Course: Regulation and control

Lecture 11: Root Locus Techniques II

Lecturer: Chalachew Werku

Quick Review: The Map We Draw

- Recap of 8 Construction Rules.
- The S-Plane: Stable LHP vs. Unstable RHP.
- The Goal: Keep poles in the LHP.

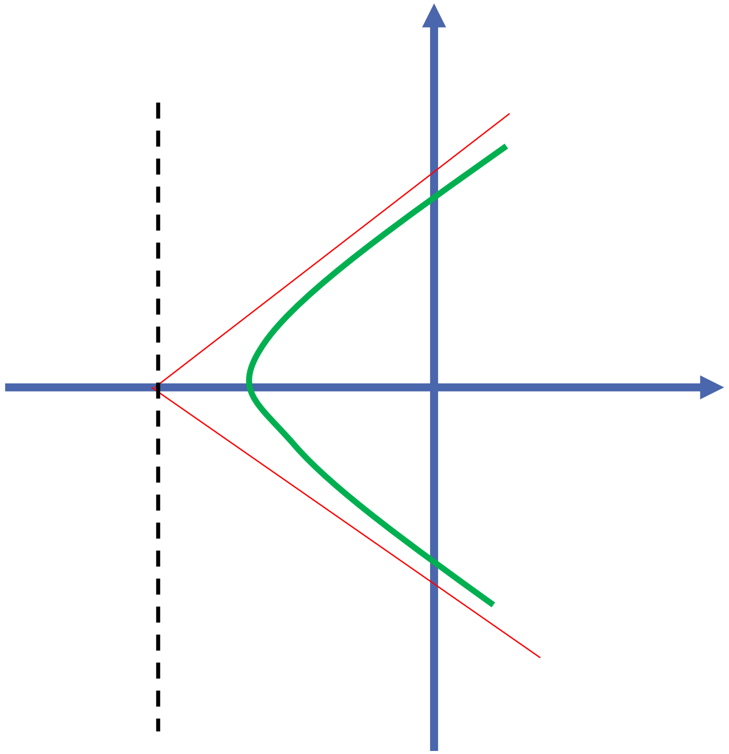

Fig 1: Construction of root locus [1]

From Drawing to Analysis

- Part I: How to draw the locus.
- Part II: What does the locus tell us?
- Key Questions:
  - Is it stable?
  - How stable is it?
  - Can we make it better?

Defining Stability

- **Stable System**: Output remains bounded for a bounded input. (Poles in LHP)
- **Marginally Stable**: Sustained oscillations. (Poles on Imaginary Axis)
- **Unstable**: Output grows without bound. (Poles in RHP)

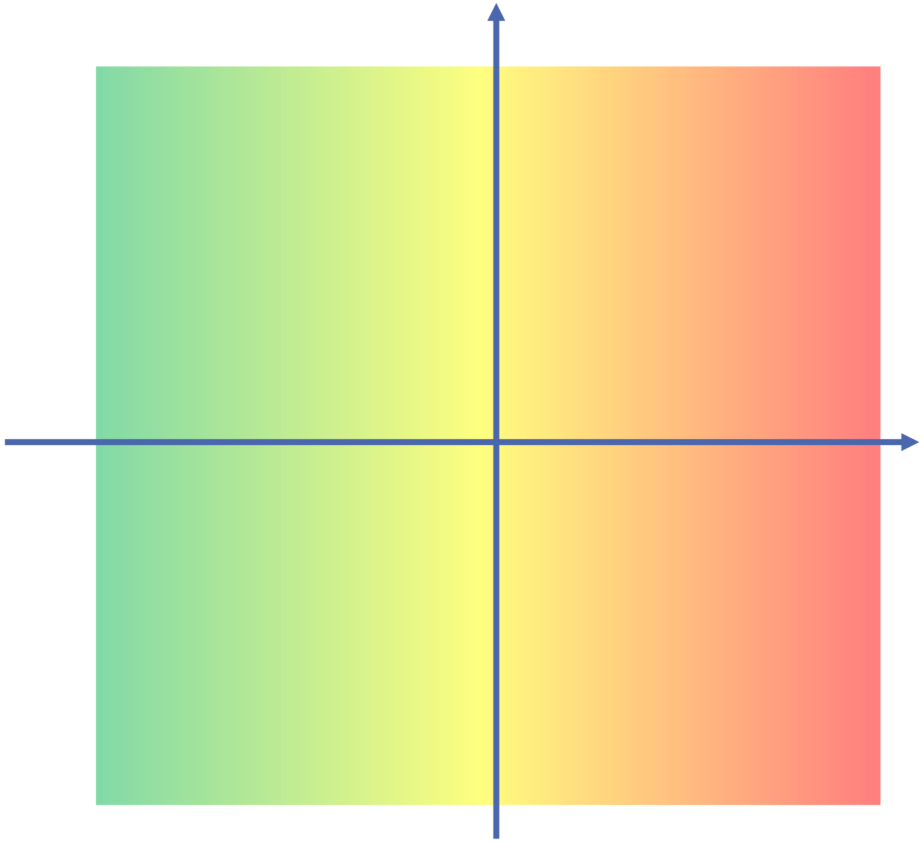

Fig 2: Stable, marginal stable and unstable regions of s-plane [2]

The Simplest Stability Analysis

- Look at the plot!
- For your chosen gain K, are all poles in the Left-Half Plane?
  - Yes → Stable.
  - No → Unstable.

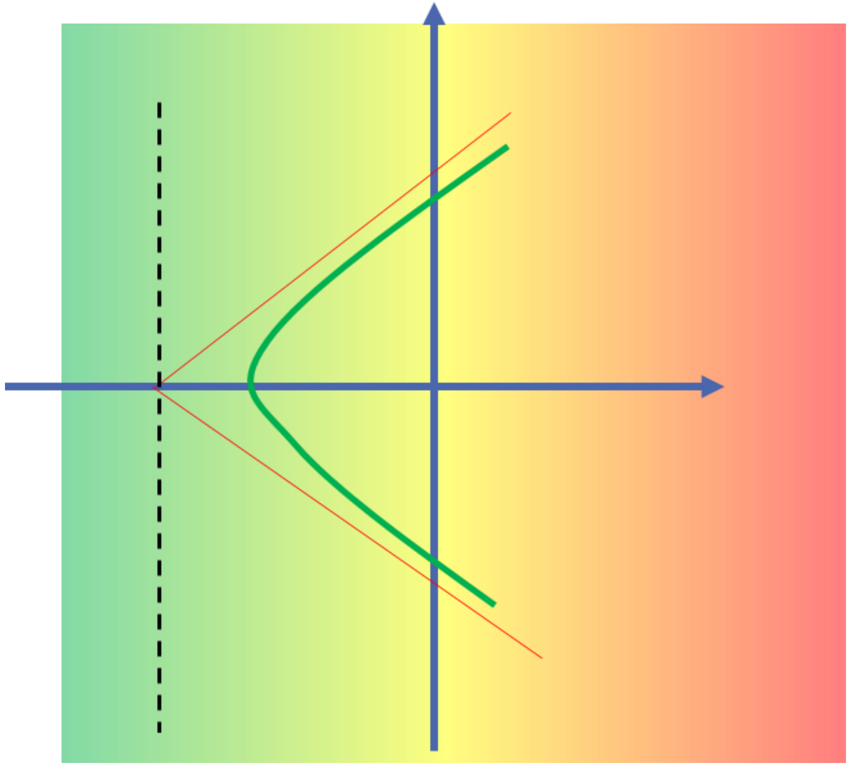

Fig 3: Movement of poles on "s-plane" as "K" varies. [3]

Finding the Range of Stable Gains

- Identify the critical gain, $K_{critical}$.
- $K_{critical}$ is the gain at the imaginary axis crossing (from Rule 8).
- Stable if: $0 < K < K_{critical}$.

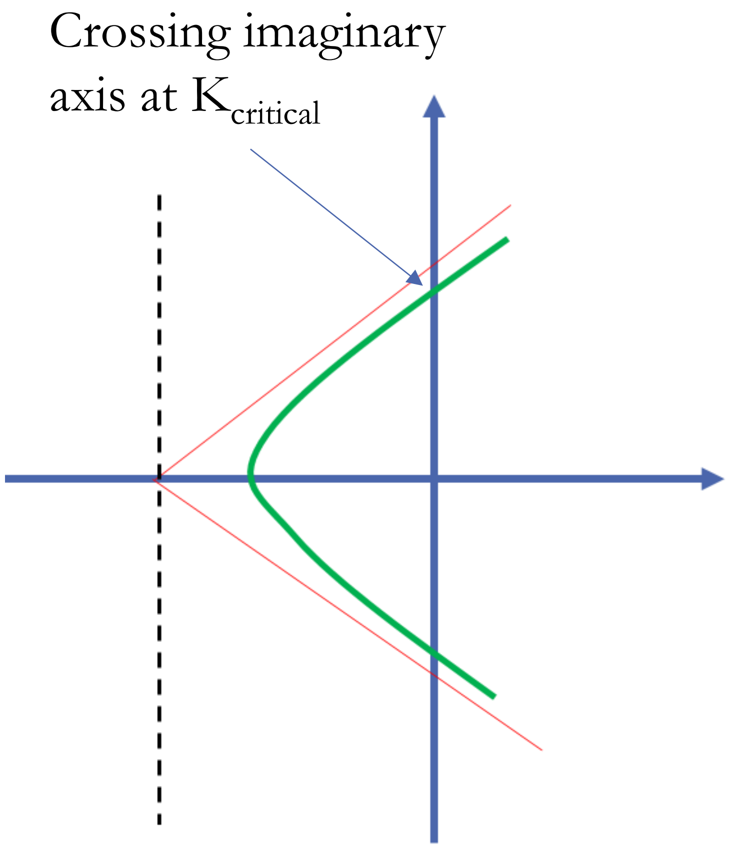

Fig 4: $K_{critical}$ -the gain at the imaginary axis crossing [4]

Relating Pole Location to Time Response

- **Real Part (σ):**

  Speed of response → Settling Time (~4/ σ)

- **Imaginary Part (ω_d):**

  Frequency of oscillation.

- **Angle from real axis(θ):**

  Damping → Overshoot.

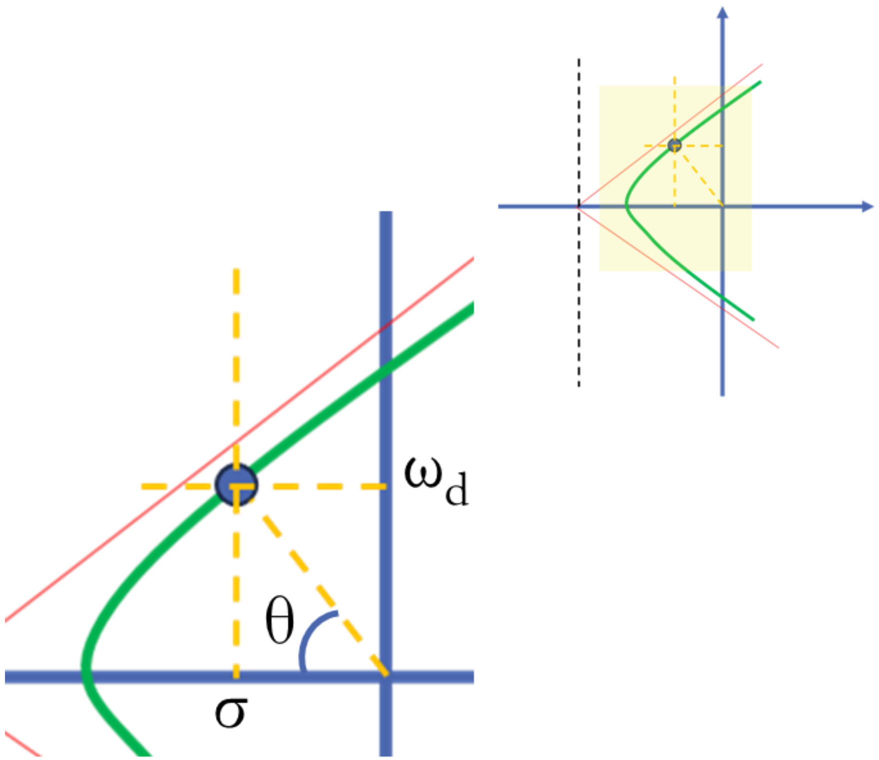

Fig 5: Relationship between pole location and time response. [5]

Lines of Constant Settling Time

- Settling Time, Ts ≈ 4 / σ

  On the s-plane: Vertical lines.

- **Example**: For Ts ≤ 2 sec, need σ ≤ -2.

  Draw a vertical line at s = -2.

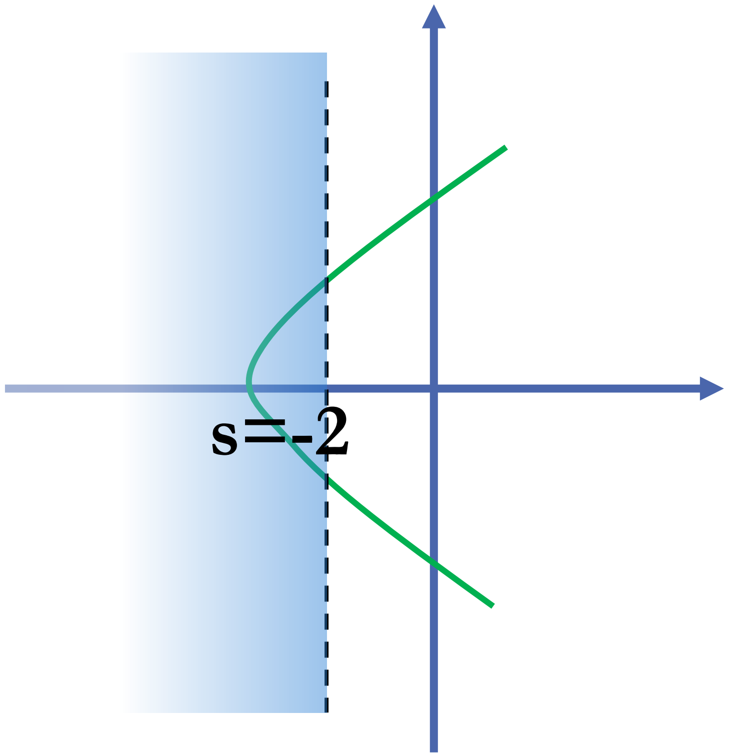

Fig 6: Regions of that satisfy given settling time. [6]

Lines of Constant Percent Overshoot

- %OS depends on Damping Ratio, ζ.
- $\zeta = \cos(\theta)$, where θ is the angle from the imaginary axis.
- On the s-plane: Radial lines.

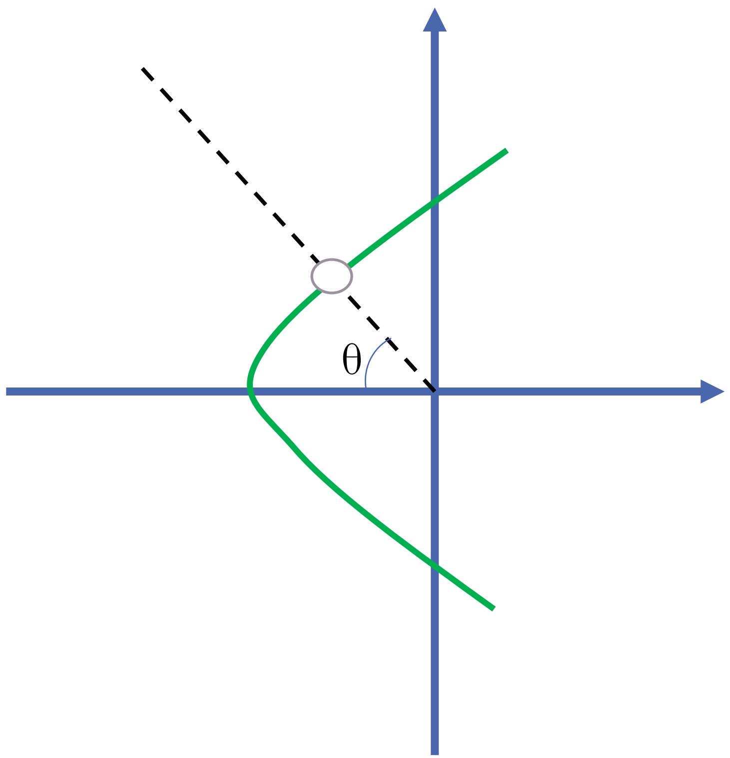

Fig 7: Line of constant damping ratio [7]

The Design Region

- Combine constraints!

  Example: %OS < 5% and Ts < 2 sec.

  θ=$\cos^{-1}(0.5)=60^0$ and $\sigma = \frac{2}{4} = 0.5$

  $$\theta \geq 60^0 \; and \; s \leq 0.5$$

- This defines a wedge-shaped region in the s-plane.
- Our goal:

  Place dominant poles inside this region.

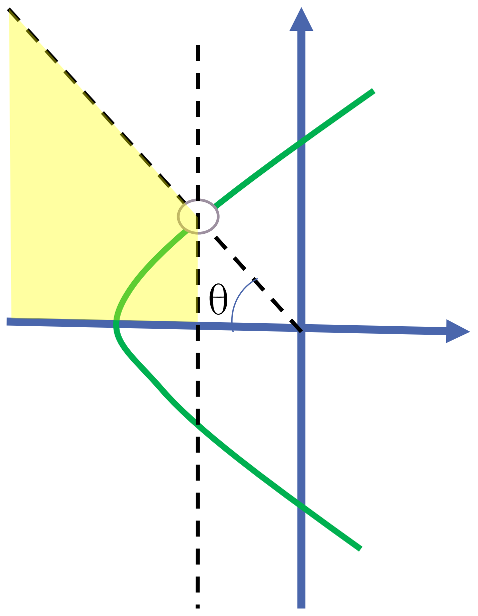

Fig 8: Desired region of pole placement. [8]

The Limit of Proportional Control

- **Problem**: The Root Locus is fixed by G(s)H(s).

  Often, no branch goes through our desired "sweet spot.“

- **Solution**: We must change the map itself! We add a compensator.

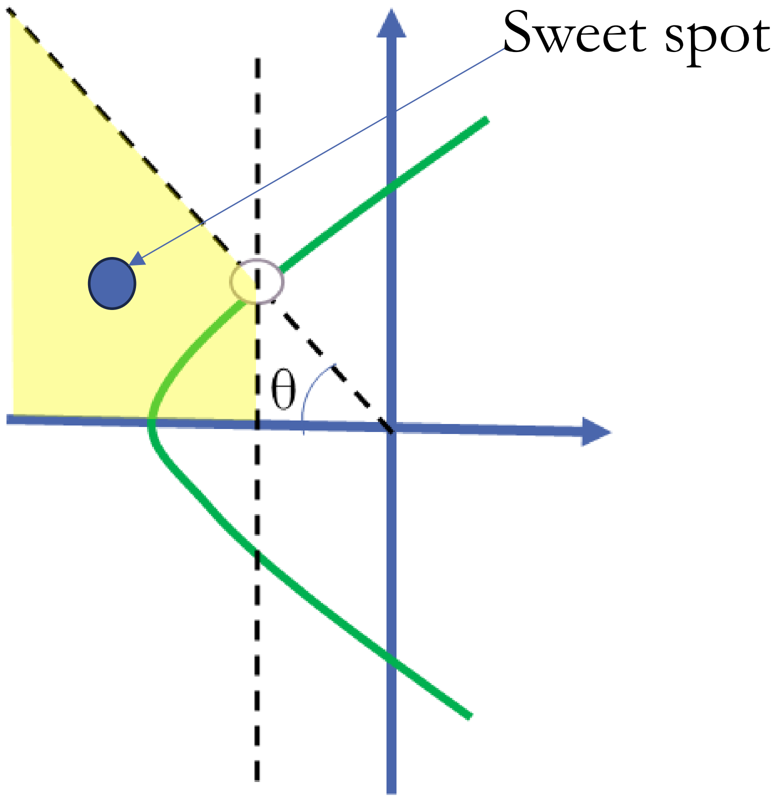

Fig 9: Desired point of pole placement. [9]

Introduction to Compensation

- **Compensator:** A block we add to the controller to improve performance.
  - It adds new Poles and Zeros to the open-loop transfer function.
  - New Poles/Zeros → New Root Locus!

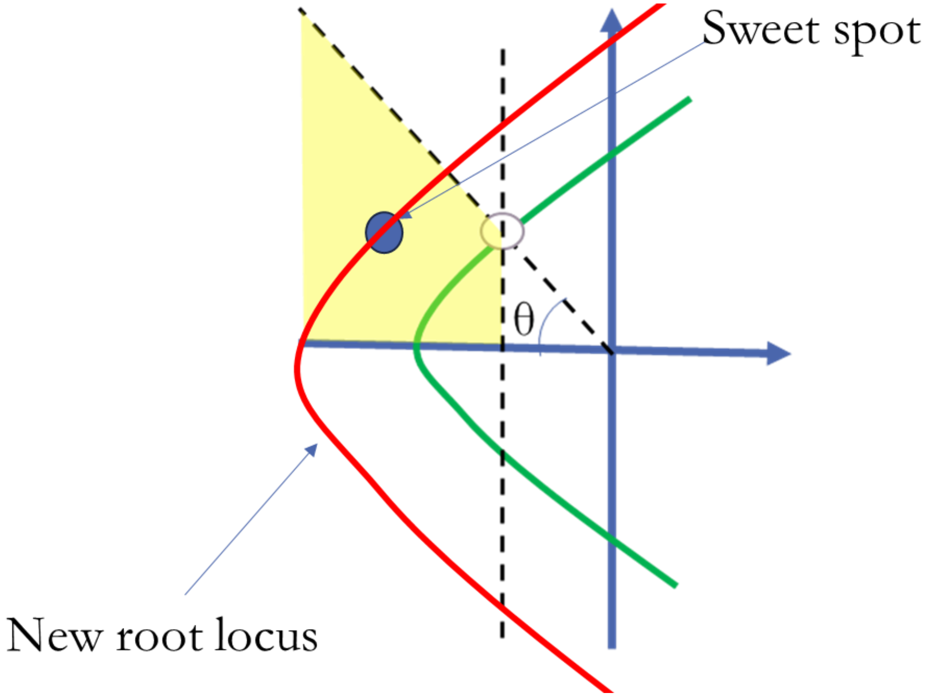

Fig 10: Root locus including the desired pole placement [10]

The Philosophy: Pole and Zero Placement

- **Adding a Zero**: Pulls the Root Locus towards it.
- **Adding a Pole**: Pushes the Root Locus away from it.
- We are like gardeners, planting trees (poles) and magnets (zeros) to shape the paths.

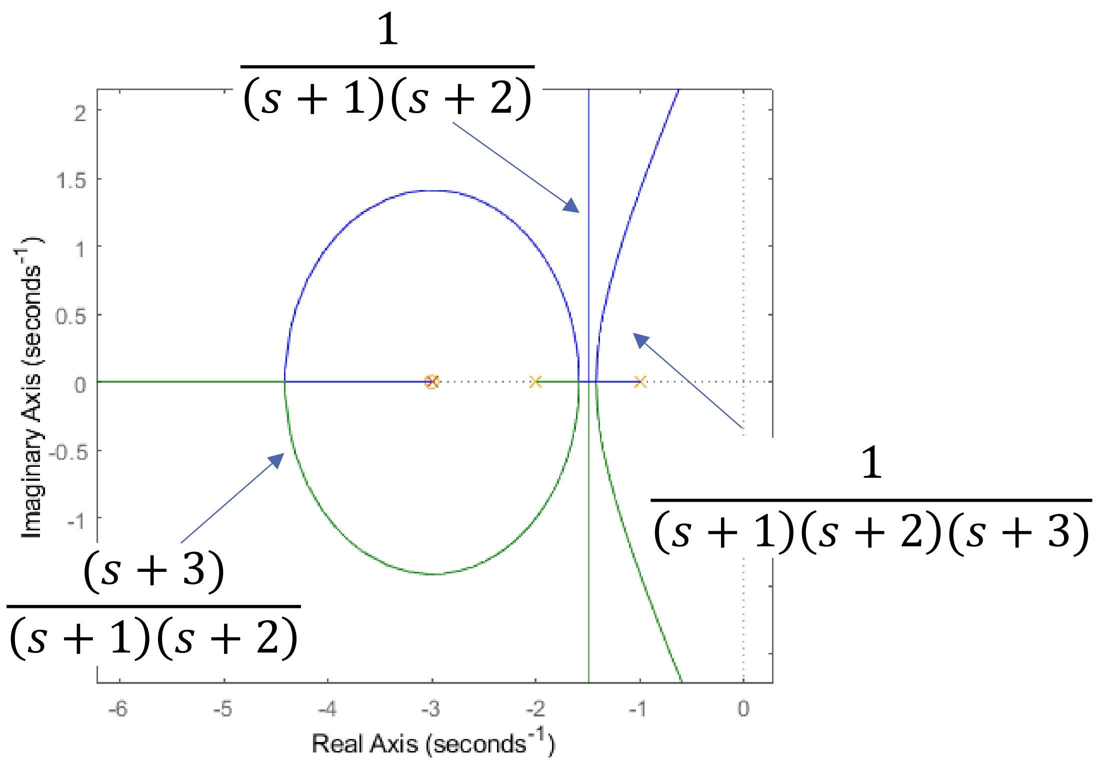

Fig 11: Effect of adding pole and zero on the root locus. [11]

Types of Compensators: Overview

- **Phase-Lead (adding zero):** Speeds up response, improves stability.
- **Phase-Lag (adding pole)**: Improves steady-state accuracy.
- **PID**: The most famous controller (Combines all).

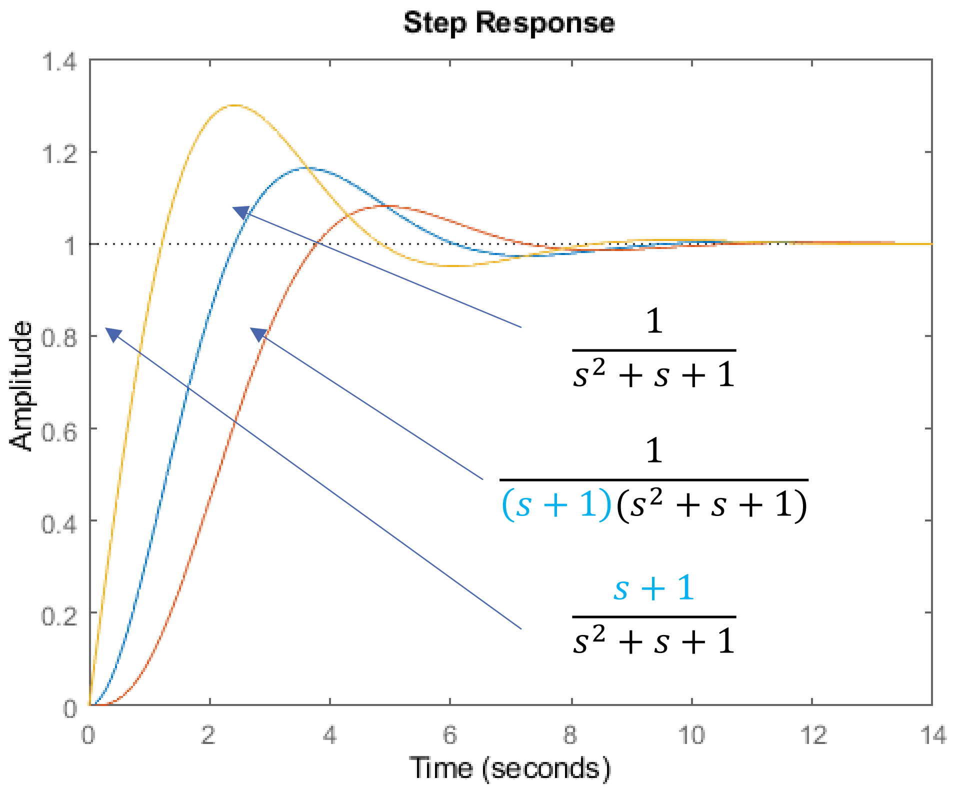

Fig 12: Effect of adding poles and zero on the system response. [12]

Phase-Lead Compensation: The Idea

- What it is: Adds a zero and a pole, with the zero closer to the origin.
- Effect: Pulls the locus to the LEFT.
- Analogy: Adding a magnet to pull a branch into the stable, fast region.

What does it mean "Phase lead"

- "Lead" refers to Phase Lead.
- It adds positive phase to the system.
- This increases the Phase Margin, making the system more robust

Phase-Lead Design Steps

1. Draw uncompensated locus.
2. Locate desired dominant pole (from specs).
3. Calculate the needed "phase angle" to be added.
4. Place the zero to provide this angle.
5. Gain calculation

Phase-Lead Worked Example

step 1 & 2

- System: K/(s(s+4))
- Spec: ζ=0.5. and Ts=1sec

Step1 & 2: convert the specs

- θ=cos(ζ)=60, radial line constant damping coefficient.

$\sigma = \frac{4}{1} = 4$, real axis point to satisfy the settling time.

- The imaginary point becomes

  $j\omega$=4*tan 60=j6.928→-4+j6.928

- The dominant pole should be placed 5x further

  P_c=-20+j6.928

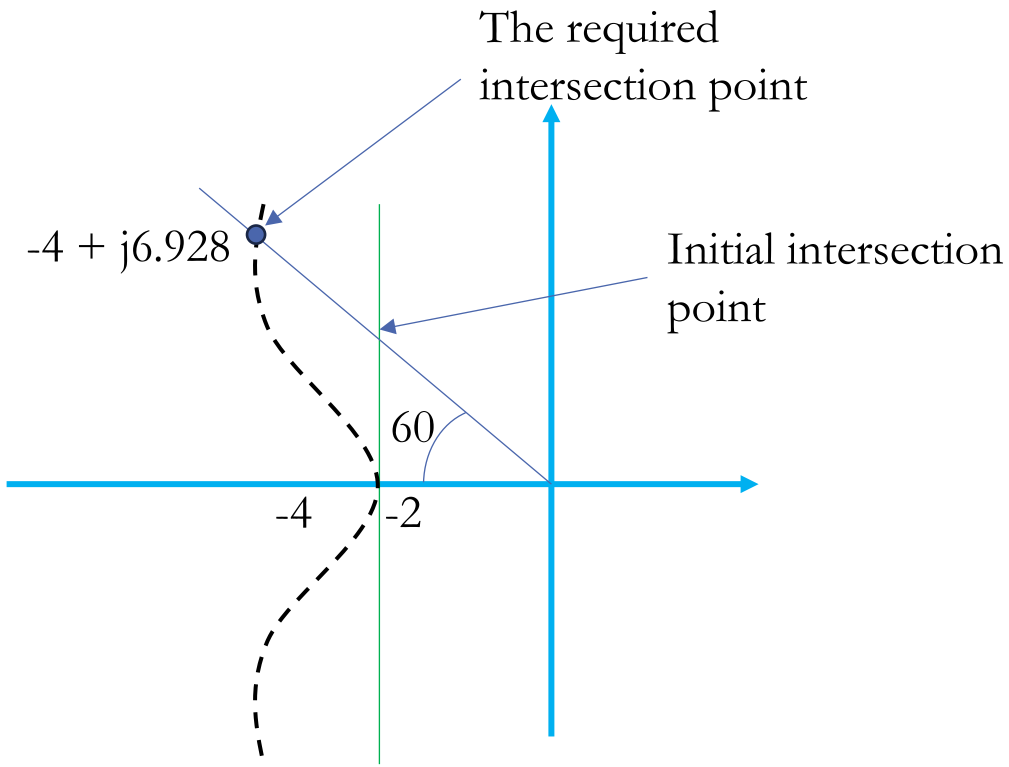

Fig 13: Desired vs initial root locus. [13]

Phase-Lead Worked Example

step 3

Step 3: Phase to Add

Angle from desired poles to system poles

$\theta_1 = \tan^{-1}(6.928/4) = 60$ (from –x axis 120 from +x axis)

$\theta_2 = \tan^{-1}(6.928/0) = 90$

$\theta_c = \tan^{-1}(6.928/16) = 23.4$

- *Angle from poles: 120° + 90°+23.4 = 233.4°*
- *Need 180° for RL*
- *Deficiency = -53.4° (Need +53.4° phase lead)*

Phase-Lead Worked Example

step 4

- **Step 4: Zero Placement**

Zero placement $\angle(s+a)=53.4$

At s = -4 + j6.928: $\angle(-4+a+j6.928)=\tan(53.4)$

$\tan(53.4) = 1.35 = 6.928/(a-4)$

a= 9.13

$$G_c=\frac{s+9.13}{s+20}$$

Phase-Lead Worked Example

step 5

Step 5: Gain calculation

At $s_1 = -4 + j6.928$:

- **Magnitudes:**

1. From pole at 0: $|s_1| = \sqrt{(4^2 + 6.928^2)} = 8$
2. From pole at -4: $|s_1 + 4| = 6.928$
3. From zero at -9.13: $|s_1 + 9.13| = \sqrt{(5.13^2 + 6.928^2)} = \sqrt{(26.3 + 48)} = 8.62$
4. From pole at -20: $|s_1 + 20| = \sqrt{(16^2 + 6.928^2)} = \sqrt{(256 + 48)} = 17.44$

Phase-Lead Worked Example

step 5: magnitude condition

- Magnitude condition
  - $K_c \frac{8.62}{17.44x8x6.928} = 1$
  - K_c=112.1
- Final compensator
  - $G_c(s) = 112.1 \frac{s+9.13}{s+20}$
- Compensated system $112.1 \frac{s+9.13}{s+20} \frac{1}{s(s+4)}$

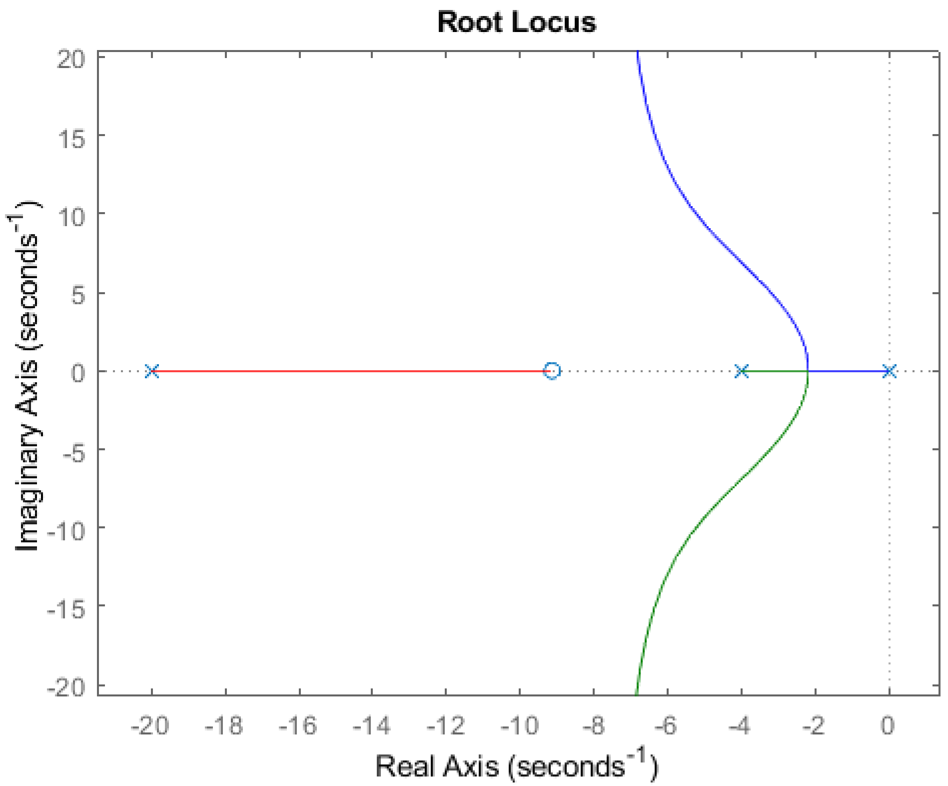

Fig 14: Root locus of compensated system. [14]

Phase-Lag Compensation: The Idea

- **What it is**: Adds a zero and a pole, with the pole closer to the origin.
- **Effect**: Does NOT change transient response much, but BOOSTS low-frequency gain.
- **Result**: Reduces steady-state error.

What does “lag” mean

- "Lag" refers to Phase Lag.
- It adds negative phase to the system.
- This is a side-effect; the main goal is to increase gain at low frequencies

Phase-Lag Design Steps

1. Design for transient response with gain K.
2. Calculate the error constant and the error.
3. If error is too large, determine the required gain boost.
4. Place lag pole/zero close to the origin to provide this boost.

Phase-Lag Worked Example

- System: K/(s(s+4))
- Spec: K_v (velocity error constant) > 20.

Step 1: Design for transient response with gain K.

- Let's choose a **dominant design point**. A common choice is the **breakaway point** where the system is critically damped ($\zeta=1$), as it provides a fast response without overshoot.

Phase-Lag Worked Example

step 1: Design for Transient Response with Gain K

- The breakaway point is calculated from:

$$\frac{1}{s} = \frac{1}{s+4} = 0 \Rightarrow \frac{2s+4}{s(s+4)} = 0 \Rightarrow s = -2$$

- At s=−2, the gain K is:
- $K=|s|\cdot|s+4|=|-2|\cdot|2|=4$
- So, for our initial design, we use K = 4.

Phase-Lag Worked Example

step 2: Calculate the Error Constant and the Error

- The velocity error constant K_v is calculated for the uncompensated system with our chosen gain K=4.

$$K_{v_{uncomp}} = \lim_{s\to 0} s\left[\frac{4}{s(s+4)}\right] = 1$$

- The steady-state error for a ramp input is:

$$e_{ss} = \frac{1}{K_v} = 1$$

- This is a **100% error**, which is unacceptably large.

Phase-Lag Worked Example

step 3: Determine the Required Gain Boost

- The specification requires K_v>20. Our current K_v is 1.
- The lag compensator works by boosting the gain at low frequencies (near s=0) without affecting the gain at higher frequencies (where our transient poles are located). The required boost β is:

$$\beta = \frac{K_{v_{spec}}}{K_{v_{uncomp}}} = \frac{20}{1} = 20$$

- We need the compensator to provide a gain boost of **20**.

Phase-Lag Worked Example

step 4: Place Lag Pole/Zero Close to the Origin

$$G_c = \frac{s + z_c}{s + p_c} \text{ and } \beta = \frac{z_c}{p_c} = 20$$

- The rule of thumb is to place the zero **close to the origin** but not so close that it causes numerical issues or excessively slow settling.
- A good starting point is to place the zero at $z_c=0.1$ → $p_c=0.005$

$$G_c = \frac{s + 0.1}{s + 0.005}$$

Phase-Lag Worked Example

step 5: Verification

- **Compensated System:**

$$G_c(s)G(s) = \frac{s+0.1}{s+0.005} * \frac{4}{s(s+4)}$$

- **Verification of Steady-State Performance:**

$$K_{v_{comp}} = \lim_{s \to 0} \frac{s+0.1}{s+0.005} * \frac{4}{s(s+4)} = 20$$

PID Control: The All-Star

- **P: Proportional (Gain K) →** Reacts to present error.
- **I: Integral →** Reacts to past error (eliminates steady-state error).
- **D: Derivative →** Reacts to future error (damps oscillations).

PID in the root locus

P-term: Just the gain K.

I-term: Adds a pole at the origin.

D-term: Adds a zero somewhere on the real axis

PI: Pole at origin + a zero.

PD: A zero.

PID: Pole at origin + two zeros

Designing with PID

- **Step 1:** Use D (or PD) to shape the locus for good transient response.
- **Step 2:** Use I (or PI) to add a pole at the origin to kill steady-state error.
- **Step 3**: Tune the gains.

Digital Implementation Note

- Modern controllers are computers.
- We must discretize our compensator design.
- Sampling rate must be fast enough (~10x closed-loop bandwidth).

Case Study 1: Position Control

- **System:** DC Motor (Poles at 0 and -1).
- **Spec:** Fast response, no overshoot.
- **Problem:** Original locus has overshoot.
- **Solution:** Add a compensator zero to pull locus to real axis.

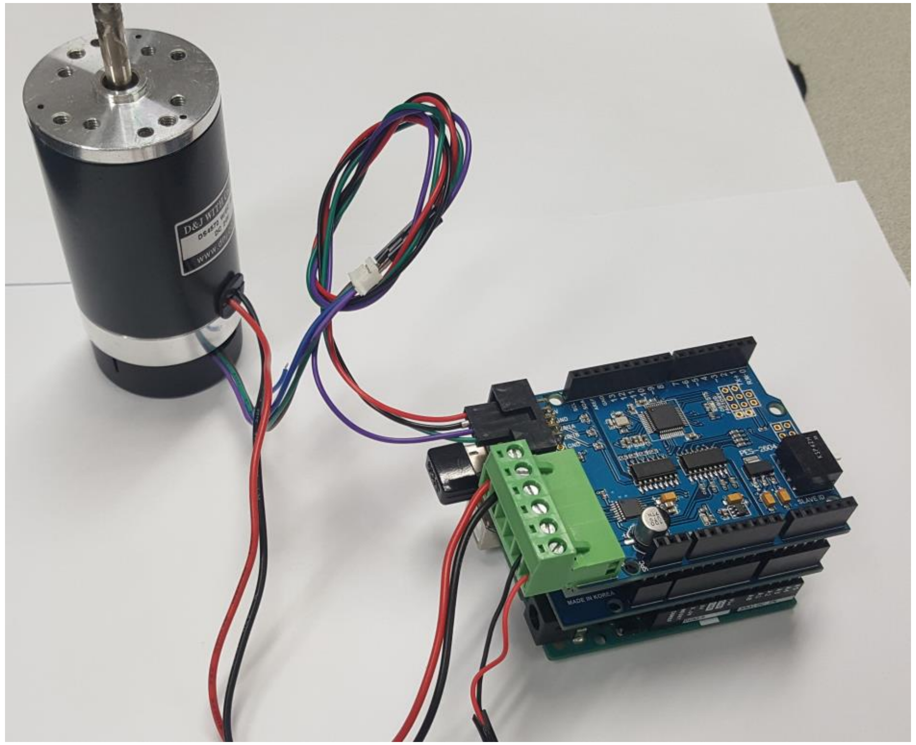

Fig 15: Controlling of DC motor [15]

Case Study 2: Temperature Control

- **System:** Slow, first-order system.
- **Spec:** Very accurate final temperature.
- **Problem:** With just gain K, there is steady-state error.
- **Solution:** Add an Integral term (PI controller).

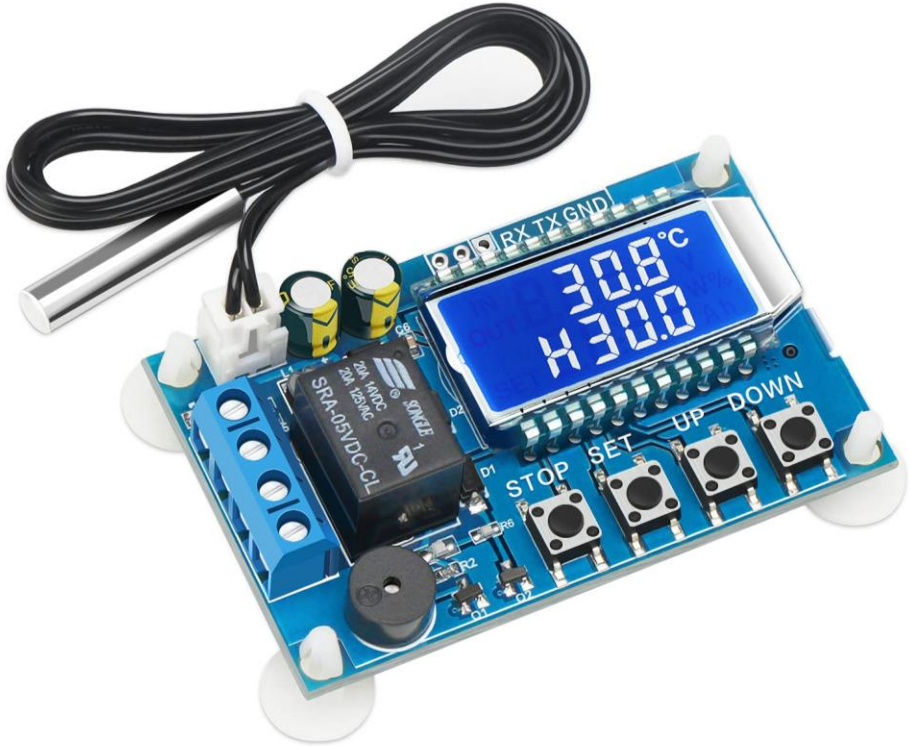

Fig 16: Temperature controller. [16]

Case Study 3: Aircraft Pitch

- **System:** Complex, higher-order, naturally unstable.
- **Spec:** Highly stable, responsive.
- **Problem:** Original poles are in the RHP!
- **Solution:** Aggressive lead/PID compensation to pull the entire locus into the LHP.

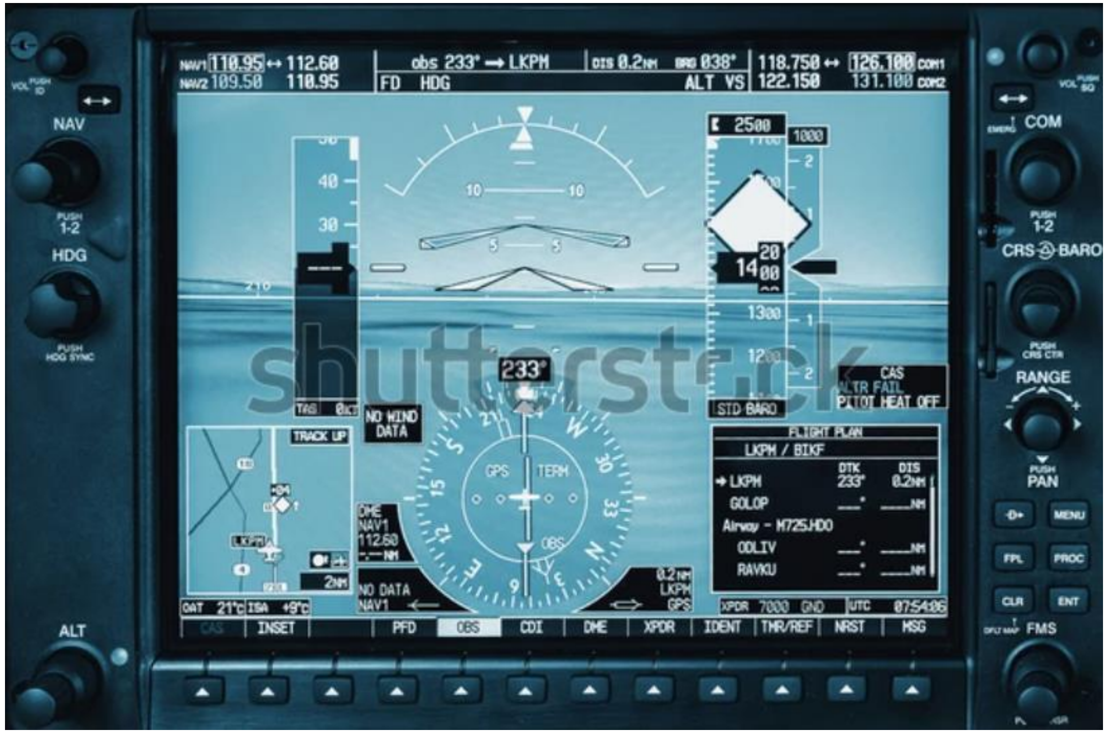

Fig 17: Aircraft pitch control module. [17]

Review: Lead vs. Lag Compensation

- **Lead:** Zero closer to origin. For speed/stability. Changes locus shape.
- **Lag:** Pole closer to origin. For accuracy. Does not change locus shape much.
- **Lead-Lag:** Combine them!

How Root Locus and Bode Plots Relate

- **Root Locus:** Shows pole locations vs. ONE parameter (K).
- **Bode Plot**: Shows frequency response vs. frequency.
- They are two different views of the same system.

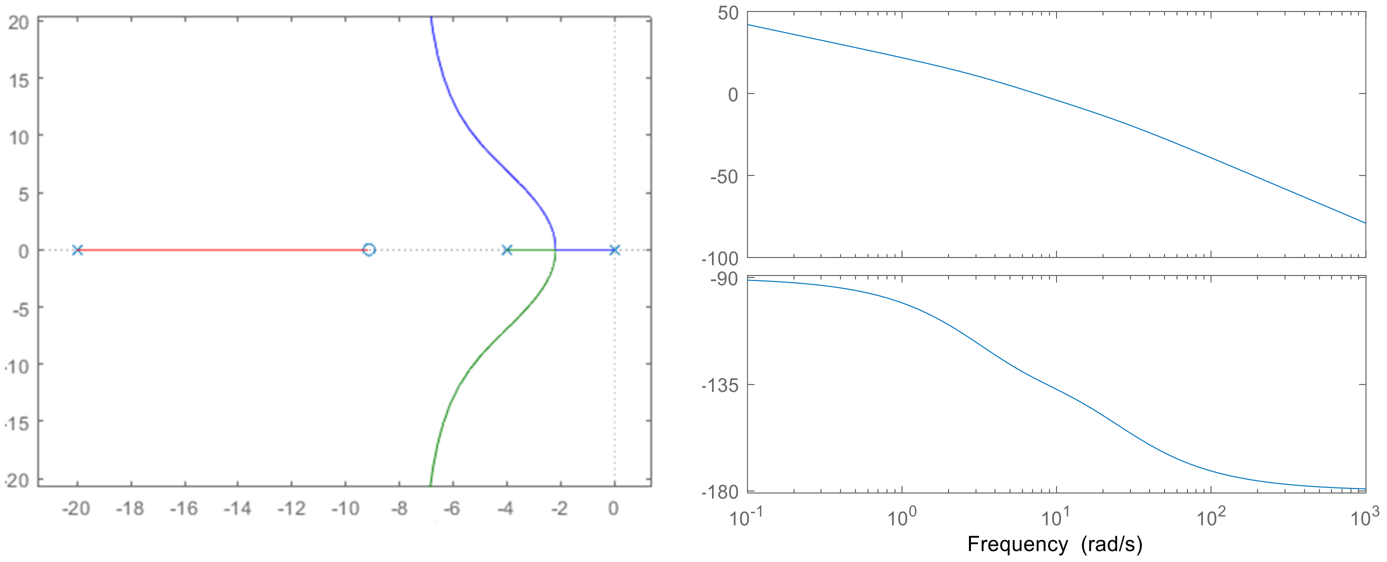

Fig 18: Root locus vs Bode plot [18]

Limitations of the Root Locus

- **Only one parameter**: It shows effect of gain K, but what if other parameters change?
- **Ignores zeros:** The locus only shows poles. Zeros also affect the response.
- **Approximate:** The drawing is a sketch.

Designing for Robustness

- Don't design for the "edge" of performance.
- Account for real-world variations: component wear, temperature changes.
- Keep poles well within the LHP.

The Iterative Nature of Design

  - **Design is a cycle:** Design → Analyze Simulate → Redesign.
  - Rarely get it perfect on the first try.
  - Use Root Locus for the initial "big picture" design.

Beyond the Basics: Advanced Concepts

- **Parameter Variation:** How does the locus change if a different parameter varies?
- **Sensitivity:** How sensitive is the system to model inaccuracies?
- **State-Space Methods:** For modern, multi-input, multi-output systems.

Root Locus in the Real World

- **Aerospace**: Flight control systems.
- **Automotive**: Cruise control, stability control.
- **Robotics**: Every joint in a robot arm.
- **Process Control**: Chemical plants, refineries.

Summary

The Analyst

- You can now read the Root Locus map.
- You can determine stability and performance.
- You understand the limits of proportional control

The Designer

- You can reshape the Root Locus using compensation.
- You know how to use Lead and Lag compensators.
- You understand the power of PID control.

The Big Picture

- Root Locus links open-loop system (poles/zeros) to closed-loop performance.
- It is a visual and intuitive method.
- It provides a deep physical insight into system behavior.

Final Conclusion

- Practice: Draw loci for many different systems.
- Use Software: Get comfortable with MATLAB/Python.
- Learn More: Frequency response, state-space, digital control.
- You have mastered a powerful engineering tool.
- You can analyze and design control systems.

References

[1] Chalachew Werku, 2025, Construction of root locus, Self-created

[2] Chalachew Werku, 2025, Stable, marginal stable and unstable regions of s-plane, Self-created

[3] Chalachew Werku, 2025, Movement of poles on "s-plane" as "K" varies, Self-created

[4] Chalachew Werku, 2025, $K_{critical}$ -the gain at the imaginary axis crossing, Self-created

[5] Chalachew Werku, 2025, Relationship between pole location and time response, Self-created

[6] Chalachew Werku, 2025, Regions of that satisfy given settling time, Self-created

[7] Chalachew Werku, 2025, Line of constant damping ratio, Self-created

[8] Chalachew Werku, 2025, Desired region of pole placement, Self-created

[9] Chalachew Werku, 2025, Desired point of pole placement, Self-created

[10] Chalachew Werku, 2025, Root locus including the desired pole placement, Self-created

[11] Chalachew Werku, 2025, Effect of adding pole and zero on the root locus, Self-created

[12] Chalachew Werku, 2025, Effect of adding poles and zero on the system response, Self-created

[13] Chalachew Werku, 2025, Desired vs initial root locus, Self-created

References

[14] Chalachew Werku, 2025, Root locus of compensated system, Self-created

[15] Controlling of DC motor, https://www.google.com/url?sa=i&url=https%3A%2F%2Fforum.phpoc.com%2Farticles%2Ftutorials%2F1264-arduino-dc-motor-controller&psig=AOvVaw0Qzv6FilCuGVW1pmHZDDgI&ust=1762068590073000&source=images&cd=vfe&opi=8997844,9&ved=0CBUQjRxqFwoTCOD1xqe30JADFQAAAAAdAAAAABAE, accessed on October 28, 2025

[16] Temperature controller, https://www.google.com/url?sa=i&url=https%3A%2F%2Fwww.amazon.ca%2FElectronic-Thermostat-Controller-DROK-Temperature%2Fdp%2FB07GQPT9VG&psig=AOvVaw1OiOodRcXRvmNJvDkfH2h_&ust=1762068713471000&source=images&cd=vfe&opi=89978449&ved=0CBUQjRxqFwoTCNjY6om40JADFQAAAAAdAAAAABAE, accessed on October 28, 2025

[17] Aircraft pitch control module. https://www.shutterstock.com/image-photo/computer-cockpit-airplane-glass-display-weather-462036739, accessed on October 28, 2025.

[18] Chalachew Werku, 2025, Root locus vs Bode plot, Self-created

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- Norman S. Nise – Control Systems Engineering (2015, Wiley)