10):
  - The two asymptotes meet. The actual value at this point is $20 \cdot \log_{10}(|1/(j+1)|) = 20 \cdot \log_{10}(1/\sqrt{2}) \approx -3$ dB.
- Final Magnitude Plot: We simply add the two parts together (0 dB + the first-order term), which just gives us the first-order term's plot since the constant is 0 dB.

Step-by-Step Sketching Example1: Bode plot Cont….

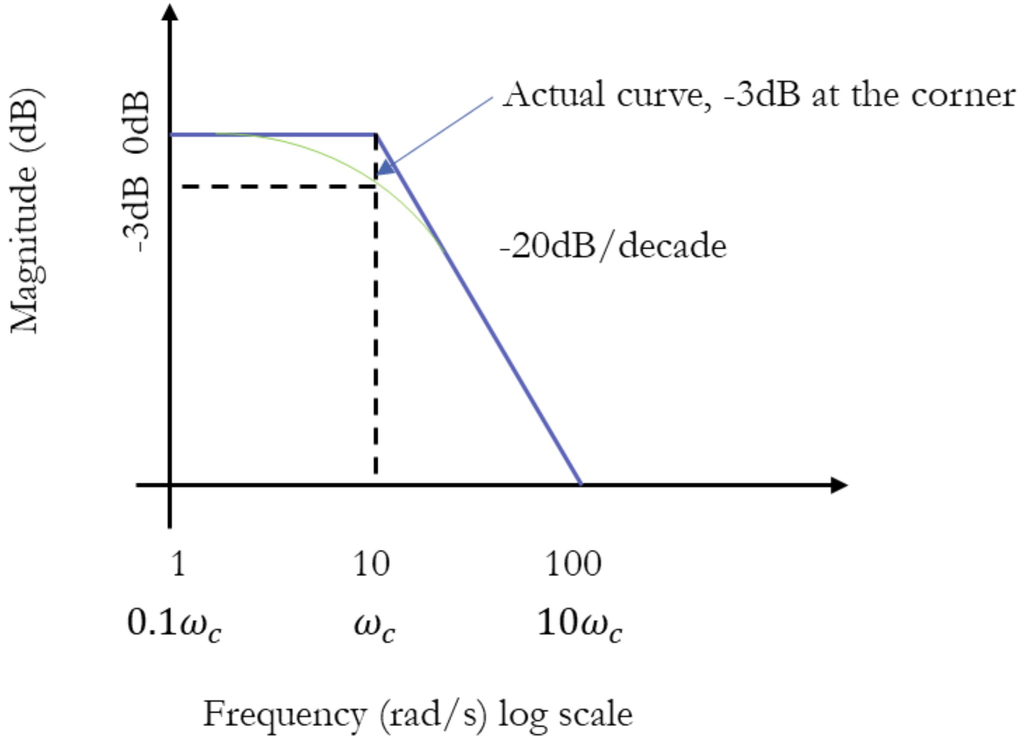

Fig 5: Approximate bode (magnitude) plot for example 1 [5]

Using MATLAB G=tf([10],[1 10]);

bode(G)

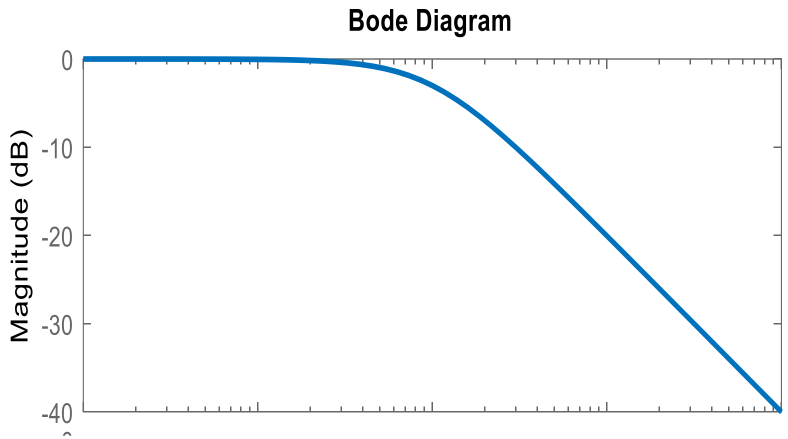

Fig 6: Bode (magnitude) plot using MATLAB for example 1 [6]

Step-by-Step Sketching Example: Bode plot Cont....

4. Draw the Phase Plot

- **Part A**: Constant Gain (K=1)

  The phase shift of a positive, real constant is 0°. It doesn't change the phase.

- **Part B**: First-Order Term [1/(0.1s + 1)]

  This term introduces a phase lag.

  The phase is given by $\tan^{-1}(\omega\tau) = \tan^{-1}(\omega/10)$.

Step-by-Step Sketching Example: Bode plot Cont….

- We plot this using key points:
- At very low frequencies ($\omega << 0.1\omega_c = 1$)

  $-\tan^{-1}(\omega/10) \approx -\tan(0) \approx 0°$

- At the corner frequency ($\omega = \omega_c = 10$)

  $-\tan^{-1}(10/10) = -\tan^{-1}(1) = -45°$

- At very high frequencies ($\omega >> 10\omega_c = 100$)

  $-\tan^{-1}(\omega/10) \approx -\tan^{-1}(\infty) \approx -90°$

- The transition is smooth: We can draw a straight-line approximation that passes through:
  - $0°$ at $\omega = 1$ (one decade below the corner)
  - $-45°$ at $\omega = 10$ (at the corner)
  - $-90°$ at $\omega = 100$ (one decade above the corner)

Step-by-Step Sketching Example: Bode plot Cont….

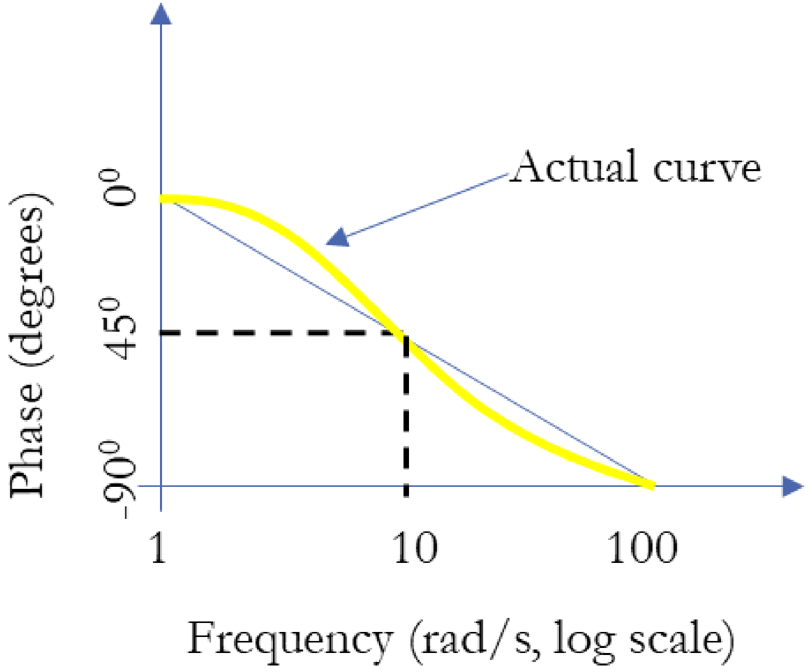

Fig 7: Approximate bode (phase) plot for example 1 [7]

Using MATLAB

```
G=tf([10],[1 10]);
bode(G)
```

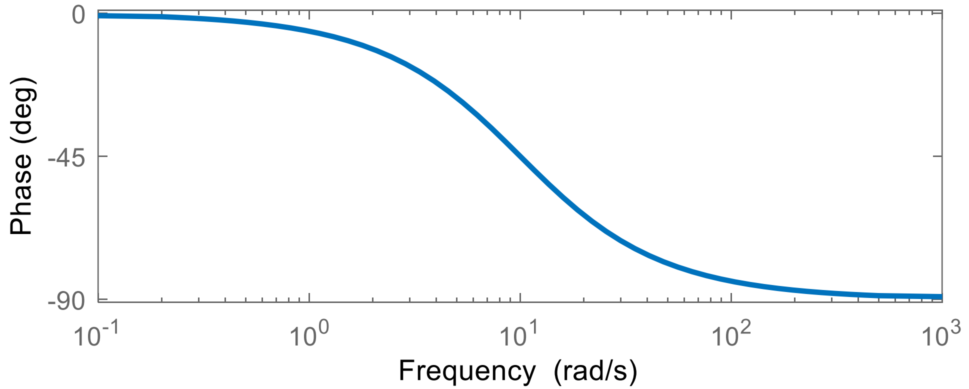

Fig 8: Bode (phase) plot using MATLAB for example 1 [8]

Interpreting the plot

- From this Bode plot, we can see that $G(s) = 10 / (s + 10)$ is a low-pass filter.
- It passes low-frequency signals (< 10 rad/s) with a gain of about 1 (0 dB) and little phase shift.
- It attenuates high-frequency signals (> 10 rad/s) with a slope of -20 dB/decade and introduces up to -90° of phase lag.
  - Phase never reaches -180° → GM = ∞ (infinite gain margin)
  - Phase never goes below -90° → PM is very large
- This confirms what we knew: a first-order system is always stable in closed-loop. GM and PM give us quantitative measures of how stable it is.

Connecting Time and Frequency Domains

- Low-Frequency Gain ↔ Steady-State Accuracy
- Bandwidth (Crossover Frequency) ↔ Speed of Response
- Phase Margin ↔ Damping and Overshoot

Practical Example: DC Motor Frequency Response

- For an armature-controlled DC motor:
- The magnitude decreases with frequency, acting as a low-pass filter.
- The phase lag increases with frequency, and is more pronounced with higher system inertia and damping.

Nyquist Plot

- A Nyquist Plot is a single curve in the complex plane that represents the frequency response G(jω).
- It plots the real and imaginary parts of G(jω) as ω goes from 0 to ∞, providing a direct visualization of gain and phase together.
- **Key Differentiator:**
    - **Bode Plot:** Uses two separate graphs (Magnitude vs. Frequency, Phase vs. Frequency).
    - **Nyquist Plot:** A single, unified curve that combines both magnitude and phase information.

Plotting the Curve

1. Evaluate G(jω) for frequencies from ω = 0 to ω → ∞.
2. For each frequency ω, calculate:
   - **Real Part (x-axis)**
   - **Imaginary Part (y-axis)**
3. The plot for ω = -∞ to 0 is the mirror image.

- **What does a point on the curve mean?**
  - Vector from origin to the point:
    - Length = Gain (Magnitude)
    - Angle = Phase Shift

Example1: Drawing a Simple Nyquist Plot

- Let's plot: G(s) = 1 / (s + 1)

Step 1: Substitute s = jω:

  - G(jω) = 1 / (jω + 1)

Step 2: Find Real & Imaginary Parts:

$$\frac{1}{j\omega + 1} * \frac{j\omega - 1}{j\omega - 1} = \frac{j\omega - 1}{j^2\omega^2 - 1}$$

$$\frac{j\omega - 1}{(-1)\omega^2 - 1} = j\left(\frac{\omega}{-\omega^2 - 1}\right) + \left(\frac{-1}{-(\omega^2 + 1)}\right)$$

- Real Part, Re(ω) = 1 / (1 + ω^2)
- Imag. Part, Im(ω) = -ω / (1 + ω^2)

Example1: Drawing a Simple Nyquist Plot

Step 3: *Calculate Key Points on the Plot*

Frequency (ω)	Real Part, Re(ω)	Imaginary Part, Im(ω)	Point (Re, Im)
$\omega = 0$	$1 / (1 + 0) = \mathbf{1}$	$-0 / (1 + 0) = \mathbf{0}$	**(1, 0)**
$\omega = 0.5$	$1 / (1 + 0.25) \approx \mathbf{0.8}$	$-0.5 / (1 + 0.25) \approx \mathbf{-0.4}$	~(0.8, -0.4)
$\omega = 1$	$1 / (1 + 1) = \mathbf{0.5}$	$-1 / (1 + 1) = \mathbf{-0.5}$	**(0.5, -0.5)**
$\omega = 2$	$1 / (1 + 4) = \mathbf{0.2}$	$-2 / (1 + 4) = \mathbf{-0.4}$	~(0.2, -0.4)
$\omega \to \infty$	$1 / (\infty) \to \mathbf{0}$	$-\omega / (\omega^2) \to \mathbf{0}$	**(0, 0)**

Example1: Drawing a Simple Nyquist Plot

Step 4: *Sketching the Plot*

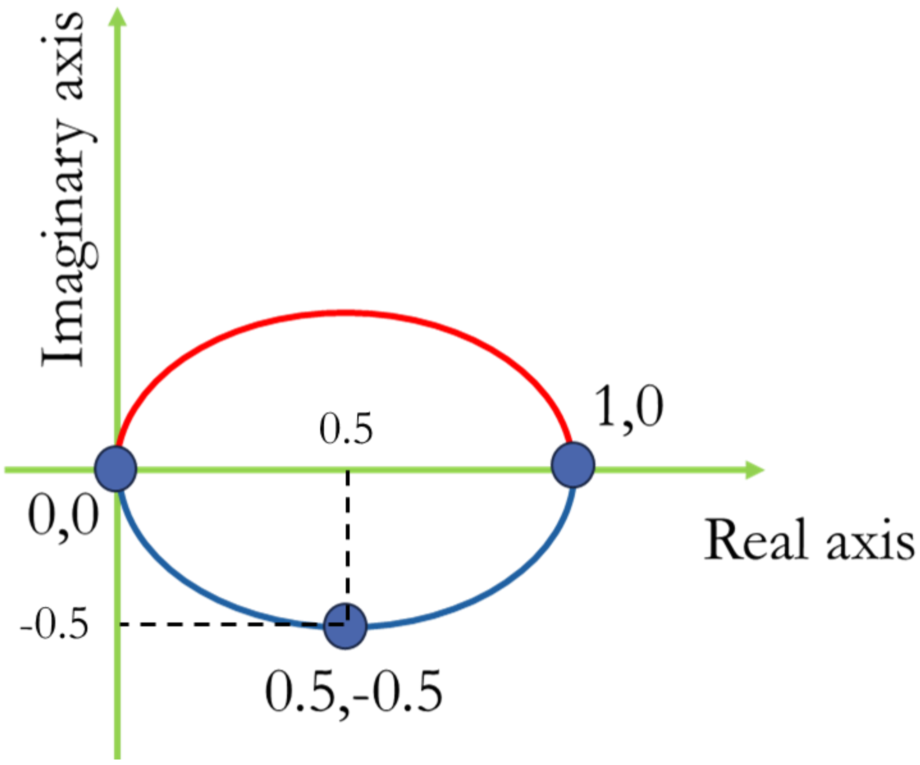

Fig 9: Nyquist plot for example 1 [9]

MATLAB code to plot

```
G = tf([1],[1 1]);
nyquist(G)
```

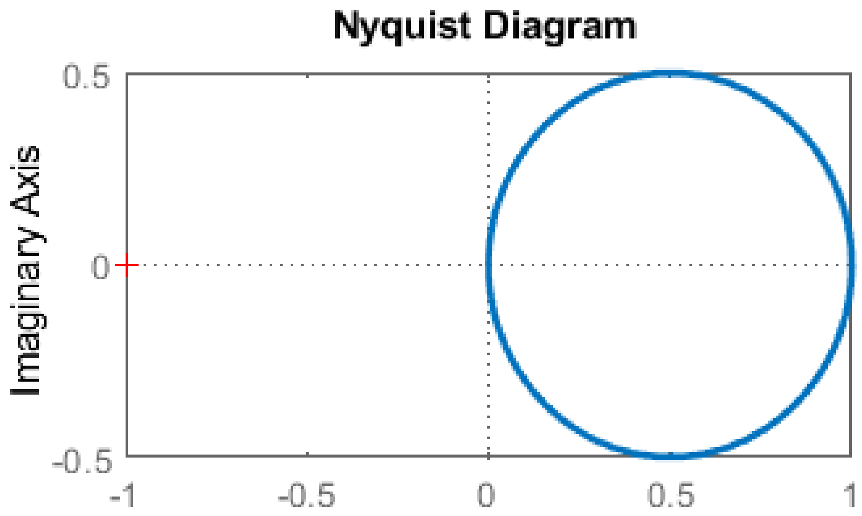

Fig 10: Nyquist plot using MATLAB for example 1 [10]

Nyquist Stability Concept

- The Nyquist Stability Criterion uses the open-loop frequency response to determine closed-loop stability.
- Stability is judged by the plot's encirclements of the critical point (-1, 0) on the real axis.
- The number and direction of encirclements relate to the number of open-loop poles in the right-half plane (RHP).

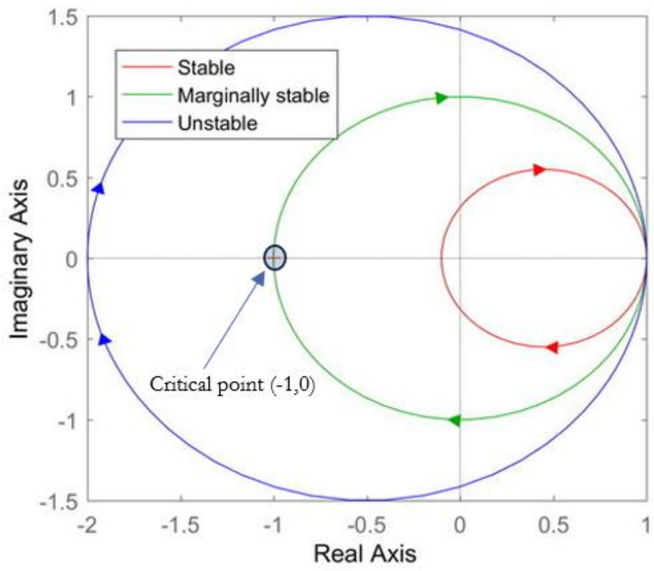

Fig 11: Nyquist stability criterion [11]

Example1: Drawing a Simple Nyquist Plot

***Step 5**: Stability Analysis using the Nyquist Criterion*

Now, let's apply the stability criterion: Z = P + N

- P (Open-loop poles in the Right-Half Plane):
    - Our open-loop system is G(s) = 1 / (s + 1).
    - It has one pole at s = -1.
    - This pole is in the Left-Half Plane (LHP).
    - Therefore, P = 0.

Example1: Drawing a Simple Nyquist Plot Cont….

***Step 5**: Stability Analysis using the Nyquist Criterion*

- N (Number of encirclements of -1):
  - Look at the plot. The critical point (-1, 0) is located at -1 on the real axis.
  - Does our plot encircle this point? No. The point (-1, 0) is to the left of our starting point at (1, 0), and the entire plot stays to the right of it.
  - Therefore, N = 0.
- Z (Closed-loop poles in the RHP):
  - Z = P + N = 0 + 0 = 0.
- **Conclusion**: Since Z = 0, the closed-loop system is stable.

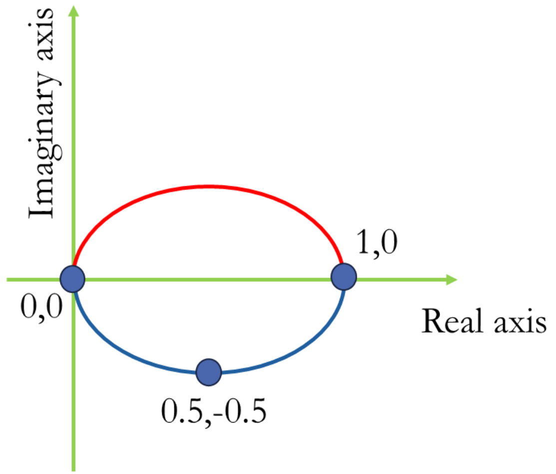

Fig 12: Nyquist plot for example 1 [9]

Intuitive Visualization of Encirclement

- Clockwise encirclements of the -1 point generally indicate potential instability.
- The specific stability condition depends on the number of open-loop RHP poles, linking the graphical result to the analytical Routh-Hurwitz method.

Nyquist Plot and Margins

- The Nyquist plot directly visualizes stability margins:
- **Gain Margin**: The reciprocal of the distance from the plot to the -1 point along the real axis.
- **Phase Margin**: The angle to the -1 point from the point where the plot crosses the unit circle ($|G(j\omega)| = 1$).

Example2 *Bode plot*

$$G(s)=\frac{6s+30}{s^4+10s^3+3s^2+50s+24}=\frac{6(s+5)}{(s+1)(s+2)(s+3)(s+4)}$$

Let's convert each term to the bode term $\frac{a}{s+b} \rightarrow \frac{a/b}{s/b+1}$

$$\frac{6(s/5+5/5)*5}{(s+1)(s/2+2/2)*2*(s/3+3/3)*3*(s/4+4/4)*4}$$

$$\frac{1.25(0.2s+1)}{(s+1)(0.5s+1)(0.33s+1)(0.25s+1)}$$

Example2 *Bode plot* Cont. . . .

$$G(j\omega)=\frac{1.25(0.2j\omega+1)}{(j\omega+1)(0.5j\omega+1)(0.33j\omega+1)(0.25j\omega+1)}$$

- The corner frequencies are 1rad/s, 2rad/s, 3rad/s, 4rad/s and 5rad/s
- For very small $\omega=0$

$G(0)=1.25 \Rightarrow dB=20\log|1.25|=1.94$

- @other corner frequencies -2.6db -10dB, -16.8dB, -22.6dB, and -27.7dB
- for very large value $\omega=100$, -105dB

Example2 *Bode plot* Cont. . . .

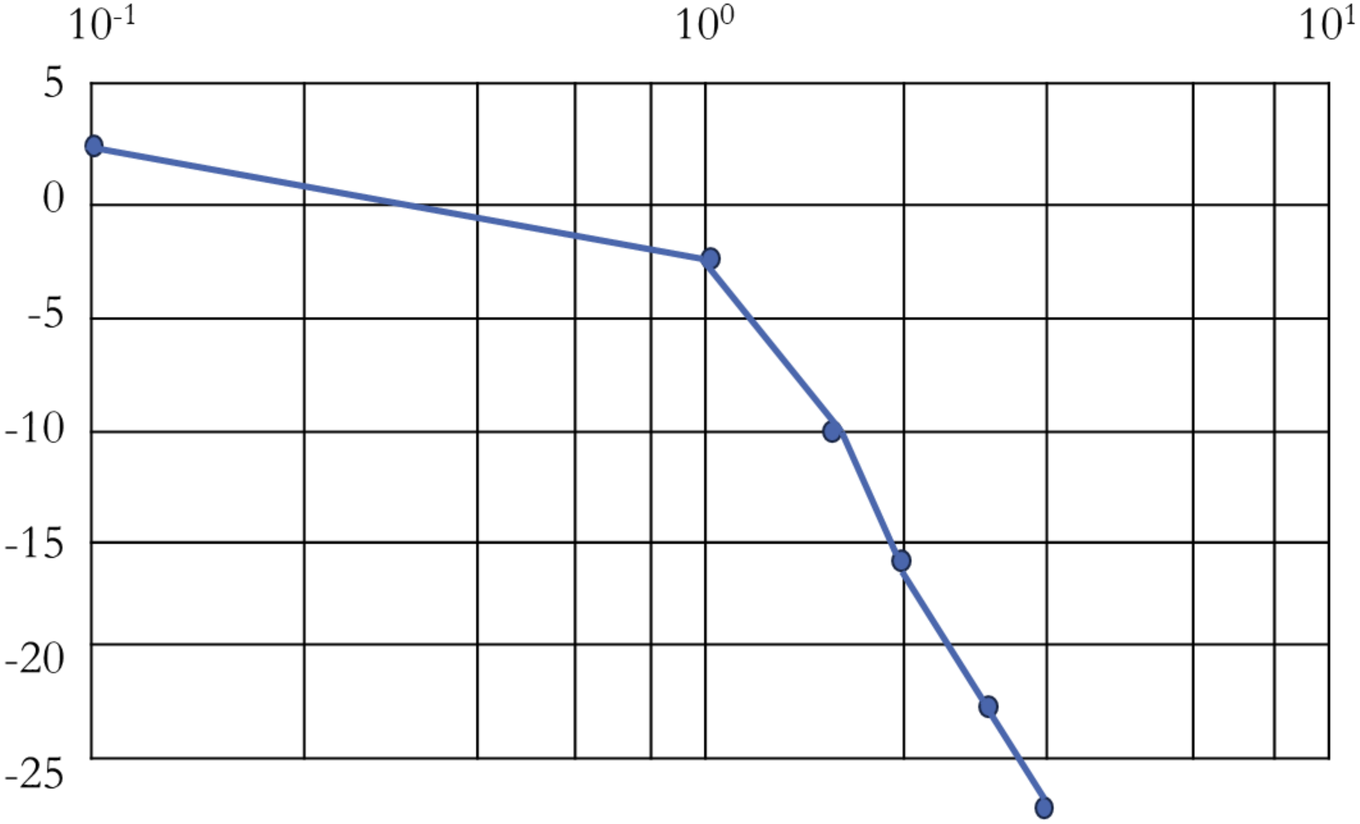

Fig 13: Bode (magnitude) plot for example 2 [12]

Example2 *Bode plot* Cont. . . .

Phase contributions:

- **Zero at ω=5**: $\tan^{-1}(\omega/5)$
- **Pole at ω=1**: $-\tan^{-1}(\omega/1)$
- **Pole at ω=2**: $-\tan^{-1}(\omega/2)$
- **Pole at ω=3**: $-\tan^{-1}(\omega/3)$
- **Pole at ω=4**: $-\tan^{-1}(\omega/4)$

$$\psi_{total} = \tan^{-1}\left(\frac{\omega}{5}\right) - \tan^{-1}\left(\frac{\omega}{1}\right) - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{3}\right) - \tan^{-1}\left(\frac{\omega}{4}\right)$$

$@\ small\ values\ of\ \omega \approx 0$

$$\psi_{total} = \tan^{-1}(0) - \tan^{-1}(0) - \tan^{-1}(0) - \tan^{-1}(0) - \tan^{-1}(0) = 0$$

Example2 *Bode plot* Cont. . . .

- @ other corner frequencies

  -93^0, -145^0, -178^0, -199^0, -212^0

- @ large frequency ≈∞ -270^0

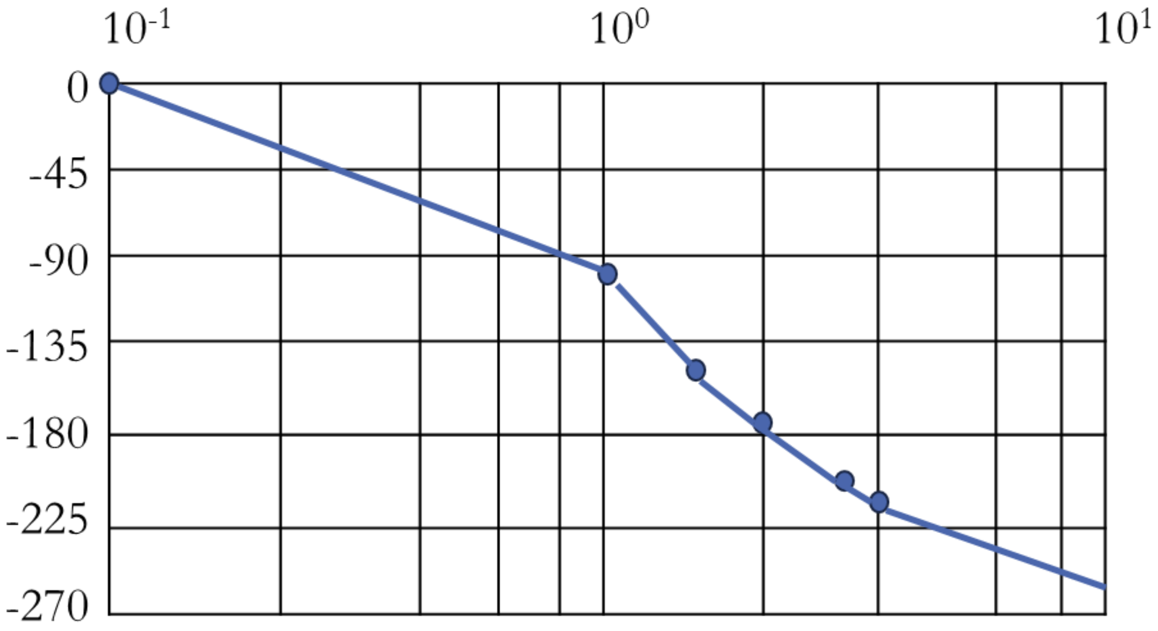

Fig 14: Bode (phase) plot for example 2 [13]

Example2 *Nyquist plot* Cont. . . .

$$G(j\omega)=\frac{6j\omega+30}{(j\omega)^4+10(j\omega)^3+35(j\omega)^2+50(j\omega)+24}=\frac{30+j6\omega}{\omega^4-j10\omega^3-35\omega+j50\omega+24}=\frac{30+j6\omega}{j(50\omega-10\omega^3)+(\omega^4-35\omega+24)}$$

$$=\frac{30+j6\omega}{j(50\omega-10\omega^3)+(\omega^4-35\omega+24)}*\frac{j(50\omega-10\omega^3)-(\omega^4-35\omega+24)}{j(50\omega-10\omega^3)-(\omega^4-35\omega+24)}=\frac{(30+j6\omega)\left(j(50\omega-10\omega^3)-(\omega^4-35\omega+24)\right)}{-(50\omega-10\omega^3)^2-(\omega^4-35\omega+24)^2}$$

$$=\frac{30\omega^4+300\omega^2+1050\omega-720}{(50\omega-10\omega^3)^2+(\omega^4-35\omega+24)^2}+j\frac{6\omega^5+60\omega^4+300\omega^3-210\omega^2-1366\omega}{(50\omega-10\omega^3)^2+(\omega^4-35\omega+24)^2}$$

Example2 *Nyquist plot* Cont. . . .

$$G(j\omega)= \frac{30\omega^4+300\omega^2-1050\omega+720}{(50\omega-10\omega^3)^2+(\omega^4-35\omega+24)^2} + j\,\frac{6\omega^5+60\omega^4+300\omega^3-210\omega^2-1366\omega}{(50\omega-10\omega^3)^2+(\omega^4-35\omega+24)^2}$$

Frequency (ω)	Real Part, Re(ω)	Imaginary Part, Im(ω)	Point (Re, Im)
ω =0	1.25	≈0	**(1.25,0)**
ω = 1	0	-0.74	**(0,-0.74)**
ω = 2	-0.26	-0.18	**(-0.26, -0.18)**
ω = 3	-0.146	0	**(-0.146, 0)**
ω = 4	-0.07	0.02	**(-0.07, 0.02)**
ω = 5	-0.04	0.02	**(-0.04, 0.02)**
ω → ∞	0	0	**(0, 0)**

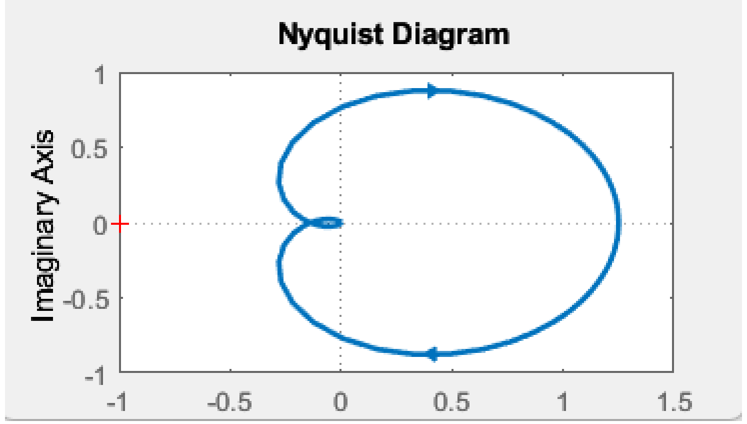

Fig 15: Nyquist plot for example 2 [14]

Real-World Application: Feedback Amplifier

- In electronics, feedback amplifiers can easily oscillate if not designed carefully.
- Frequency domain analysis (Bode/Nyquist) is used to ensure sufficient Phase Margin, preventing oscillation by shaping the gain and phase response.

Summary

- Bode and Nyquist plots are powerful, complementary tools for frequency domain analysis.
- Together, they reveal critical information about stability, speed, and accuracy.
- Looking Ahead: This foundation sets the stage for more advanced topics in Frequency Domain Analysis II, such as controller design techniques.

References

[1] Chalachew Werku, 2023, Frequency input and response, Self-created

[2] Audio equalizer, https://www.shutterstock.com/image-photo/three-golden-equalizer-knobs-bass-260nw-94352236.jpg accessed on October 13, 2025

[3] Bode plot, https://encrypted-tbn0.gstatic.com/images?q=tbn:ANd9GcQvk31ym9rOsjuW0cP2h2j4VYe0gB1kWK8Ncg&s, accessed on October 13, 2025

[4] Filter responses, “generate an image to show "Low-Pass, High-Pass, and Band-Pass Characteristics" on frequency response of control system” prompt, Gemini, Google, 13 Oct. 2025

[5] Chalachew Werku, 2025, Approximate bode (magnitude) plot for example 1, Self-created

[6] Chalachew Werku, 2025, Bode (magnitude) plot using MATLAB for example 1, Self-created

[7] Chalachew Werku, 2025, Approximate bode (phase) plot for example 1, Self-created

[8] Chalachew Werku, 2025, Bode (phase) plot using MATLAB for example 1, Self-created

[9] Chalachew Werku, 2025, Nyquist plot for example 1, Self-created

[10] Chalachew Werku, 2025, Nyquist plot using MATLAB for example 1, Self-created

[11] Nyquist stability criterion, https://www.researchgate.net/publication/323755499/figure/fig6/AS:633761817427970@1528112000727/Nyquist-curves-for-unstable-blue-marginally-stable-green-and-stable-red-systems.png, accessed on October 15, 2025

[12] Chalachew Werku, 2025, Bode (magnitude) plot for example 2, Self-created

[13] Chalachew Werku, 2025, Bode (phase) plot for example 2, Self-created

[14] Chalachew Werku, 2025, Nyquist plot for example 2, Self-created