



# Course: Regulation and control

Lecture 10: Root Locus Techniques I

Lecturer: Chalachew Werku

# What is Root Locus?

- **Definition:** A graphical method for analyzing how the poles of a closed-loop system move in the s-plane as a single parameter (usually gain  $K$ ) is varied from 0 to infinity.
- **Key Point:** It visualizes the migration of system poles.

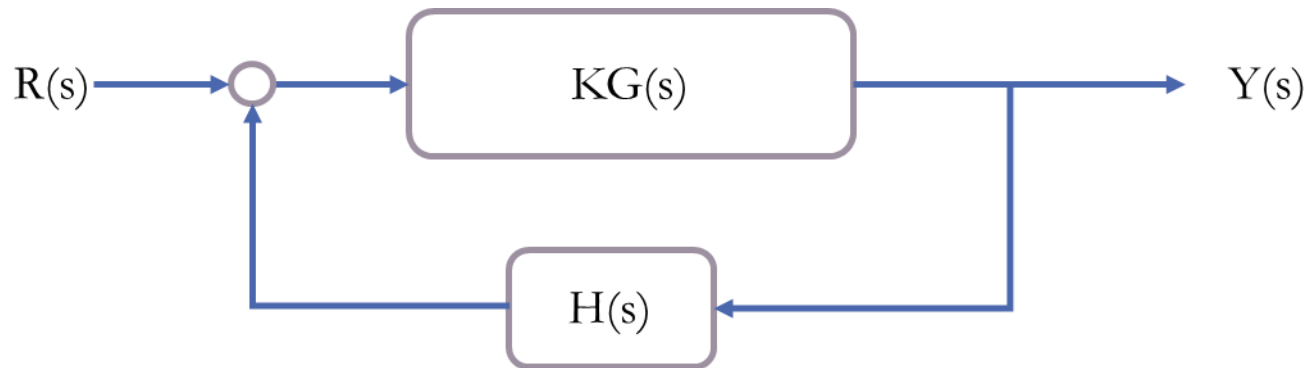


Fig 1: Closed loop block diagram [1]



# Why Use Root Locus?

- **Visualize System Behavior:** See pole trajectories at a glance.
- **Avoid Repetitive Calculations:** No need to solve the characteristic equation repeatedly.
- **Key Benefits:** Determine stability, Predict transient response, Design controller gain.

K	Pole 1	Pole 2
0	-1	-2
2	$-1.5-1.32j$	$-1.5+1.32j$
6	$-1.41j$	$1.41j$
10	$0.5-2.18j$	$0.5+2.18j$

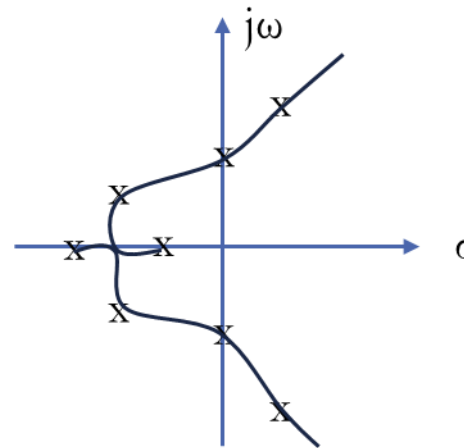


Fig 3: Tracing root locus [3]

# Key Applications

- **Control System Design:** Tuning controllers for desired performance.
- **Stability Analysis:** Ensuring systems do not oscillate uncontrollably.
- **Performance Tuning:** Adjusting for speed of response and overshoot.



Fig 4: Applications of root locus (Robotics) [4]

# Closed-Loop System Block Diagram

- Definitions:

$R(s)$ =Input,

$Y(s)$ =Output,

$G(s)$ =Plant,

$H(s)$ =Sensor, and

$K$ =Gain.

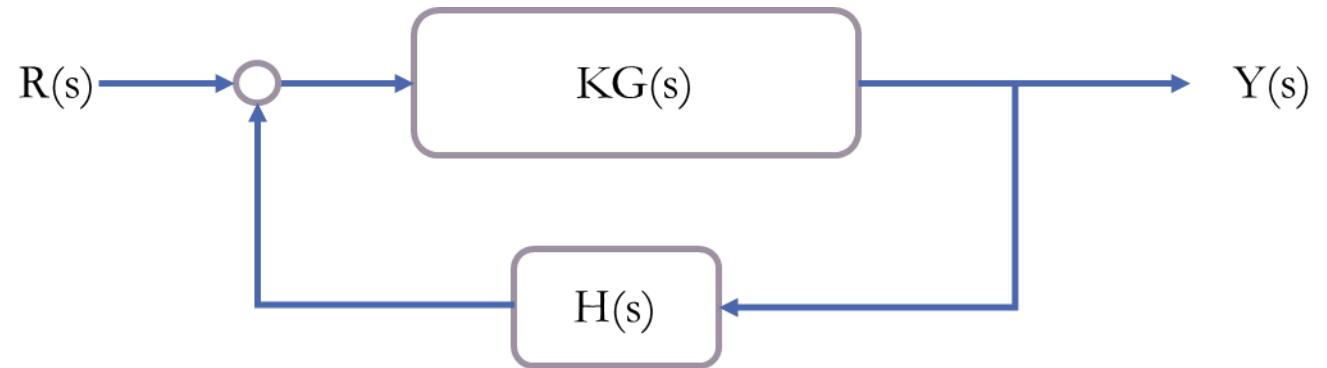


Fig 5: Closed loop block diagram [1]

# Characteristic Equation: $1 + KG(s)H(s) = 0$

- The Fundamental (Characteristics) Equation

$$1 + KG(s)H(s) = 0$$

- It defines the poles of the closed-loop system.
- Rearranged:  $KG(s)H(s) = -1$

# Pole Locations & System Response

- **Left:** Stable and further left faster response.
- **Right:** Unstable
- **Imaginary Axis:** Sustained oscillations.
- Angle from real axis:  
Damping/Overshoot.

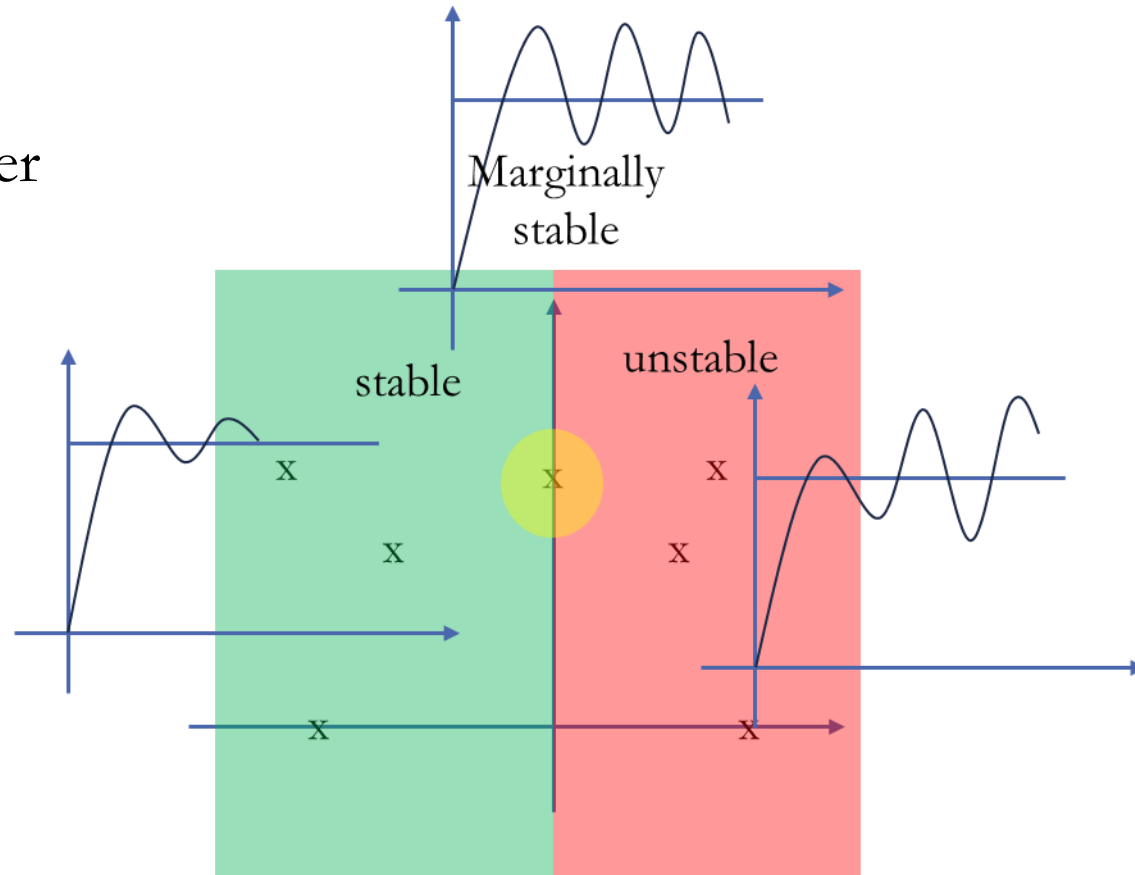


Fig 6: Effect of pole location on system stability and response [5]

# Effect of Gain Variation

- Low K: Poles start at open-loop poles.
- High K: Poles end at open-loop zeros or infinity.
- Key Message: Gain change moves poles, changing response.

$$G(s)H(s) = K \frac{s^2 + 8s + 15}{s^2 + 4s + 8}$$

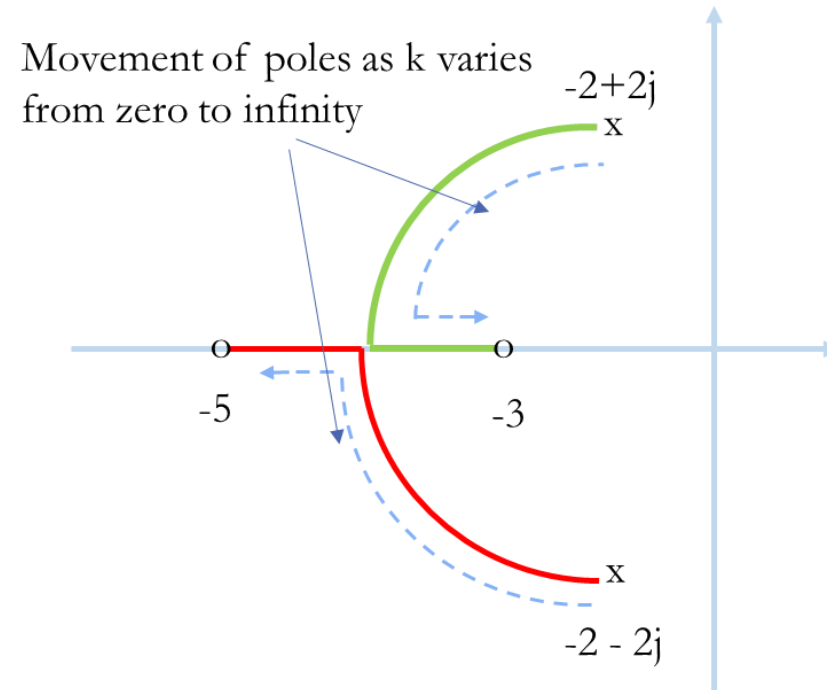


Fig 7: Movement of pole as K varies [6]

# Review: Open-Loop vs Closed-Loop Poles

- **Open-Loop Poles:** Poles of  $G(s)H(s)$ .  
The system without feedback.
- **Closed-Loop Poles:** Poles of the full system with feedback. Roots of  $1+kG(s)H(s)=0$ .
- **Analogy:** Natural talent (Open-Loop) vs. Performance with a coach (Closed-Loop).

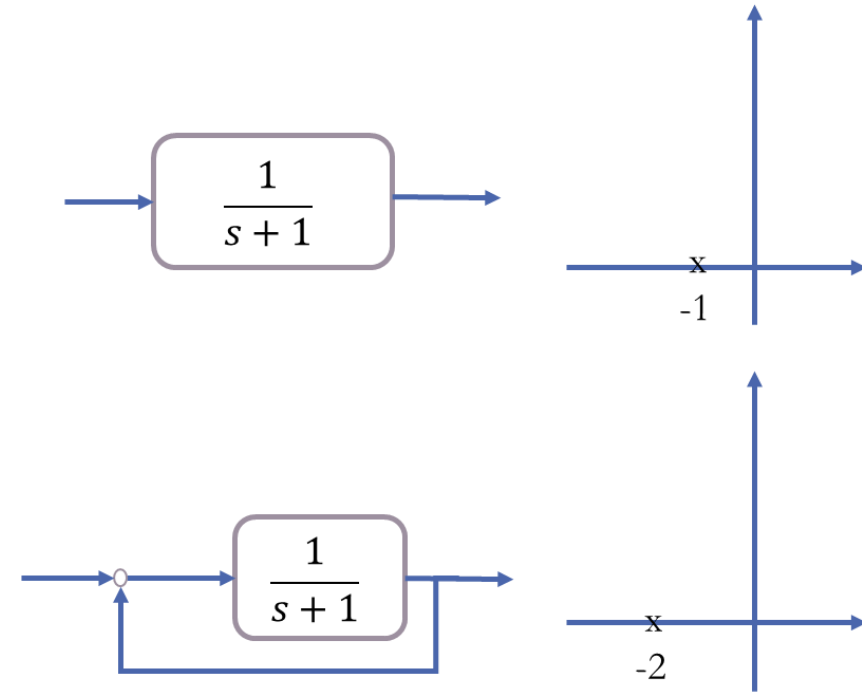


Fig 8: Open and closed loop system with their respective pole locations. [7]

# Angle Condition

- **The Rule:**  $\angle G(s)H(s) = \pm 180^\circ(2k+1)$

where  $k=0,1,2,\dots$

- **Meaning:** For a point 's' to be on the root locus, the sum of angles from all open-loop poles and zeros to that point must be an odd multiple of  $180^\circ$ .

**Equation:**  $\Sigma(\text{Angles from Zeros}) - \Sigma(\text{Angles from Poles}) = 180^\circ + 360^\circ k$

# Angle Condition Visualization

Checking whether a point “s” is on the root locus or not

Equation:

$$\Sigma(\text{Angles from Zeros}) - \Sigma(\text{Angles from Poles}) = 180^\circ + 360^\circ k$$

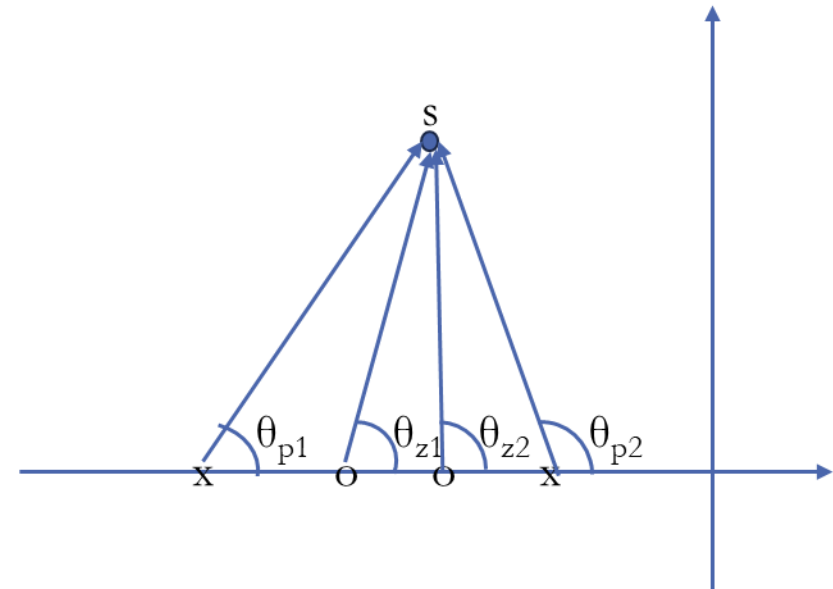


Fig 9: Angle condition [8]

# Magnitude Condition

- **The Rule:**  $|K G(s)H(s)| = 1$

$$\text{Rearranged: } K = \frac{1}{|G(s)H(s)|}$$

- **Meaning:** The gain  $K$  at a point ' $s$ ' on the locus is the reciprocal of the product of the lengths of the vectors from the open-loop poles to ' $s$ ' divided by the product of the lengths from the open-loop zeros to ' $s$ '.

# Magnitude Condition Application

- $K = \frac{1}{|G(s)H(s)|}$
- $K = \frac{l_{p1} * l_{p2}}{l_{z1} * l_{z2}}$

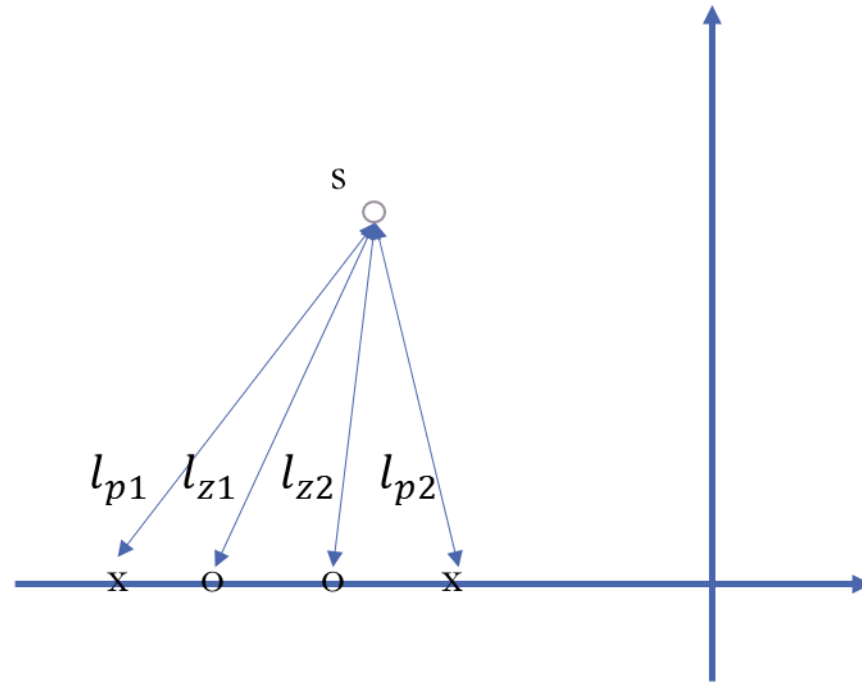


Fig 10: Magnitude condition [9]

# Practice Problem: Verify if a point lies on root locus

## *angle condition*

- Given:  $G(s)H(s) = \frac{k}{s(s+2)}$ .
  - A specific test point  $s = -1 + j$ .
- Task:
  1. Plot poles/zeros and the point.
  2. Draw vectors.
  3. Calculate angle sum.
  4. Check if it equals  $180^\circ$ .

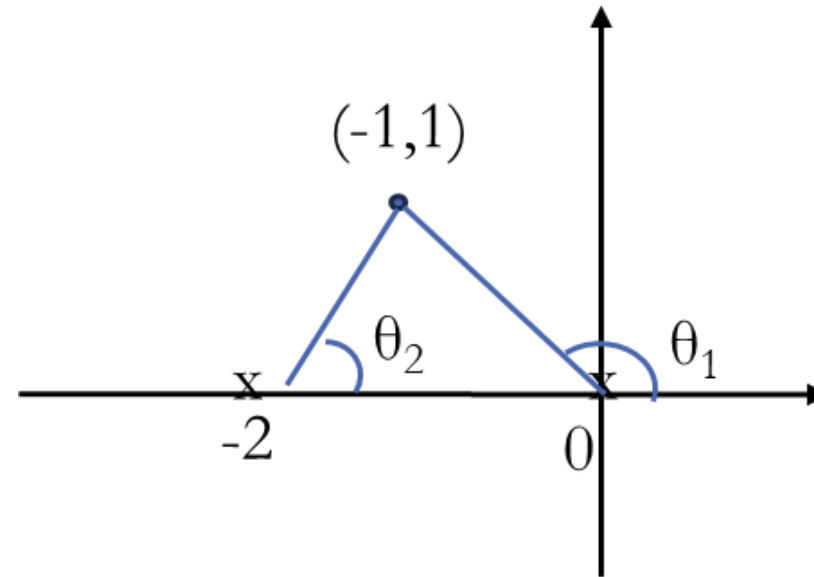


Fig 11: Angle condition plot [10]

# Rule 1: Symmetry

- **The Rule:** The root locus is symmetric about the real axis.
- **Reason:** Complex poles always occur in conjugate pairs.
- **Implication:** We only need to draw the upper half; the lower half is a mirror image.

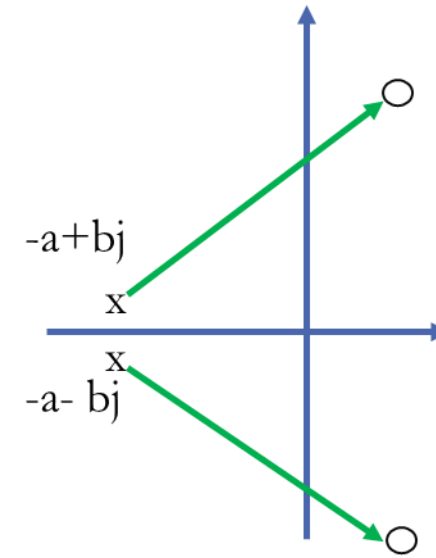


Fig 12: Root locus is symmetry [11]

## Rule 2: Starting & Ending Points

- **Starting Points ( $K=0$ ):** The root locus starts at the open-loop poles.
- **Ending Points ( $K \rightarrow \infty$ ):** The root locus ends at the open-loop zeros.

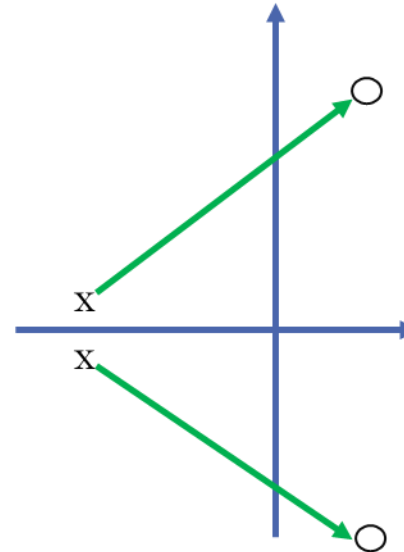
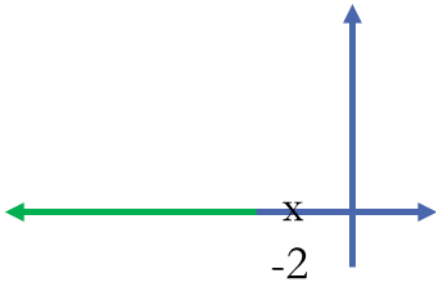


Fig 13: Root locus emanating from poles (X) and terminating at zeros (O). [12]

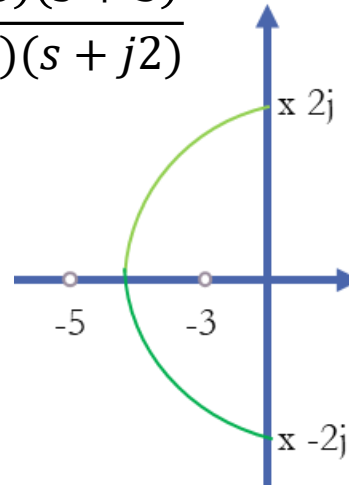
# Rule 2 Examples

$$\frac{k}{s + 2}$$



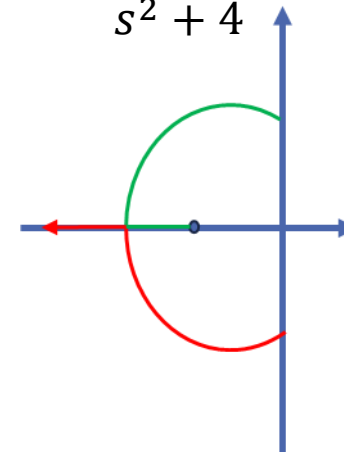
1 poles, 0 zeros (ends at infinity)

$$\frac{k(s + 3)(s + 5)}{(s - j2)(s + j2)}$$



2 pole, 2 zero

$$\frac{k(s + 3)}{s^2 + 4}$$



2 poles, 1 zero (2 branches end at infinity)

Fig 14: Root locus starting and ending points [13]

## Rule 3: Number of Branches

- **The Rule:** The number of branches of the root locus is equal to the number of poles of the open-loop transfer function.
- **Reason:** The characteristic equation has a number of roots equal to the number of poles.

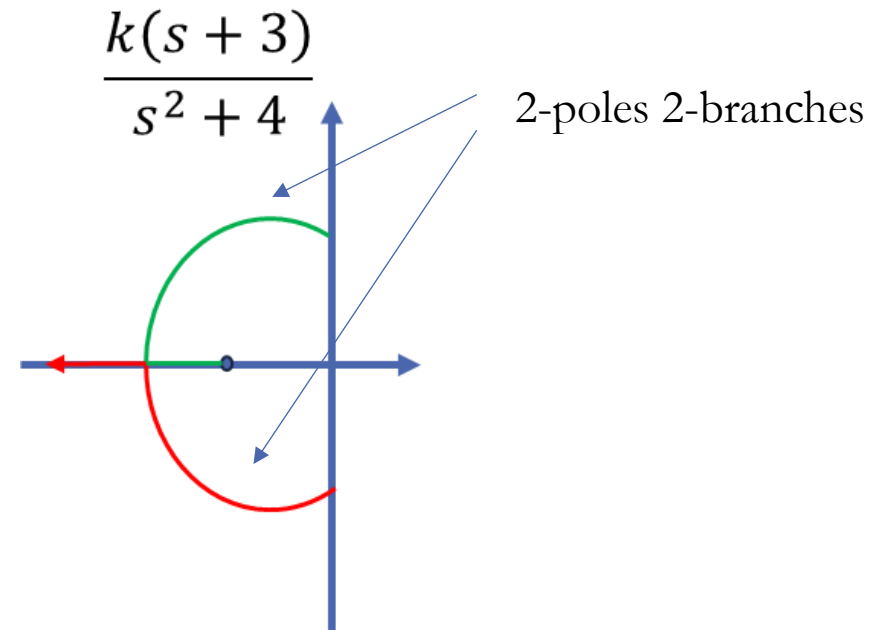
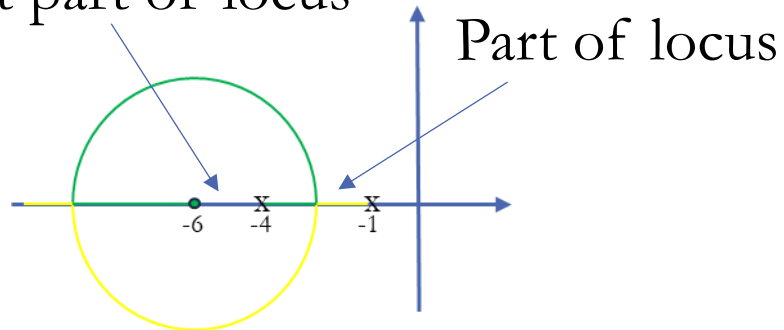


Fig 15: Pole number vs branches [14]

## Rule 4: Real Axis Segments

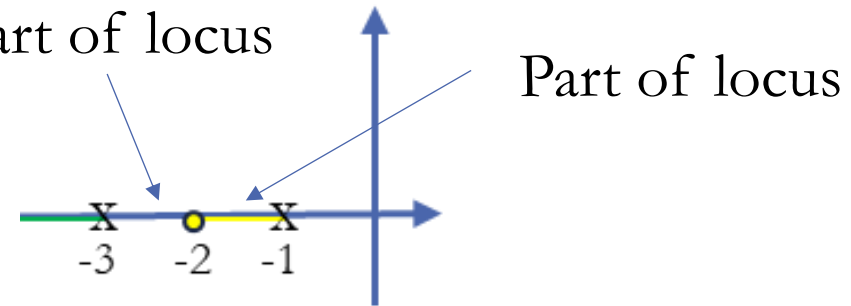
- **The Rule:** A point on the real axis is part of the root locus if the number of open-loop poles and zeros to the right of that point is odd.

Not part of locus



**Example 1:** Poles at -4, -1; Zero at -6.  
Locus left of -6 and between -1 and -4.

Not part of locus



**Example 2:** Poles at -3, -1; Zero at -2.  
Locus between -1 and -2 and left of -3

Fig 16: Real axis segments [15]

# Interactive Exercise: Identify real axis segments

- Given: A pole-zero plot on the real axis.
- Task: "Shade the portions of the real axis that are part of the root locus."

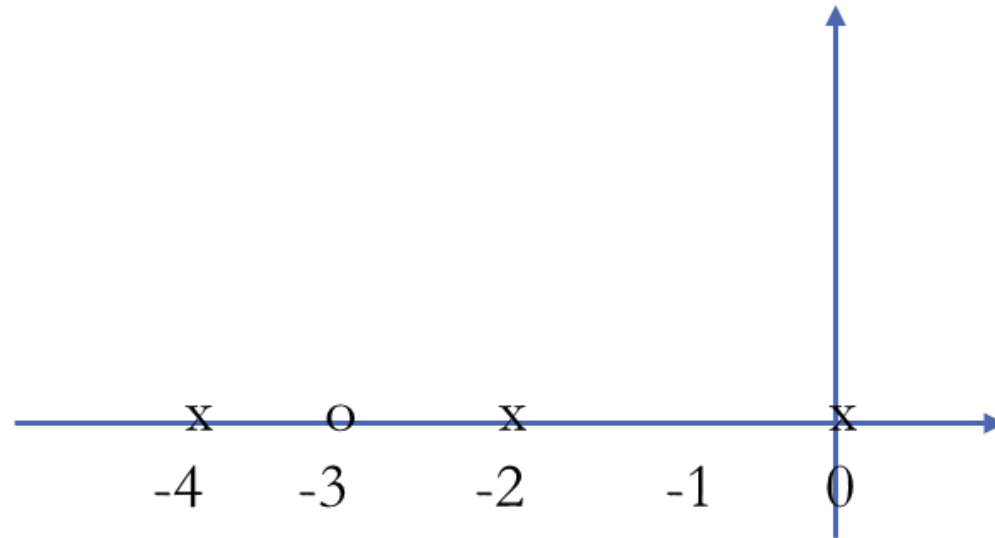


Fig 17: Exercise: identify real axis segments [16]

# Interactive Exercise: Identify real axis segments

*solution*

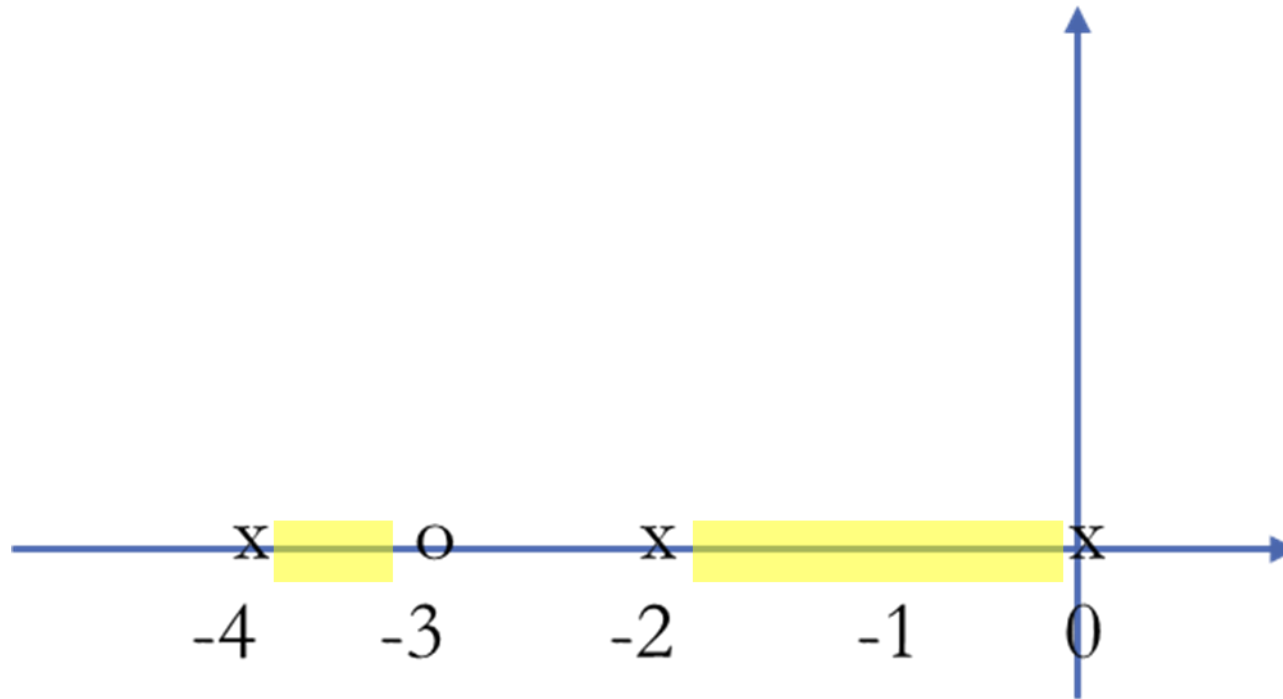


Fig 18 Solution: identify real axis segments [17]

# Case Study 1: Second-order system

- System: Simple second-order system  $\frac{1}{s(s+2)}$
- Task: Apply Rules 1-4 to sketch the root locus.
  1. Plot poles (0, -2). No zero
  2. Apply Rule 4 (locus on real axis between 0 and -2).
  3. Two branches start at 0 and -2 and ends at infinity

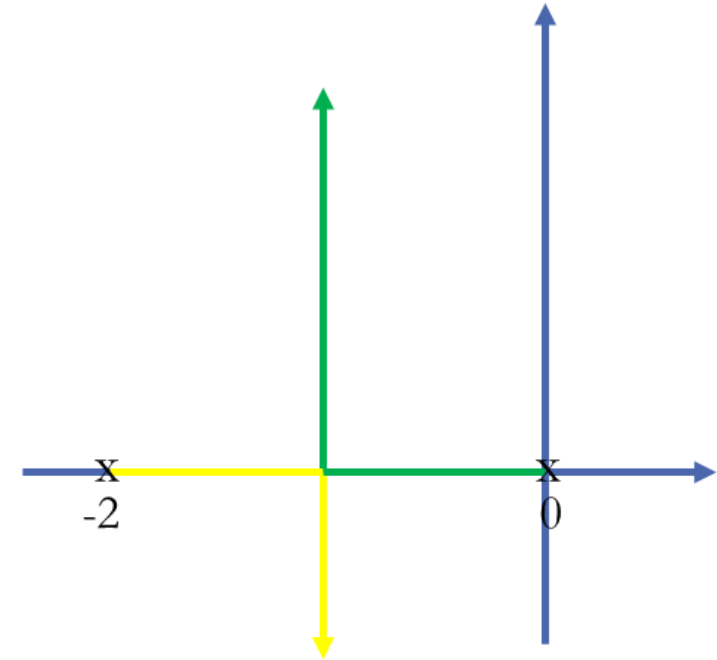


Fig 19: Sketching the root locus, first order system [18]

## Case Study 2: Third-order system

- System: A third-order system  $\frac{1}{s(s+1)(s+3)}$
- Task: Apply Rules 1-4.
  1. Plot poles (0, -1, -3).
  2. Rule 4: Locus on real axis between 0 and -1, and left of -3. 3.
  3. Three branches start at the three poles.

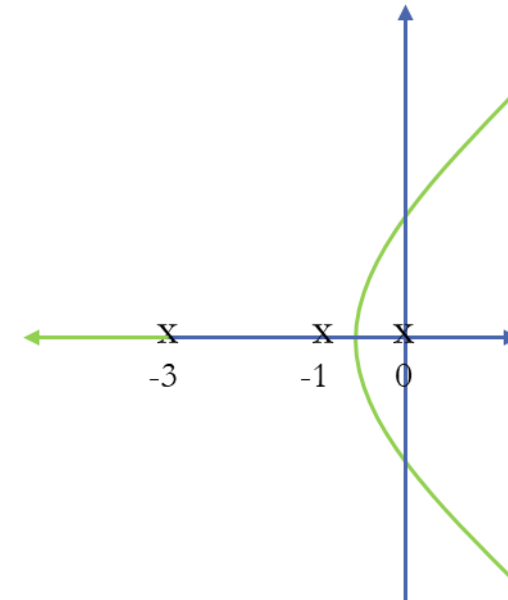


Fig 20: Sketching the root locus, third order system [19]

## Quick Review: Rules 1-4 summary

- **Rule 1:** Symmetry about real axis.
- **Rule 2:** Starts at poles ( $K=0$ ), ends at zeros ( $K=\infty$ ).
- **Rule 3:** number of Branches = number of Poles.
- **Rule 4:** Real axis segment is on locus if number of poles + zeros to the right is odd.

## Rule 5: Asymptotes for Large K

- **Concept:** Branches that go to infinity approach straight-line asymptotes.
- **Center of Asymptotes ( $\sigma$ ):**

$$\sigma = \frac{\Sigma \text{Poles} - \Sigma \text{Zeros}}{n - m}$$

- **Angle of Asymptotes ( $\theta$ ):**

$$\theta = \pm 180^\circ \frac{(2k+1)}{n-m} \text{ for } k=0,1,2\dots$$

**Where:**  $n$  = number of poles,  $m$  = number of zeros.

# Asymptote Center and Angle Calculation

- **Example:** For a system with poles at 0, -1, -3 and no zeros:

**Centroid:**

$$\sigma = \frac{0 + (-1) + (-3)}{3 - 0} = -\frac{4}{3} = -1.33$$

**Angles:**

$$\theta = \pm 180^\circ \frac{2k + 1}{3 - 0}$$

For  $k=0$ :  $\theta=60^\circ$ ,  $k=1$ :  $\theta=180^\circ$ ,  $k=2$ :  $\theta=300^\circ$  (or  $-60^\circ$ ).

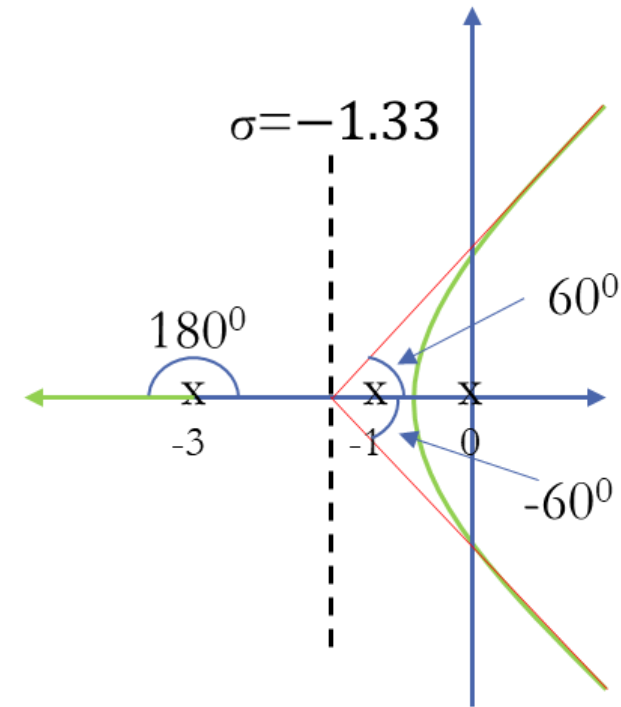


Fig 21: Asymptote center and angle calculation [20]

# Asymptote Examples

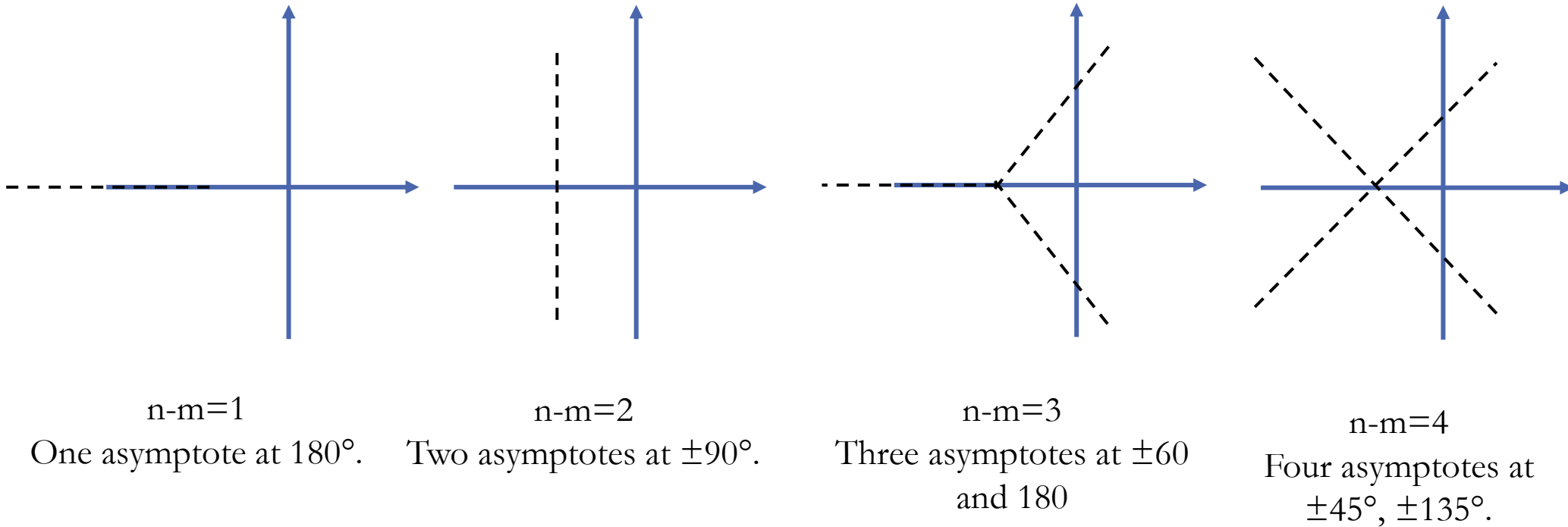


Fig 22: Common asymptote configuration. [21]

## Rule 6: Breakaway/Break-in Points

- **Concept:** Points where root locus branches leave (breakaway) or join (break-in) the real axis.

**Calculation:** Solve  $\frac{\partial K}{\partial s} = 0$  from the characteristic equation.

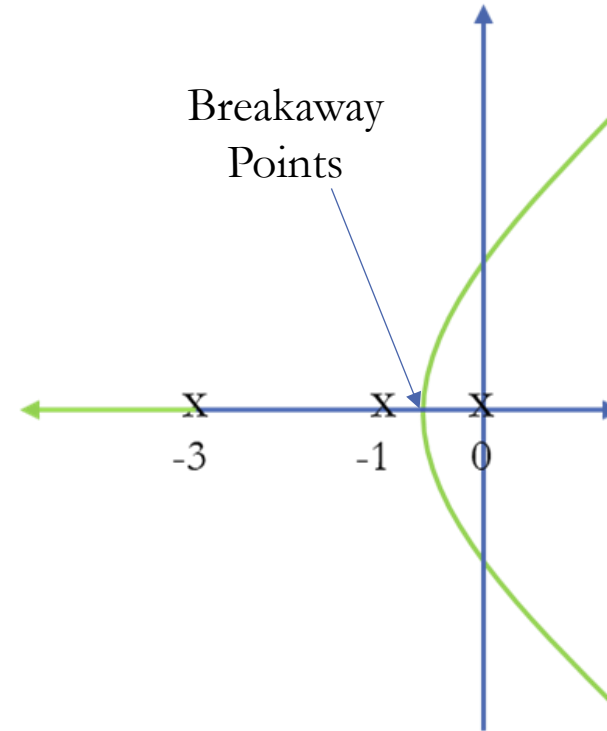


Fig 23: Breakaway point [22]

# Break in or breakaway point

- **Between Two Poles:** There must always exist a Break-Away point on the real axis segment between two consecutive open-loop poles.
- **Between Two Zeros:** There must always exist a Break-In point on the real axis segment between two consecutive open-loop zeros.
- **Between Pole & Zero:** A break point may or may not exist on the segment between a pole and a zero on the real axis. Its existence and type must be verified by solving  $dK/ds = 0$ .
- **Key Principle:** These points arise from the locus "repelling" from poles and "attracting" to zeros as the gain  $K$  increases.

# Breakaway Point Calculation Methods

- **Method 1 (Calculus):**

1. Write  $K = -1/G(s)H(s)$ .
2. Differentiate with respect to  $s$ .
3. Set  $dK/ds = 0$  and solve for  $s$ .

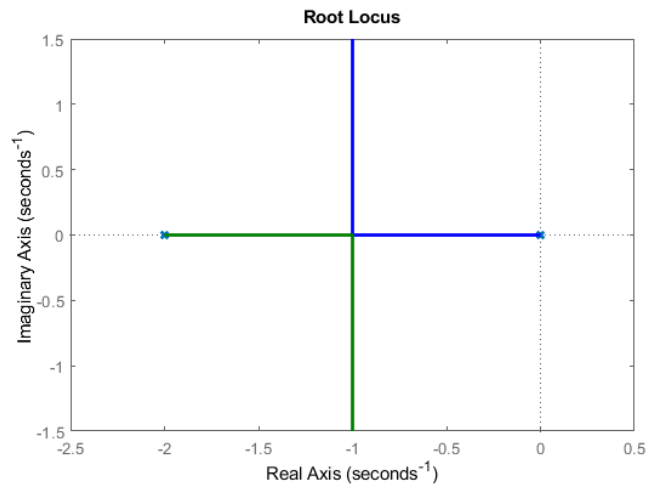
- **Method 2 (Characteristic Eqn):**

1. Write  $1+KG(s)H(s)=0$ .
2. Solve for  $K$ .
3. Differentiate, set  $dK/ds=0$ .

# Breakaway Point Examples

**Example 1:** System with poles at 0 and -2.

$$1 + k \frac{1}{s(s+2)} = 0 \quad \frac{dk}{ds} = -2s - 2 = 0$$
$$k = -s^2 - 2s = 0 \quad s = -1$$



**Example 2:** System with poles at -1, -2 and a zero at -3, -5. Finding a break-in and breakaway point.

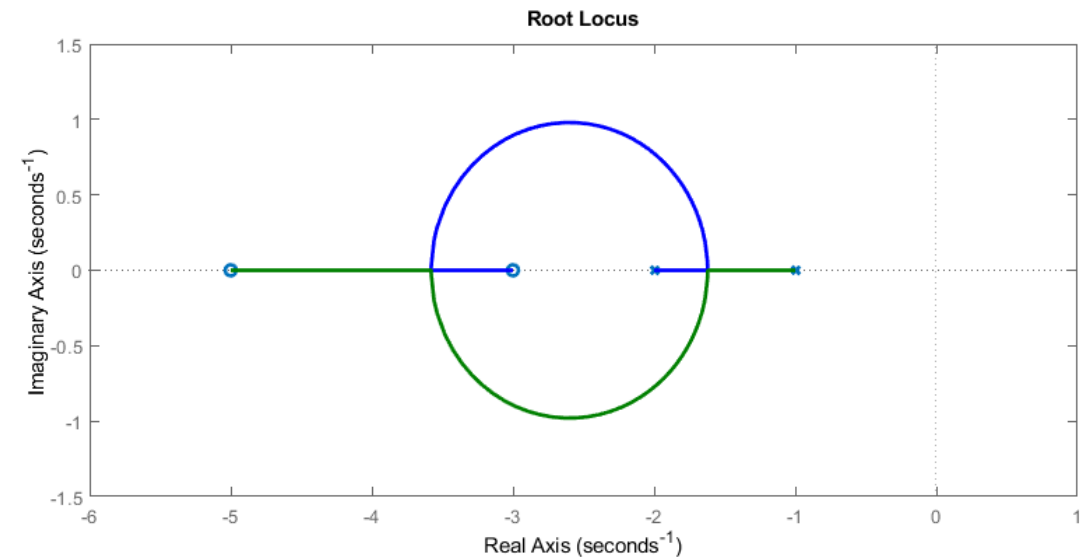


Fig 24: Break-in and breakaway points example. [23]

## Rule 7: Departure/Arrival Angles

- **Concept:** The angle at which a root locus branch leaves a complex pole (departure angle) or arrives at a complex zero (arrival angle).
- **Calculation:** Use the Angle Condition, solving for the unknown departure/arrival angle.

# Departure/ Arrival Angle Calculation

- **Formula:**

$\varphi_d = 180^\circ - \Sigma(\text{Angles from other poles to this pole}) + \Sigma(\text{Angles from zeros to this pole}).$

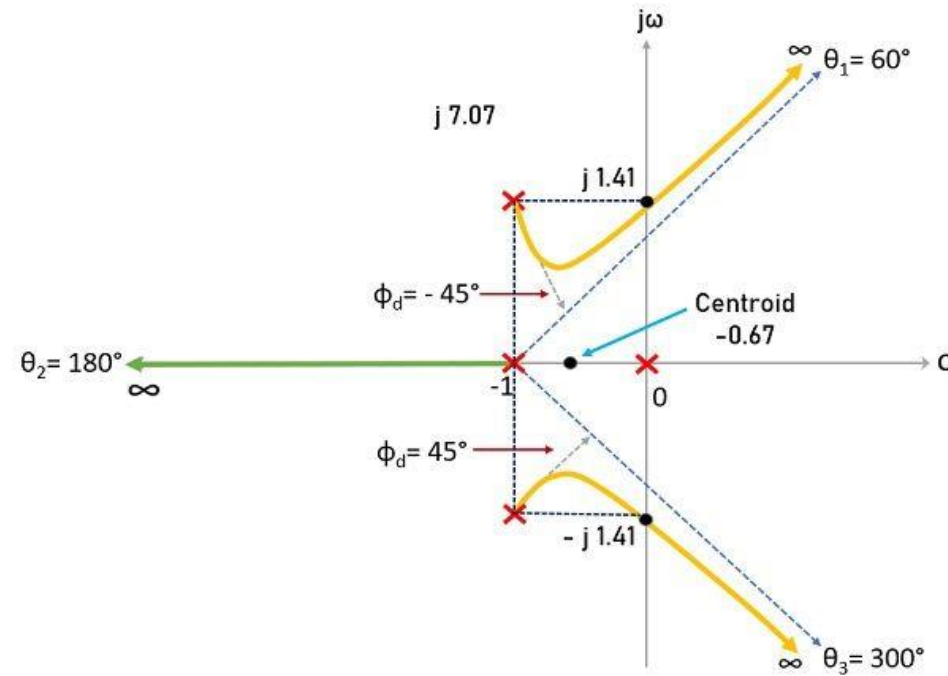


Fig 25: Angle of departure and arrivals [24]

## Rule 8: Imaginary Axis Crossings

- **Concept:** Finding the value of gain  $K$  where the root locus crosses the imaginary axis into the Right-Half Plane (instability).
- **Method 1:** Use the Routh-Hurwitz Criterion on the characteristic equation.
- **Method 2:** Substitute  $s=j\omega$  into the characteristic equation and solve for  $K$  and  $\omega$ .

# Examples

## *angles of departure and arrival*

$$G(s) = \frac{1}{s(s^2 + 2s + 2)} \text{ and } H(s) = 1$$

$$G(s)H(s) = \frac{1}{s(s^2 + 2s + 2)} = \frac{1}{s(s+1-j1)(s+1+j1)}$$

We need only angle of departures (i.e conjugate poles), from  $s = -1 \pm j1$

$$\varphi_{s=-1+j1} = 180 - \left[ \tan^{-1} \left( \frac{1}{-1} \right) + \tan^{-1} \left( \frac{2}{0} \right) \right] = 180 - [135 + 90] = -45^\circ$$

$$\varphi_{s=-1-j1} = 180 - \left[ \tan^{-1} \left( \frac{-1}{-1} \right) + \tan^{-1} \left( \frac{-2}{0} \right) \right] = 180 - [225 + (-90)] = 45^\circ$$

# Examples

## *imaginary crossing*

$$1 + KG(s)H(s) = 1 + K \frac{K}{s(s^2 + 2s + 2)} = 0$$

$$s^3 + 2s^2 + 2s + K = 0$$

$s^3$	1	2
$s^2$	2	K
$s^1$	$\frac{4 - K}{2}$	0
$s^0$	K	0

$$\frac{4 - K}{2} = 0$$
$$K = 4$$

@K=4, auxiliary equation b

$$A(s) = 2s^2 + K = 2s^2 + 4$$

$$2s^2 + 4 = 0$$

$$2s^2 = -4,$$

$s = \pm j\sqrt{2}$  imaginary crossing

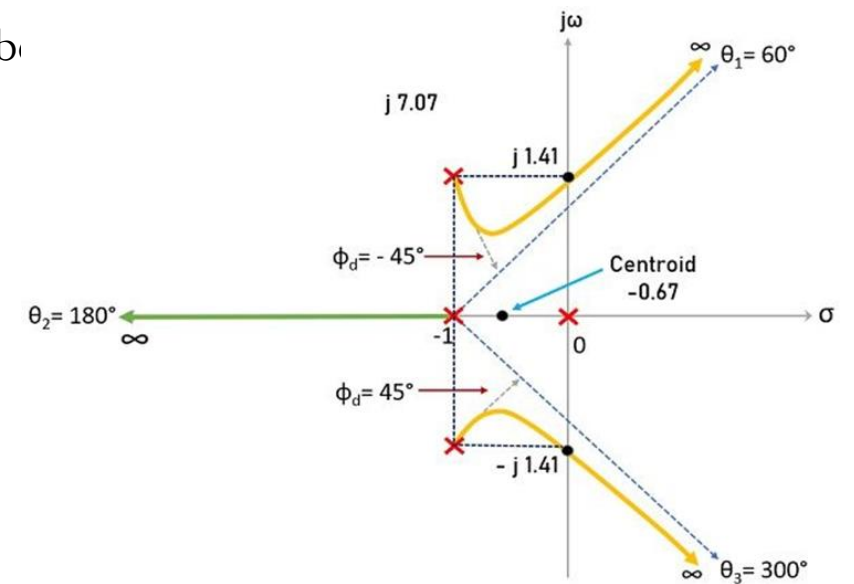


Fig 26: Angle of departure and imaginary crossing point [24]

# Dominant Poles Concept

- What are Dominant Poles? The poles closest to the imaginary axis.
- **Why they matter:** They dominate the transient response of the system. Slower poles have a longer-lasting effect.

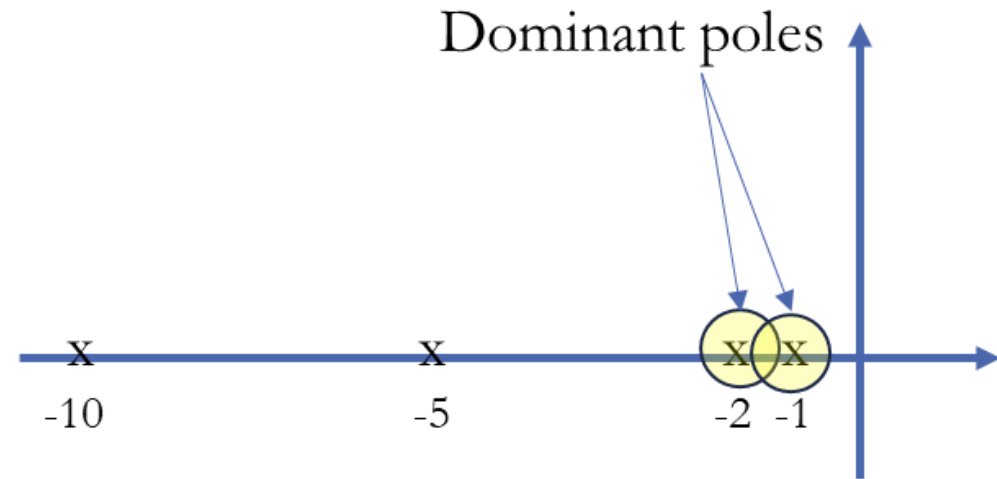


Fig 27: Dominant poles [25]

# Why Dominant Poles Matter

- **Transient Response:** The slowest poles (dominant poles) determine the settling time and the nature of the oscillations.
- **Approximation:** We can often approximate a 4th-order system by its dominant 2nd-order pair.

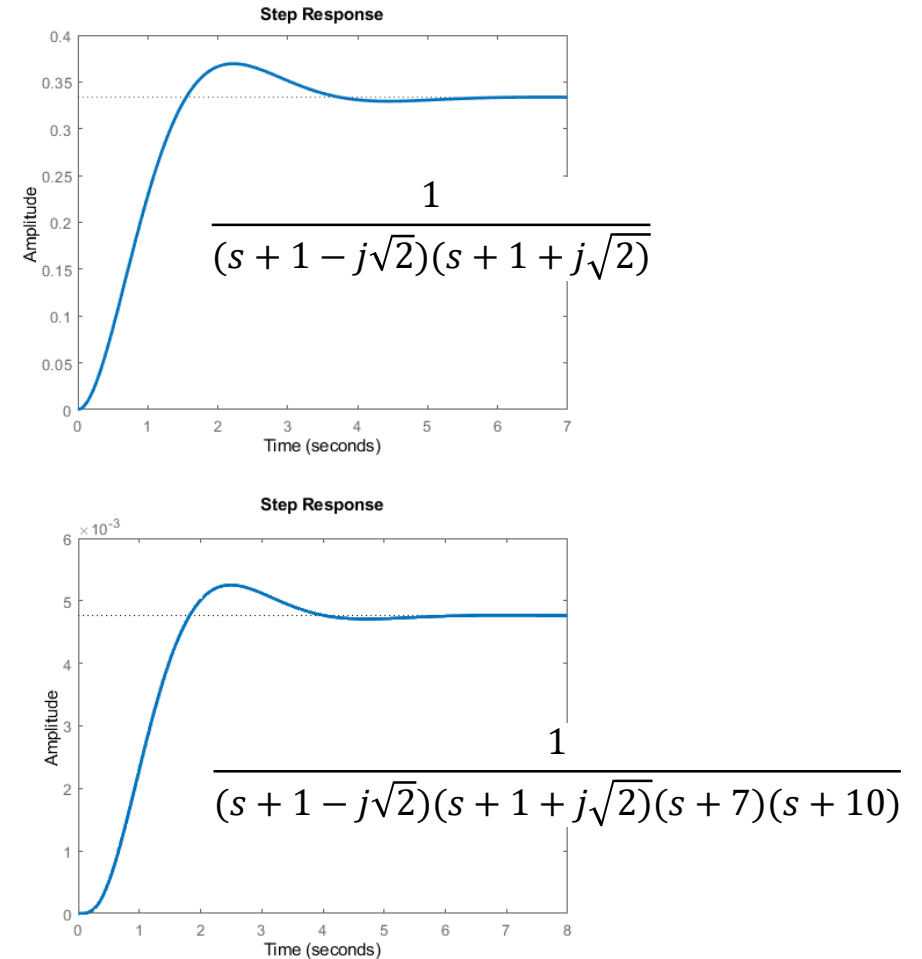


Fig 28: Effects of dominant poles [26]

# Second-Order Approximation

- **Concept:** Using the characteristics ( $\zeta$ ,  $\omega_n$ ) of the dominant complex pole pair to approximate the response of a higher-order system.
- **Method:**
  1. Identify the dominant pole pair on the root locus.
  2. Calculate  $\zeta$  and  $\omega_n$  from their location.
  3. Use second-order system formulas to predict  $T_s$ ,  $T_r$ , %OS.
- **Caution:** This is invalid if a zero is very close to the dominant poles.

# Dominant Pole Examples

- Example 1: A system with poles at  $-1 \pm 2j$ ,  $-10$ ,  $-20$ . The  $-1 \pm 2j$  poles are dominant. The response looks like a  $\zeta=0.45$ ,  $\omega_n=2.23$  rad/s system.
- Example 2: A system with poles at  $-1 \pm 2j$  and a zero at  $-1.5$ . The zero negates the dominant pole approximation.

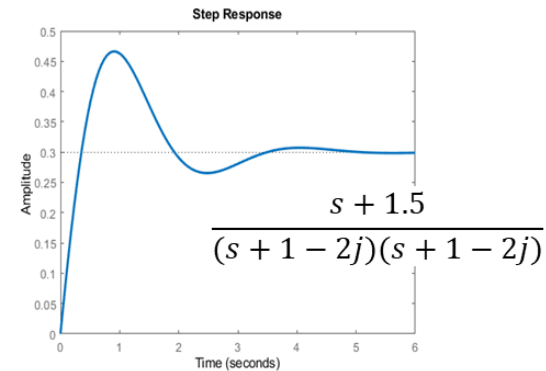
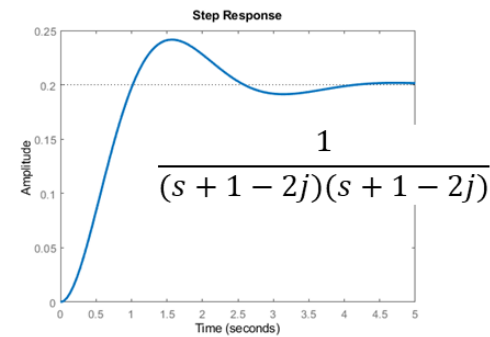
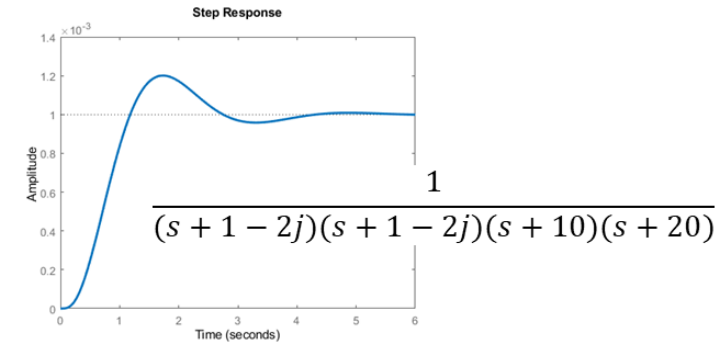


Fig 29: Effects of dominant poles and zero [27]

# Summary & Key Takeaways

- Root Locus: A map of closed-loop pole paths vs. gain  $K$ .
- Governed by: Angle and Magnitude conditions.
- 8 Construction Rules: Symmetry, Start/End, Branches, Real-Axis, Asymptotes, Breakaway, Angles, Imaginary Crossings.
- Design Insight: Use dominant poles and second-order approximation for tuning.

# References

- [1] Chalachew Werku, 2025, Closed loop block diagram, Self-created.
- [2] Root locus construction, [https://media.springernature.com/lw685/springer-static/image/chp%3A10.1007%2F978-3-030-93445-3\\_5/MediaObjects/521794\\_2\\_En\\_5\\_Fig8\\_HTML.png](https://media.springernature.com/lw685/springer-static/image/chp%3A10.1007%2F978-3-030-93445-3_5/MediaObjects/521794_2_En_5_Fig8_HTML.png), accessed on October 21, 2025.
- [3] Chalachew Werku, 2025, Tracing root locus, Self-created.
- [4] Applications of root locus (Robotics), [https://www.google.com/url?sa=i&url=https%3A%2F%2Fwww.evsint.com%2Frobot-maintenance-guide%2F&psig=AOvVaw09mNDuTEidjNo3GnQbJpQK&ust=1761557503511000&source=images&cd=vfe&opi=89978449&ved=0CBUQjRxqFwoTCMD2\\_6LHwZADFQAAAAdAAAAABAE](https://www.google.com/url?sa=i&url=https%3A%2F%2Fwww.evsint.com%2Frobot-maintenance-guide%2F&psig=AOvVaw09mNDuTEidjNo3GnQbJpQK&ust=1761557503511000&source=images&cd=vfe&opi=89978449&ved=0CBUQjRxqFwoTCMD2_6LHwZADFQAAAAdAAAAABAE), accessed on October 21, 2025.
- [5] Chalachew Werku, 2025, Effect of pole location on system stability and response, Self-created.
- [6] Chalachew Werku, 2025, Movement of pole as K varies, Self-created.
- [7] Chalachew Werku, 2025, Open and closed loop system with their respective pole locations, Self-created.

# References

Cont. . . .

- [8] Chalachew Werku, 2025, Angle condition, Self-created.
- [9] Chalachew Werku, 2025, Magnitude condition, Self-created.
- [10] Chalachew Werku, 2025, Angle condition plot, Self-created.
- [11] Chalachew Werku, 2025, Root locus is symmetry, Self-created.
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- [13] Chalachew Werku, 2025, Root locus starting and ending points, Self-created.
- [14] Chalachew Werku, 2025, Pole number vs branches, Self-created.
- [15] Chalachew Werku, 2025, Real axis segments, Self-created.
- [16] Chalachew Werku, 2025, Exercise: identify real axis segments, Self-created.
- [17] Chalachew Werku, 2025, Solution: identify real axis segments, Self-created.
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