Course: Regulation and control

Lecture 9: Frequency Domain Analysis II

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Recap: The Pillars of Frequency Domain Analysis

- Bode Plot: Logarithmic plots of Magnitude (dB) and Phase (degrees) vs. Frequency (rad/s).
- Nyquist Plot: Parametric plot of the open-loop frequency response in the complex plane.
- Nyquist Stability Criterion: Determines closed-loop stability by counting encirclements of the -1 point.
- The Critical Point: The location (-1 + 0j) represents a gain of 1 and a phase shift of -180 degrees.

Beyond Stability: The Concept of Robustness

- **Absolute Stability**: A binary yes/no question (Is the system stable?).
- **Relative Stability**: A quantitative measure of **how close** the system is to instability.
- **Robustness**: The system's ability to maintain stability and performance in the face of:
  - Model Uncertainties (e.g., unmodeled dynamics, parameter variations)
  - Real-World Variations (e.g., manufacturing tolerances, temperature changes, wear and tear)
- **Margins as a Safety Buffer**: Gain and Phase Margins provide this crucial measure of relative stability.

Visualizing the Margins on the Bode Plot

- **Gain Crossover Frequency** (ω_{gc}): The frequency where the magnitude plot crosses 0 dB.
- **Phase Margin**: The additional phase lag required at the gain crossover frequency to bring the system to the verge of instability. Measured as the vertical distance from the phase curve to -180 degrees at ω_{gc}.
- **Phase Crossover Frequency** (ω_{pc}): The frequency where the phase plot crosses -180 degrees.
- **Gain Margin**: The additional gain required at the phase crossover frequency to bring the system to the verge of instability. Measured as the vertical distance from the magnitude curve to 0 dB at ω_{pc}.

Visualizing the Margins on the Nyquist Plot

- **The Critical Point**: The -1 point on the negative real axis remains our landmark for instability.
- **Gain Margin on Nyquist**: The reciprocal of the distance from the plot to the -1 point along the negative real axis.
  - If the plot crosses the real axis at -0.5, Gain Margin = 1/0.5 = 2 (which is 6 dB).
- **Phase Margin on Nyquist**: The angle between the negative real axis and the line connecting the origin to the point where the plot crosses the unit circle ($|G| = 1$).
- This point is the Nyquist equivalent of the gain crossover frequency.

Numerical Example 1: Calculating Margins from a Transfer Function

- System: G(s) $=\frac{100}{s(s+2)(s+10)}$

- **Step 1**: Convert to Frequency Response:

$$G(j\omega) = \frac{100}{j\omega(j\omega+2)(j\omega+10)}$$

- **Step 2**: Find Phase Crossover Frequency (ω_{pc}) by setting phase = -180 degrees

$$PM = \tan^{-1}\left(\frac{0}{100}\right) - tan^{-1}\left(\frac{1}{0}\right) - tan^{-1}\left(\frac{\omega_{pc}}{2}\right) - tan^{-1}\left(\frac{\omega_{pc}}{10}\right) = -180$$

Numerical Example 1: Calculating Margins from a Transfer Function Cont....

$$PM = 0 - 90 - tan^{-1}\left(\frac{\omega_{pc}}{2}\right) - tan^{-1}\left(\frac{\omega_{pc}}{10}\right) = -180$$

$$tan^{-1}\left(\frac{\omega_{pc}}{2}\right) + tan^{-1}\left(\frac{\omega_{pc}}{10}\right) = 90$$

$$\tan(\tan^{-1} a + \tan^{-1} b) = \tan(90) \;\; where\; a = \frac{\omega_{pc}}{2} \;\; and\; b = \frac{\omega_{pc}}{10}$$

$$\frac{a+b}{1-ab} = \frac{number}{0} \Rightarrow 1 - ab = 0$$

$$1 = ab \Rightarrow \frac{\omega_{pc}}{2} * \frac{\omega_{pc}}{10} = 1 \Rightarrow \omega_{pc}^2 = 20$$

$$\omega_{pc} = 2\sqrt{5} = 4.47\text{rad/s}$$

Numerical Example 1: Calculating Margins from a Transfer Function Cont….

Step 3: Calculate Gain Margin from magnitude at ω_{pc}.

$$G(j\omega) = \frac{100}{j\omega(j\omega+2)(j\omega+10)}$$

$$\Rightarrow G(j4.47) = \frac{100}{j4.47(j4.47+2)(j4.47+10)}$$

$$|G(j4.47)| = \frac{100}{\sqrt{4.47^2}\sqrt{4.47^2+2^2}\sqrt{4.47^2+10^2}} = 0.417$$

GM=-20log0.417=7.6

Numerical Example 1: Calculating Margins from a Transfer Function Cont.…

- **Step 4:** Find Gain Crossover Frequency (ω_{gc}) by setting magnitude = 1 (0 dB).

$$|G(j\omega_{gc})| = 1 = \frac{100}{\sqrt{\omega_{gc}^2}\sqrt{\omega_{gc}^2+2^2}\sqrt{\omega_{gc}^2+10^2}} \Rightarrow \sqrt{\omega_{gc}^2}\sqrt{\omega_{gc}^2+2^2}\sqrt{\omega_{gc}^2+10^2}=100$$

$$\omega_{gc} = 2.8 rad/s$$

- **Step 5:** Calculate Phase Margin from phase at ω_{gc}.

$$PM = 180 + \tan^{-1}\left(\frac{0}{100}\right) - tan^{-1}\left(\frac{1}{0}\right) - tan^{-1}\left(\frac{2.8}{2}\right) - tan^{-1}\left(\frac{2.8}{10}\right)=19.9$$

GM and PM on Bode plot

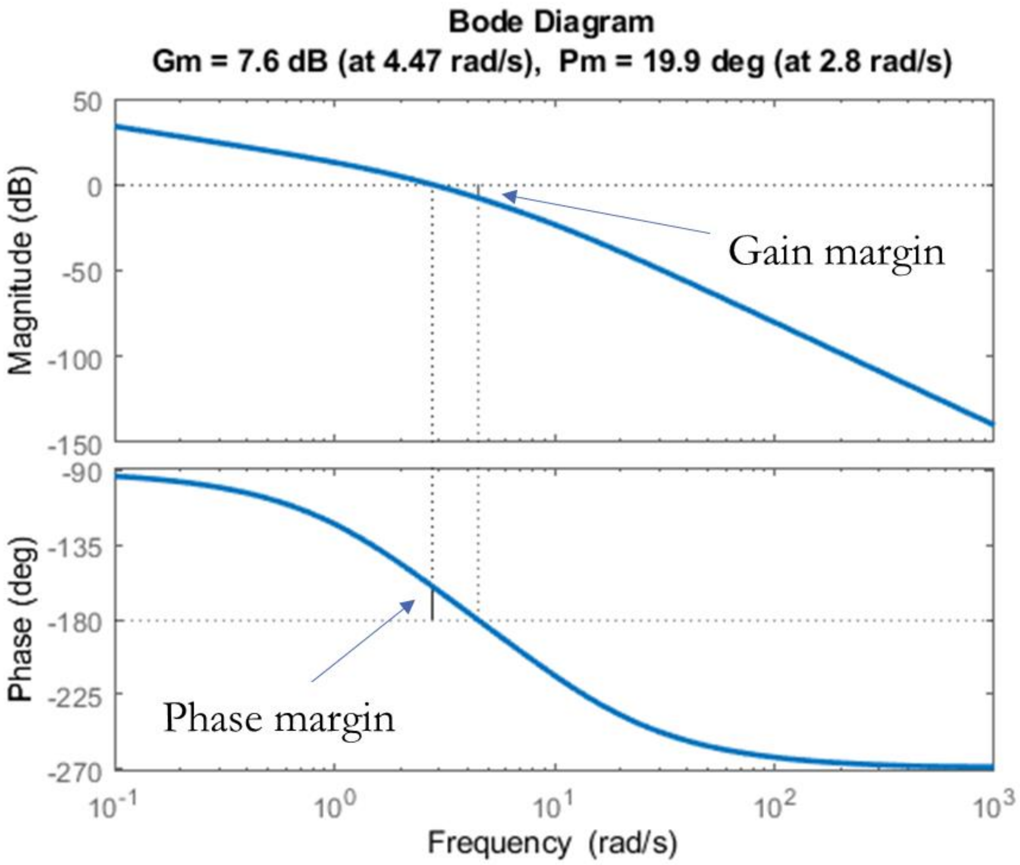

Fig 1: Bode plot for the example 1 using MATLAB [1]

- **Gain Margin**: The vertical distance from the magnitude curve to 0 dB at ω_{pc}.
- **Phase Margin**: The vertical distance from the phase curve to -180 degrees at ω_{gc}.

GM and PM on Nyquist plot

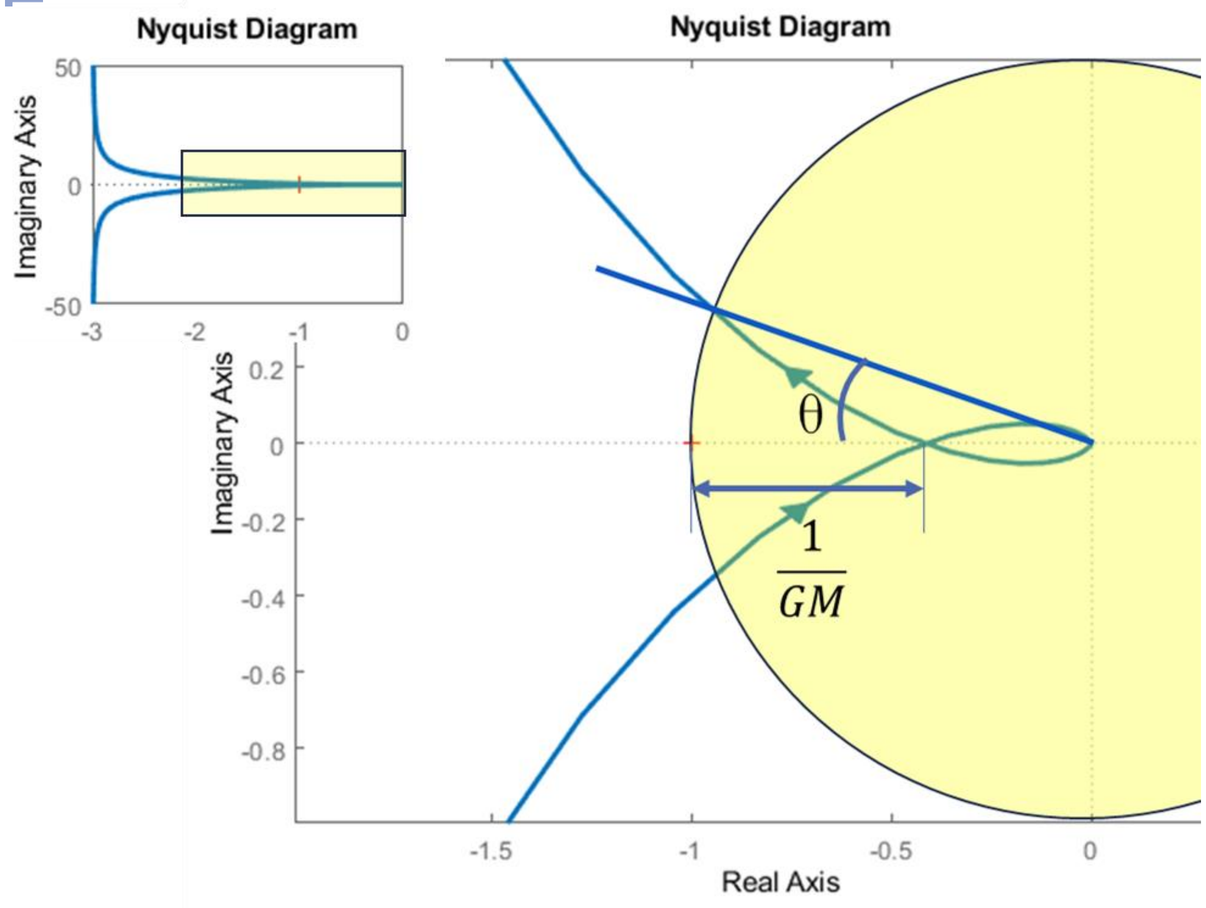

Fig 2: Nyquist plot for the example 1 using MATLAB [2]

- **GM**: The reciprocal of the distance from the plot to the -1 point along the negative real axis.
- **PM**: The angle (θ) between the negative real axis and the line connecting the origin to the point where the plot crosses the unit circle ($|G| = 1$)

Interpreting Margin Values: How Much is Enough?

Phase Margin (PM):

- < 0 degrees: Unstable
- 0-30 degrees: Highly oscillatory, poor robustness (Large Overshoot)
- 30-60 degrees: Good design target (Adequate damping, reasonable overshoot)
- > 60 degrees: Very sluggish, slow response (Over-damped)

Gain Margin (GM):

- < 0 dB: Unstable
- 3-6 dB: Risky, low robustness
- 6-12 dB: Common design range
- > 12 dB: Very conservative

Real-World Analogy: The Shopping Cart Test

- The System: A wobbly shopping cart with a stuck wheel.
- The Control Task: Pushing the cart in a straight line.
- Phase Lag: The delay between you seeing the cart drift and you applying a corrective push.
- Instability (-1 point): The cart oscillates wildly left and right until it tips over.
- Phase Margin: How much longer your reaction time could be before the cart becomes uncontrollable.
- Gain Margin: How much harder you could push the cart before the oscillations become uncontrollable.

Bandwidth: The Speedometer of Your Control System

- **Definition:** The frequency range over which the system can satisfactorily track an input. Often defined as the gain crossover frequency ω_{gc} for closed-loop, or the -3 dB frequency for open-loop.
- **Closed-Loop Bandwidth:** The frequency where the closed-loop magnitude drops to -3 dB (about 70.7%) of its DC value.
- **Interpretation:** A high bandwidth means the system can respond to fast-changing inputs. A low bandwidth means it can only respond to slow commands.
- **Analogy:** The audio bandwidth of a speaker - a woofer has low bandwidth, a tweeter has high bandwidth.

Resonant Peak: The Bump in the Road

- **Definition:** The maximum value of the closed-loop frequency response magnitude, typically occurring near the gain crossover frequency.
- **Relation to Damping:** The resonant peak M_p is directly related to the damping ratio ζ of the dominant closed-loop poles.
  - High Resonant Peak -> Low Damping -> Large Overshoot and Oscillations in time domain.
  - Low/No Resonant Peak -> High Damping -> Smooth, sluggish response.
- **Design Guideline:** For a good transient response, the resonant peak is typically kept below 1.5 (or 3.5 dB).

Crossover Frequencies: The Crossroads of Gain and Phase

- **Gain Crossover Frequency (ω_{gc}):** The frequency where $|G(j\omega)| = 1$ (0 dB). Primary determinant of closed-loop bandwidth and speed of response.
- **Phase Crossover Frequency (ω_{pc}):** The frequency where $\angle G(j\omega) = -180$ degrees. The location where the system is most vulnerable to instability from gain variations.
- **Relationship**: For a stable, well-designed system, ω_{gc}, should be less than ω_{pc}.
- **Importance**: These frequencies are the specific "coordinates" on the Bode plot where we measure our stability margins.

The Loop-Shaping Paradigm: Bode's Insights

- **Fundamental Principle**: We can design a controller by "shaping" the open-loop frequency response (the Bode plot) to achieve desired closed-loop performance.
- **Low-Frequency Gain**: Determines steady-state accuracy (e.g., tracking constant references, rejecting constant disturbances). High gain is desired.
- **Crossover Region (around ω_{gc}):** Determines transient response, stability, and robustness. The slope of the magnitude plot here is critical.
- **High-Frequency Roll-off**: Determines sensitivity to high-frequency noise and unmodeled dynamics. Steep roll-off is desired.
- The -20 dB/decade slope at crossover is a classic design goal for good phase margin.

Ideal Open-Loop Shape for Robust Performance

- Low-Freq: High gain, slope of -20 dB/decade (from an integrator) or 0 dB (from a proportional controller).
- Mid-Freq (Crossover): A clean -20 dB/decade slope through the crossover region for good phase margin.
- High-Freq: Steep roll-off (-40 dB/decade or more) for noise attenuation.
- "Bode's Step": A technique to achieve this by flattening the -40 dB/decade slope to -20 dB/decade before crossover.

Sensitivity Functions: Quantifying Robustness

- **Complementary Sensitivity T(s):** Output / Reference. $T(s) = G(s)C(s) / (1 + G(s)C(s))$. Governs reference tracking and is related to the resonant peak.
- **Sensitivity S(s):** Output / Disturbance (at output). $S(s) = 1 / (1 + G(s)C(s))$. Governs disturbance rejection and robustness to model uncertainties.
- The Relationship: **$S(s) + T(s) = 1$**. You cannot make both small at the same frequency.
- Design Trade-off: Make $|S(j\omega)|$ small at low frequencies for good disturbance rejection. Make $|T(j\omega)|$ small at high frequencies for noise attenuation.

Stability Robustness: The Disk Margin

- **Limitation of Classical Margins**: Gain and Phase Margin are "one-at-a-time" tests. Real-world uncertainties can change both gain and phase simultaneously.
- **Simultaneous Variations**: What if gain increases and phase decreases at the same time?
- Disk Margin: A more general robustness measure. Defines a circular region (a disk) around the -1 point that the Nyquist plot must not enter.
- Interpretation: If the Nyquist plot stays outside this disk, the system remains stable for any combination of gain and phase variations within the disk's boundaries.

Introduction to Frequency Domain Compensators

- The Need: Our plant's natural Bode plot rarely has the ideal shape. We use compensators (controllers) to modify it.
- **Lead Compensator**: Approximates a PD controller. Provides phase lead (positive phase). Used to improve phase margin and speed of response.
- **Lag Compensator**: Approximates a PI controller. Provides high gain at low frequencies. Used to improve steady-state error without affecting stability.
- **Lead-Lag Compensator**: A combination to get the benefits of both.

Lead Compensator

- Transfer Function: $G_{lead}(s) = K * (T s + 1) / (\alpha T s + 1)$, where $\alpha < 1$.
- Frequency Effect:
  - Magnitude: A "bump" or shelf of height $20\log10(1/\alpha)$ dB.
  - Phase: A positive phase peak. Maximum phase occurs at $\omega_{max} = 1 / (T \sqrt{\alpha})$.
- Design Goal: Place ω_{max} at the new gain crossover frequency to inject phase and increase phase margin.
- Side Effect: Increases high-frequency gain (can amplify noise).

Lag Compensator

- Transfer Function: G_lag(s) = K * (T s + 1) / (β T s + 1), where β > 1.
- Frequency Effect:
  - Magnitude: Attenuation at high frequencies. Low-freq gain = K, High-freq gain = K/β.
  - Phase: A negative phase lag. The phase lag is significant around the corner frequencies.
- Design Goal: Place the compensator's high-frequency attenuation below the crossover frequency. This allows you to increase the low-frequency gain K to reduce steady-state error, without moving the crossover frequency or harming phase margin.
- Key: The phase lag must occur at a frequency much lower than ω_{gc} to avoid reducing phase margin.

Case Study: DC Motor Position Control

- Plant: $G_{motor}(s) = K / (s (J s + b))$ - an integrator and a first-order lag. Performance Specs:
  a. Zero steady-state error to a step input.
  b. Settling time < 2 seconds.
  c. Phase Margin > 45 degrees.
- Design Steps:
  1. Use a Lag compensator (or integral action) to ensure zero steady-state error.
  2. Use a Lead compensator to achieve the desired phase margin and speed of response (bandwidth).
  3. Verify design via Bode plot and step response simulation.

Practical Design Procedure: Step-by-Step

- Step 1: Determine performance specifications (steady-state error, settling time, overshoot).
- Step 2: Translate specs to frequency domain (required low-frequency gain, bandwidth, phase margin).
- Step 3: Obtain the plant frequency response (Bode plot of G(s)).
- Step 4: Shape the loop gain using compensators:
  - Use lag compensation to improve low-frequency gain if needed.
  - Use lead compensation to achieve desired phase margin at crossover.
- Step 5: Verify design using both frequency and time domain analyses.

Connecting Frequency and Time Domains: The Complete Picture

- **Low-Frequency Gain ↔ Steady-State Accuracy**: High low-freq gain means small steady-state error.
- **Bandwidth (ω_{gc}) ↔ Speed of Response**: Higher bandwidth means faster settling time.
- **Phase Margin ↔ Damping and Overshoot**: Higher phase margin means less overshoot and better damping.
- **Resonant Peak (M_p) ↔ Oscillatory Behavior**: Larger resonant peak means more ringing and oscillation.
- **Gain Margin ↔ Robustness to Gain Variations**: Larger gain margin means more tolerance to gain changes.

Effect of Time Delays on Stability Margins

- **Time Delay Transfer Function**: e^{-sT} where T is the delay time.
- **Frequency Response**: Magnitude = 1 (0 dB) for all frequencies, Phase = $-\omega T$ radians.
- **Critical Effect**: Adds phase lag that increases linearly with frequency, without affecting gain.
- **Impact on Margins**: Reduces phase margin directly. Can turn a stable system unstable.
- **Rule of Thumb**: The phase lag from time delay at ω_{gc} should be less than the desired phase margin.

Non-Minimum Phase Systems: A Special Challenge

- **Definition**: Systems with zeros in the right-half plane (RHP).
- Strange Behavior: Initially respond in the "wrong" direction before correcting.
- Frequency Domain Signature: Additional phase lag beyond what the magnitude plot suggests.
- Practical Examples: Aircraft altitude control, some chemical processes.
- Design Implication: More difficult to control, require lower bandwidth for same stability margins.

Multivariable Frequency Response: Brief Introduction

- MIMO Systems: Multiple inputs and multiple outputs.
- Extension of Concepts: Gain and phase margins generalize to multivariable systems.
- Singular Values: Replace scalar gain - we now have a range of gains depending on direction.
- Multivariable Nyquist Criterion: Based on determinant of return difference matrix.
- Practical Approach: Analyze each input-output pair, but beware of interactions.

Frequency Domain System Identification

- Concept: Determine a system model from frequency response measurements.
- Experimental Method: Apply sinusoidal inputs at different frequencies, measure gain and phase at each frequency.
- Advantages: Directly gives Bode plot, works well with noisy measurements, reveals resonant modes.
- Applications: Vibration analysis, aircraft flutter testing, process control.
- Connection to Design: The identified model can be used directly for loop-shaping design.

Practical Tips for Bode Plot Interpretation

- Dominant Poles: Look for break frequencies that cause significant phase changes - these usually correspond to dominant dynamics.
- Check Slope at Crossover: The -20 dB/decade rule is your friend for good phase margin.
- Beware of Close Poles/Zeros: They can cause rapid phase changes that are easy to miss in hand sketches.
- Use Asymptotes First: Sketch the asymptotic Bode plot first, then refine.
- Always Check Both Margins: A system can have good phase margin but poor gain margin, or vice versa.

Software Tools for Frequency Domain Analysis

- MATLAB/Simulink: Comprehensive control systems toolbox with bode(), margin(), nyquist() functions.
- Python: SciPy and Matplotlib for basic analysis, specialized control libraries for advanced work.
- Commercial Packages: LabVIEW, MatrixX, others for specific industries.

Software Tools for Frequency Domain Analysis Cont. …

- Key Features to Look For:
  - Automatic margin calculation
  - Interactive loop shaping
  - Time-frequency correlation tools
- Best Practice: Use software to verify hand designs, not replace understanding.

Common Design Pitfalls and How to Avoid Them

- Ignoring High-Frequency Dynamics: Unmodeled fast dynamics can cause instability.
- Over-reliance on Software: Blindly trusting margin calculations without understanding the physics.
- Fighting Fundamental Limits: Trying to achieve bandwidths that are physically impossible.
- Poor Sensor/Actuator Selection: The best controller can't compensate for bad hardware.
- Solution: Understand your physical system, use appropriate models, validate with experiments.

Advanced Topics: Robust Control and H-infinity Methods

- H-infinity Control: Explicitly designs for robustness to model uncertainties.
- Design Approach: Shapes the sensitivity functions directly to meet performance specifications.
- Mixed Sensitivity Problem: Balances performance (S small) with robustness and control effort.
- Application Areas: Aerospace, precision manufacturing, systems with large uncertainties.
- Connection to Classical Methods: Extends the margin concepts we've learned to more general uncertainty descriptions.

Summary: The Frequency Domain Design Toolkit

- Key Tools: Bode plots, Nyquist plots, gain/phase margins, sensitivity functions.
- Design Philosophy: Shape the open-loop frequency response to achieve desired closed-loop performance.
- Fundamental Trade-offs: Performance vs. robustness, tracking vs. noise rejection, bandwidth vs. stability.
- Practical Process: Translate specs → analyze plant → design compensators → verify performance.
- Next Steps: Practice with real examples, explore software tools, study advanced robust control methods.

References

[1] Chalachew Werku, 2025, Bode plot for the example 1 using MATLAB, Self-created

[2] Chalachew Werku, 2025, Nyquist plot for the example 1 using MATLAB, Self-created

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