Course: Regulation and control

Lecture 13: Controller Design II – Compensators

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Course Outline & Learning Objectives

- Understand the limitations of PID control.
- Learn frequency-domain design using Bode plots.
- Master the design of Lead and Lag Compensators.

Recap: Limitations of PID

- PID is a "one-size-fits-most" solution.
- What if we need precise shaping of the closed-loop response?
- What if the performance requirements are very specific?

Introducing Compensators

- Idea: Design a controller $G_c(s)$ with a specific frequency-domain purpose.
- We can "shape" the open-loop frequency response $G_c(s)G(s)$ to meet precise stability and performance specs.

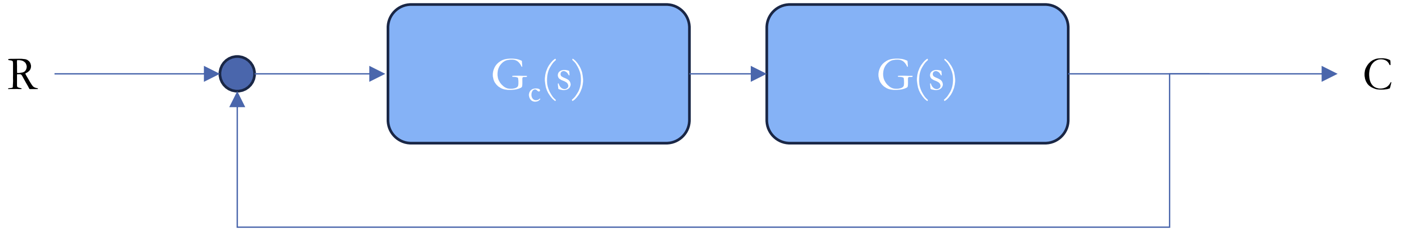

Fig 1: Closed-loop system with compensator [1]

The Design Platform: Bode Plots

- Review:
  - **Magnitude Plot**: dB vs. log(ω)
  - **Phase Plot**: Degrees vs. log(ω)

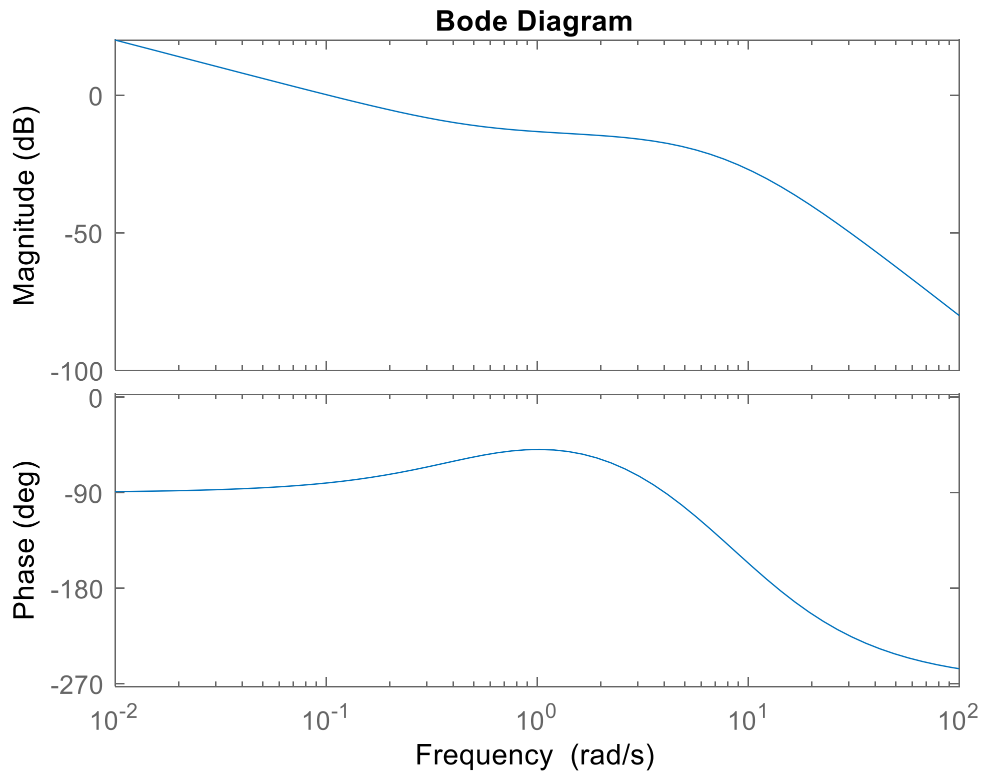

Fig 2: Bode plot, Magnitude vs Phase plot [2]

Recap Gain Margin (GM) and Phase Margin (PM)

- **Gain Margin (GM):** How much gain can increase before instability.
- **Phase Margin (PM):** How much phase lag can increase before instability.
- Indicators of Robustness.

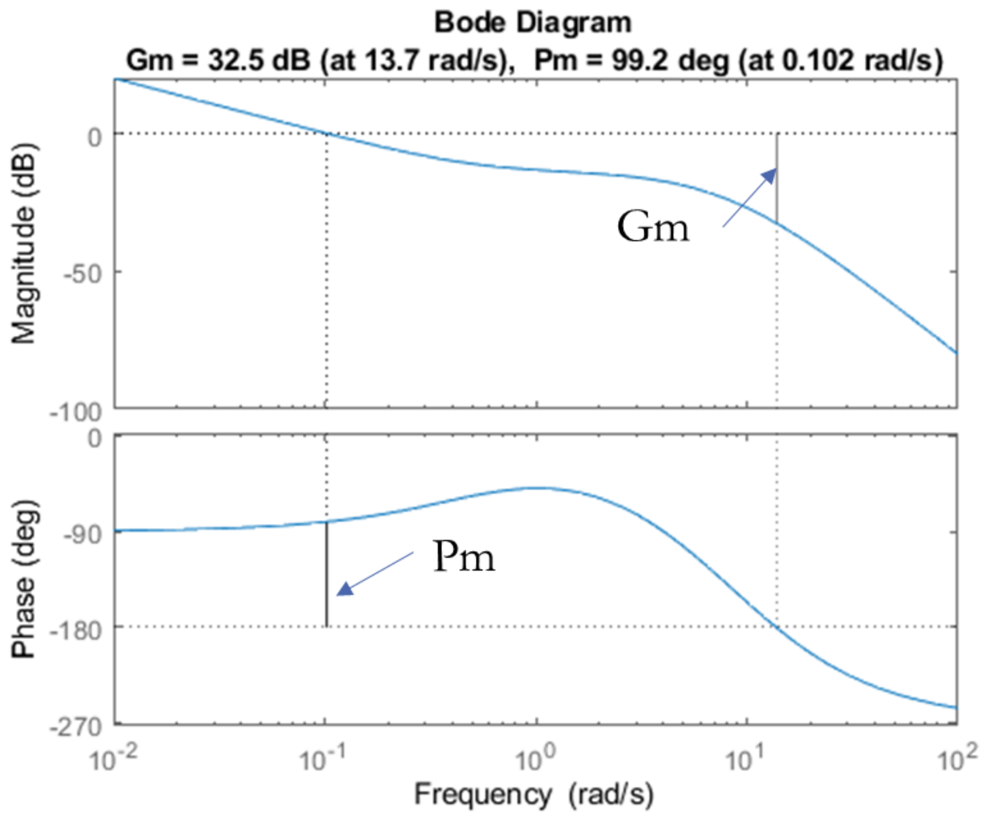

Fig 3: Gain margin vs Phase margin [3]

Relating GM/PM to Time Response

- Higher Phase Margin → Less Overshoot
- Higher Gain Crossover Frequency (ω_{gc}) → Faster Response

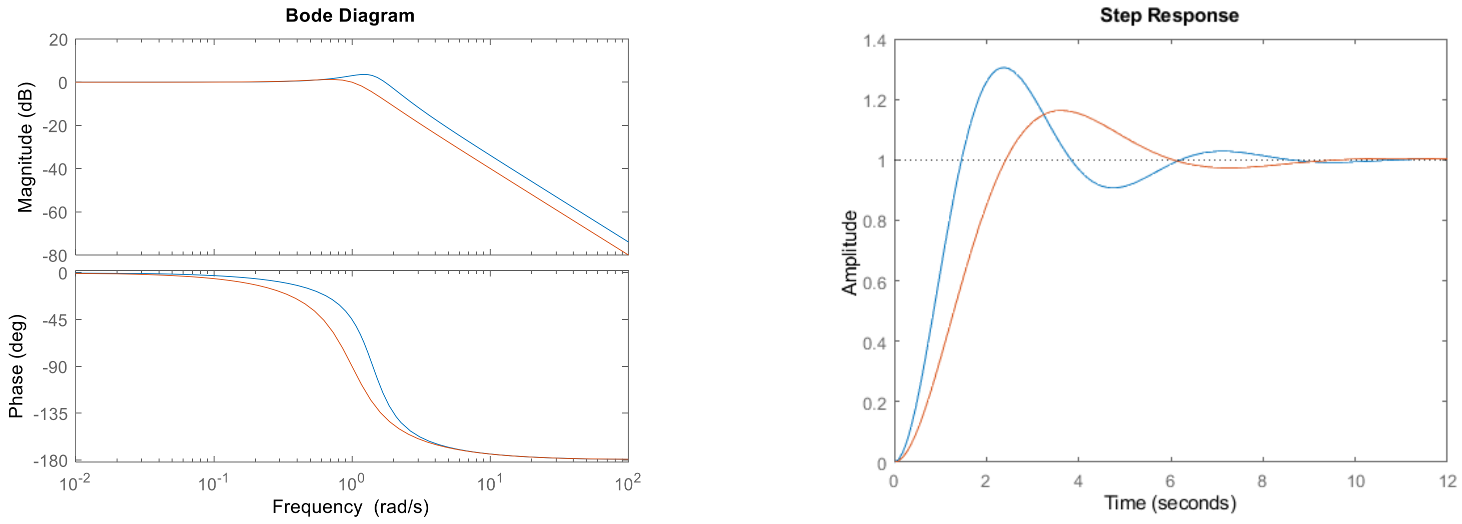

Fig 4: Frequency vs time response of a system [4]

The Concept of "Loop Shaping"

- We design the compensator $G_c(s)$ so that the open-loop transfer function

$L(s) = G_c(s)G(s)$ has a desirable Bode plot.

- We "shape" the L(s) curve to achieve desired PM, ω_{gc}, and steady-state error.
- Un-compensated
- Compensated

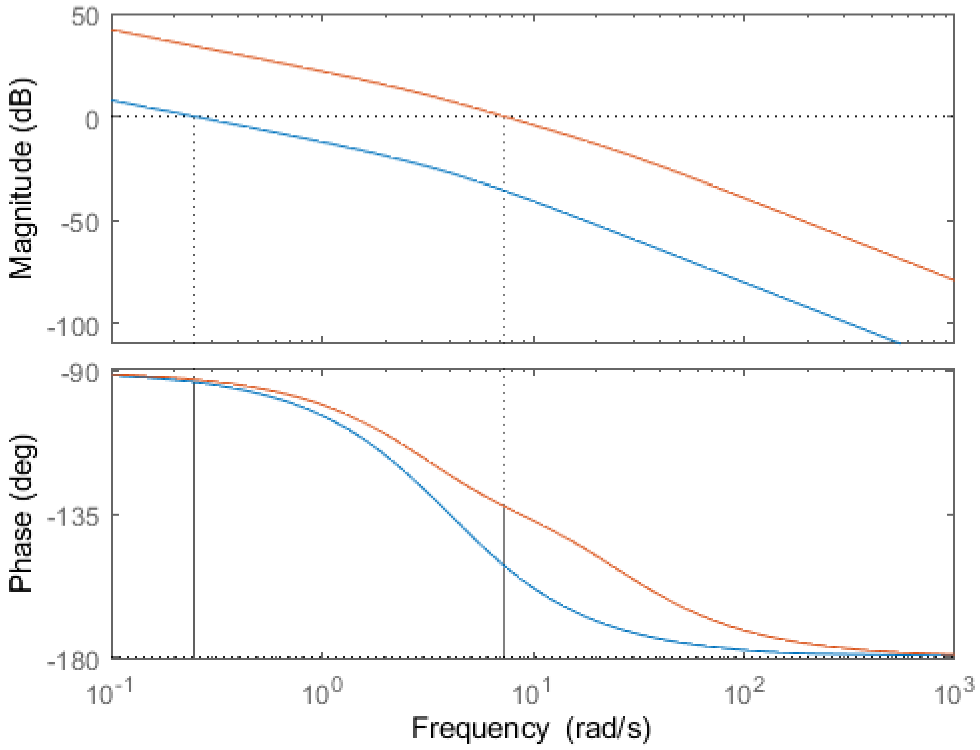

Fig 5: Loop-shaping (bode plot) [5]

Performance Specifications in Frequency Domain

- **Desired Phase Margin (PM):** For sufficient damping (e.g., PM > 45°).
- **Gain Crossover Frequency (ω_{gc}):** For desired speed.
- **Steady-State Error**: Dictated by low-frequency gain (e.g., high gain for small error).

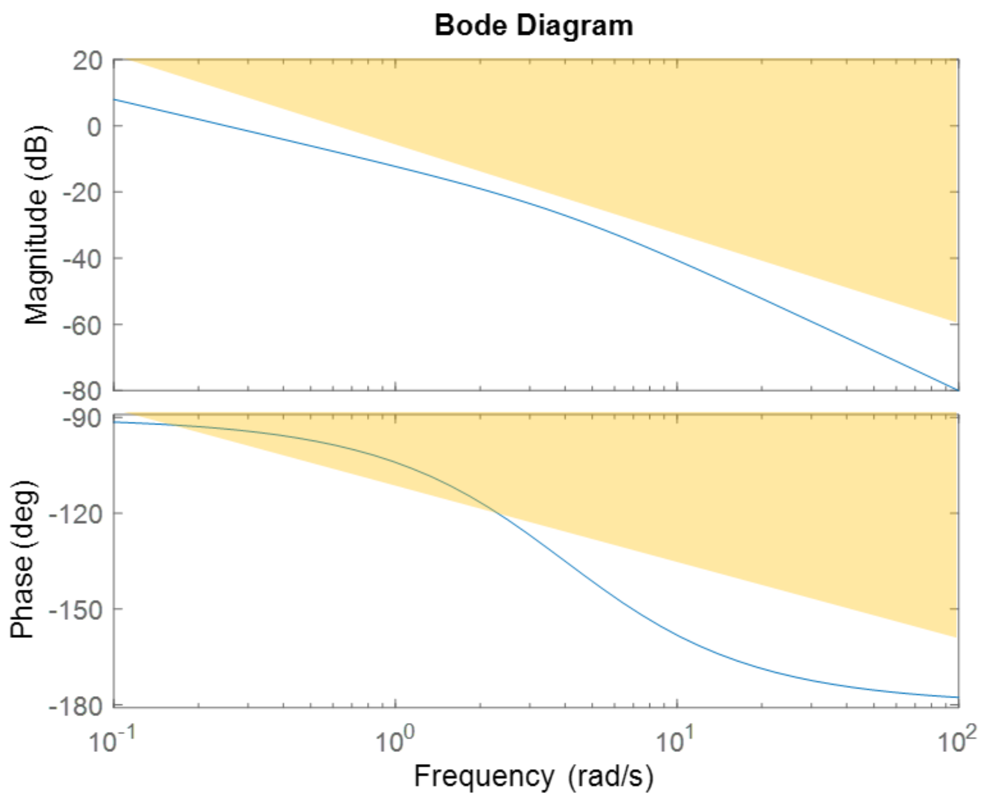

Fig 6: Desired region in bode plot [6]

The General Form of a Lead/Lag Compensator

$$G_c = K_c \frac{(s+z)}{s+p}$$

- The locations of the Zero (z) and Pole (p) define the type.
- **Lead Compensator:** |z| < |p| (Zero before Pole).
    - Adds positive phase → Improves PM.
- **Lag Compensator:** |z| > |p| (Pole before Zero).
    - Increases low-frequency gain → Reduces steady-state error.

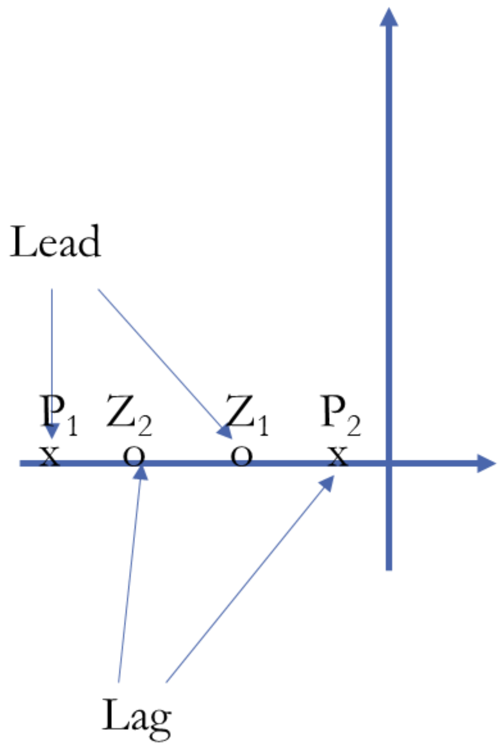

Fig 7: Pole and zero locations of compensators. [7]

Quick Quiz

- For a compensator $G_c(s)=\frac{s+1}{s+10}$, is it a lead or a lag? Justify.

Answer:

Lead. Because the zero is at s = -1 and

the pole is at s = -10.

$|z| = 1 < |p| = 10.$

The Lead Compensator

- A lead compensator is a type of controller used in feedback control systems to improve the transient response (how fast the system responds) and stability.
- It does this by adding positive phase to the system's open-loop frequency response, effectively "pushing" the phase curve upward around the gain crossover frequency.
- The name "lead" comes from the fact that the output phase of the compensator leads the input phase for sinusoidal signals.

Lead Compensator Goal

- **Primary Goal:** Improve Stability → Increase Phase Margin.
- **Secondary Goal:** Increase Speed → Raise Gain Crossover Frequency (ω_{gc}).

The Standard Form

- $G_c(s) = K_c\alpha \frac{sT+1}{\alpha sT+1}, 0 < \alpha < 1$
  - α: Determines the amount of phase lead.
  - T: Determines the frequency where max phase occurs.

Bode Plot of a Standalone Lead Compensator

- **Magnitude:** High-pass filter characteristic.
- **Phase:** A "phase bump" centered at a specific frequency.

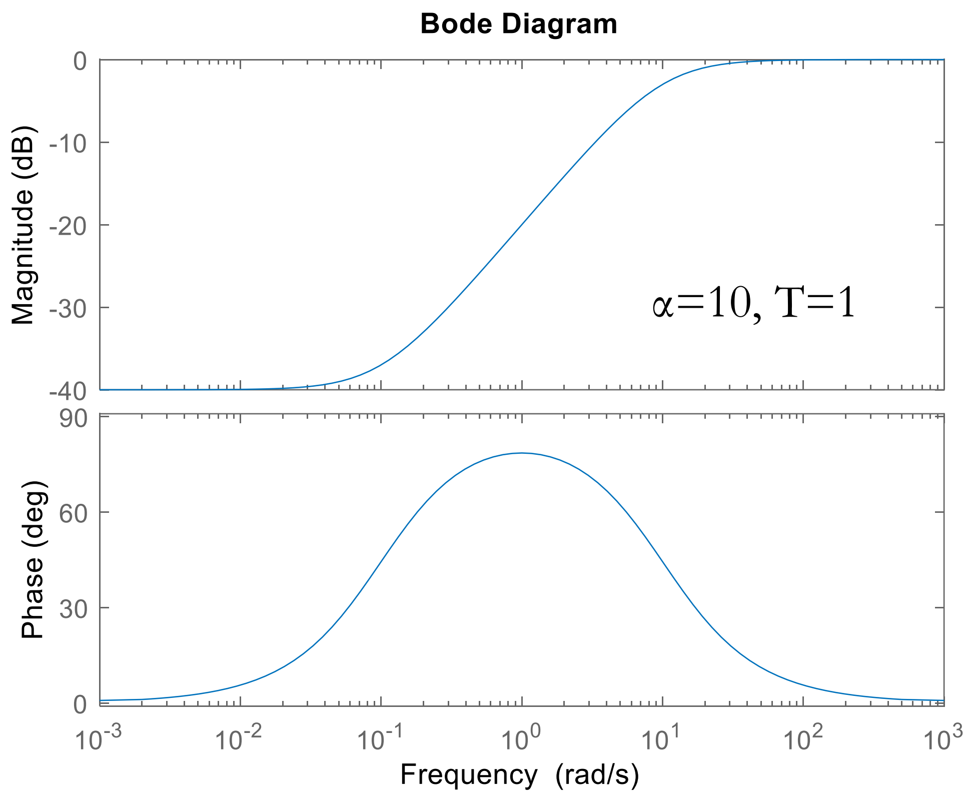

Fig 8: Standalone lead-compensator [8]

Key Parameters

- **α (alpha)**: Sets the maximum phase lead φ_m. Larger α → larger φ_m.
- **T (Time Constant):** Sets the frequency ω_m where the max phase occurs.

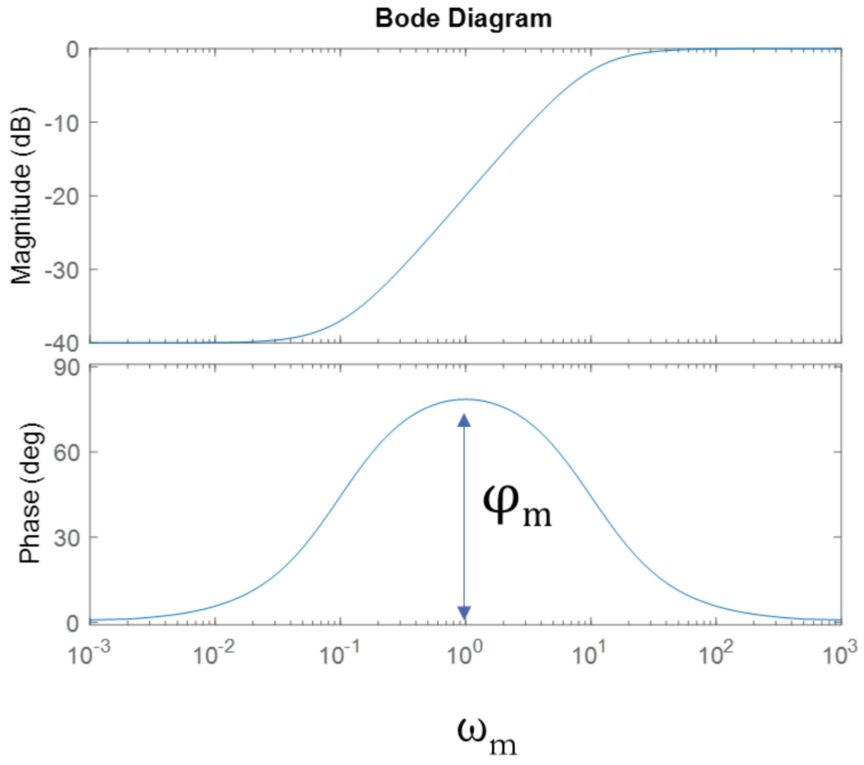

Fig 9: Key parameters on lead-compensator [9]

Formula for Maximum Phase Lead and Frequency of Max Phase

Maximum Phase Lead

$$\phi_m = \sin^{-1}\left(\frac{1-\alpha}{1+\alpha}\right)$$

- This is the maximum possible phase boost the compensator can provide

Frequency of Max Phase

$$\omega_m = \frac{1}{\sqrt{\alpha}T}$$

- This is the center of the "phase bump".

The Design Procedure for a Lead Compensator

- Step-by-Step List

  1. Determine required phase lead φ_m.
  2. Calculate α from φ_m.
  3. Place ω_m at the new gain crossover frequency.
  4. Determine T and then K_c.

Step1 and 2

Step 1: Determine required ϕ_m

- Required φ_m = Desired PM - Current PM + (5°-15° Safety Factor)
- The compensator must provide this extra phase.

- Step 2: Calculate α from ϕm
  - Use the formula: $\alpha = \dfrac{1 - \sin(\varphi_m)}{1 + \sin(\varphi_m)}$
    - Derived from $\varphi_m = \sin^{-1}\left(\frac{1-\alpha}{1+\alpha}\right)$

Step3 and 4

Step 3: Place ω_m at the new gain crossover frequency.

- Find the frequency where the uncompensated magnitude is $1/\sqrt{\alpha}$.
- This frequency will become the new ω_{gc}, and we set ω_m equal to it.

Step 4: Determine T and then Kc

- From $\omega_m = \frac{1}{T\sqrt{\alpha}}$, calculate T.
- Set K_c to achieve the required DC gain for steady-state error, or
- set or $K_c\alpha=K$, $K_c= 1$ initially and adjust if needed.

Example 1

lead-compensator

- Given: G(s) = 1 / s(s+2)
  - Spec: Achieve PM ≥ 85°.

Solution

From Bode plot of G(jω), phase at gain crossover

$\omega_{gc} \approx 0.486$rad/sec is about -103.7^0, so $PM_{current} \approx 76.3^0$.

Required additional phase lead:

- $\phi_m = PM_{des} - PM_{current} + 5^0 \approx 85 - 76.3 + 5 = 13.7^0$

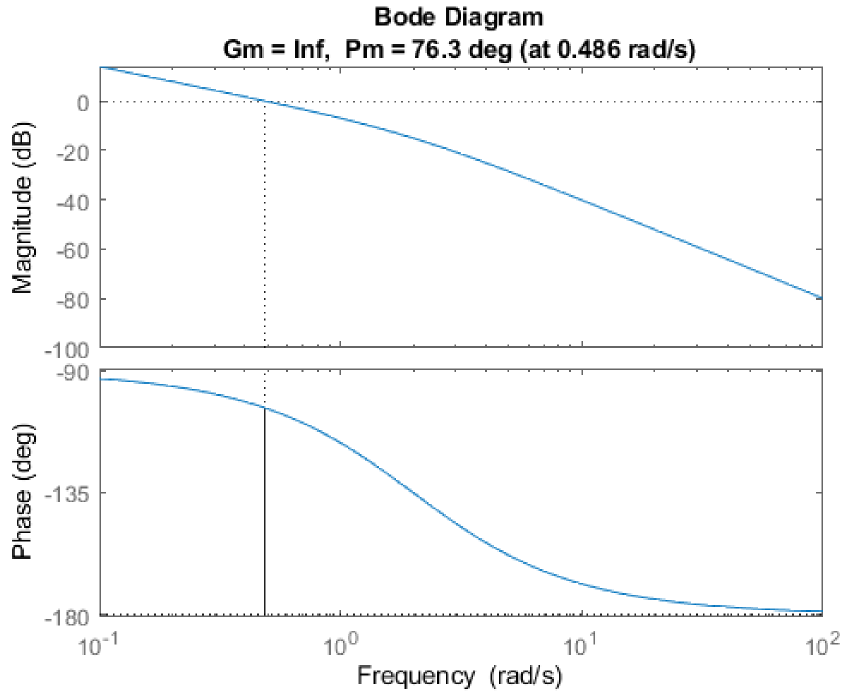

Fig 10: Uncompensated bode plot for example 1 [10]

Example 1 Cont....

lead-compensator

Calculate α from φ_m

$$\alpha = \frac{1 + sin\phi_m}{1 - sin\phi_m} = \frac{1 + sin13.7^0}{1 - sin13.7^0} \approx 0.617$$

ω_{gc-new} =?

$$\left|G\left(j\omega_{gc-new}\right)\right| = \frac{1}{\sqrt{a}} = \frac{1}{\omega\sqrt{\omega^2 + 4}} = 0.785, \qquad \omega_{gc-new} = 0.626\, rad/sec$$

Example 1 Cont….

lead-compensator

$$\omega_m = \frac{1}{T\sqrt{\alpha}}$$

$$\Rightarrow T = \frac{1}{\omega_m\sqrt{\alpha}} = \frac{1}{0.626 * \sqrt{0.617}} = 2.037$$

$$G_c(s) = K_c\alpha\frac{sT+1}{\alpha sT+1}$$ setting $K_c\alpha=1$

$$= 1 * \frac{2.037s + 1}{0.617 * 2.037s + 1} = \frac{2.037s + 1}{1.257s + 1}$$

$$G_c(s)G(s) = \frac{2.037s+1}{1.257s+1} * \frac{1}{s(s+2)}$$

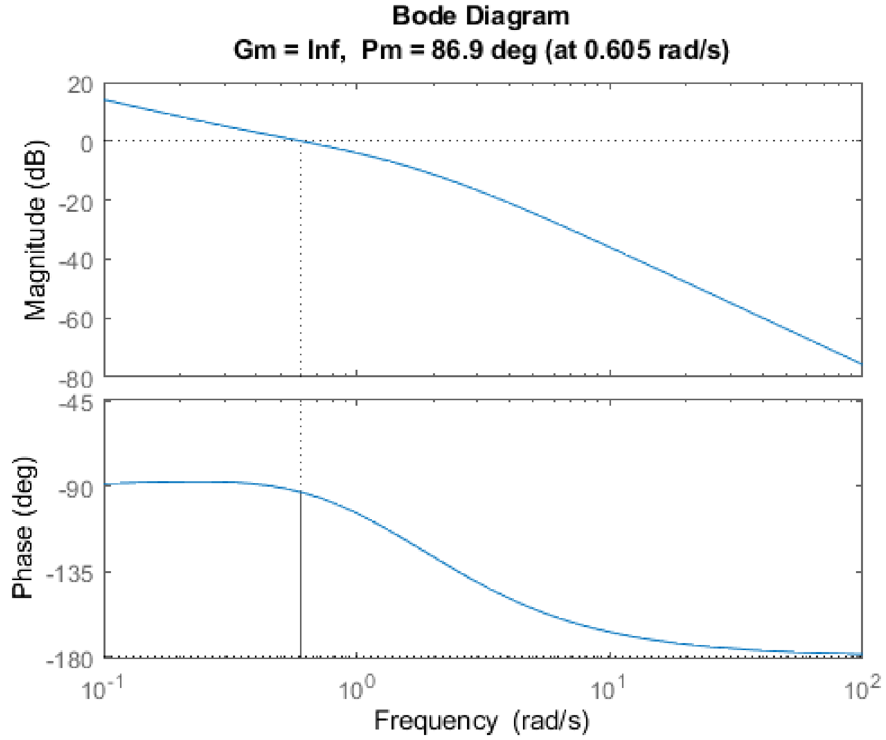

Fig 11: Compensated bode plot for example 1 [11]

Summary of lead compensators

- **Effects on step response:** Reduced overshoot and faster settling time.
- **Limitations**: Amplifies high-frequency noise (due to its high-pass nature), Not effective for systems with already low phase margins at high frequencies.
- **We use it when**: more Phase Margin is needed, the system is too sluggish and you need a faster response, the system has excessive overshoot.

The Lag Compensator

- A lag compensator is a type of controller used in feedback control systems to improve steady-state accuracy (reduce steady-state error) without significantly affecting the transient response.
- It does this by increasing the low-frequency gain of the open-loop system.
- The name "lag" comes from the fact that the output phase of the compensator lags behind the input phase for sinusoidal signals.

Lag Compensator Goal

- Primary Goal → Improve Steady-State Accuracy.
- Secondary Goal → Allow for higher low-frequency gain without affecting stability.
- The Standard Form
  - $G_c(s) = K_c\beta \frac{Ts+1}{\beta Ts+1}, \beta > 1$
  - β: Determines the amount of low-frequency gain boost.
  - T: Sets the corner frequency.

Bode Plot of a Standalone Lag Compensator

- **Magnitude:** Low-pass filter characteristic. Gain is K_c at high frequency and K_c/β at low frequency.
- **Phase:** Adds negative phase (phase lag).

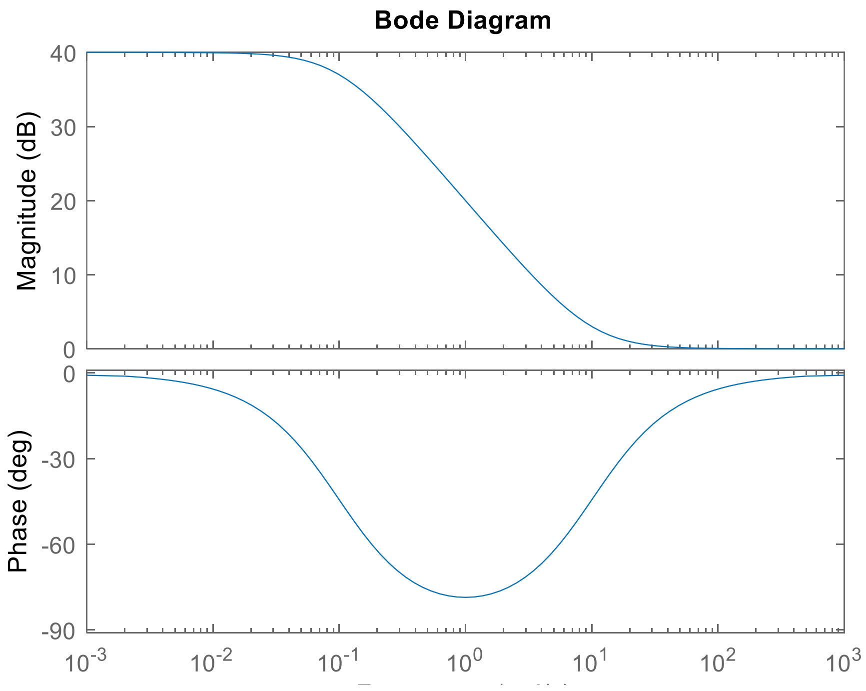

Fig 12: Standalone lag-compensator [12]

Key Parameters

- **β-** Sets the high-frequency attenuation

$$\frac{1}{\beta}$$

- The low-frequency gain boost.

$$\frac{K_c}{\beta}$$

- **T-**Sets the frequency of the corner

$$\omega = \frac{1}{T} \; and \; \omega_m = \frac{1}{T\sqrt{B}}$$

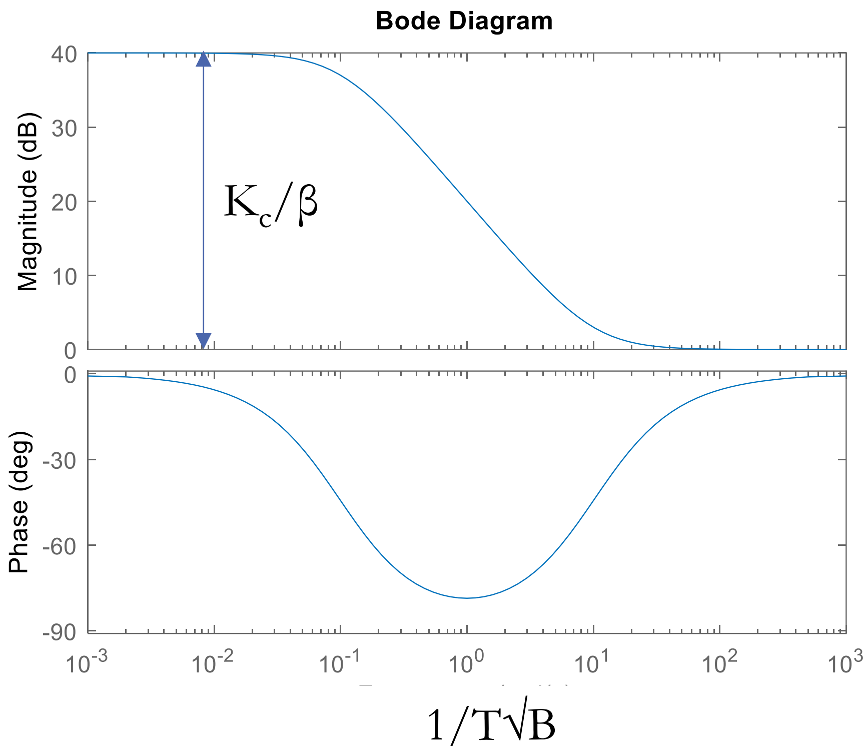

Fig 13: Key parameters on lag-compensator [13]

The Design Procedure for a Lag Compensator

- Step-by-Step List:
  1. Determine K_c from steady-state error requirement.
  2. Plot Bode of K_c*G(s), find frequency where PM is Spec PM + Safety.
  3. Place lag zero/pole well below this new crossover.
  4. Determine β from required attenuation.

Step 1: Determine K_c from SSE

- Use error constants: K_p, K_v, K_a.

SSE (e_{ss})	Static error constants
$\frac{1}{1+K_p}$	$K_p = \lim_{s\to 0} G(s)$
$\frac{1}{K_v}$	$K_p = \lim_{s\to 0} sG(s)$
$\frac{1}{K_a}$	$K_p = \lim_{s\to 0} s^2G(s)$

- e.g., For ramp input, $K_v = \lim_{s\to 0} sG(s)$ *and* $e_{ss} = 1/K_v$ Set K_c to achieve the required K_v.

Step2 and 3

Step 2: Find New Crossover with Safety

- Plot L(s) = K_c * G(s).
- Find frequency ωgc_new where Phase = -180° + (Desired PM + 5°-15°).
- This safety factor accounts for the phase lag added by the compensator

Step 3: Place Zero and Pole

- Place the zero at $\omega_z = 1/T = 0.1 * \omega_{gc\_new}$.
- This ensures the lag compensator adds minimal phase lag at the new crossover frequency.
- The pole is then at $\omega_p = 1/(\beta T)$.

Step 4: Determine β

- At ω_{gc_new}, the uncompensated gain $|K_cG(j\omega_{gc\_new})|$ must be attenuated to 0 dB.
- So, set

$$\beta = \frac{1}{\left|K_c G\left(j\omega_{gc-new}\right)\right|}$$

Example 2

lag compensator

- Design a lag compensator for a unity feedback system where the plant is:

  $G(s)=K/s(s+2)$

- Design Requirements:
  - Steady-State Error: $e_{ss} \leq 0.05$ for a unit ramp input.
  - Phase Margin: $PM \geq 45°$.

Example 2

lag compensator: Solution

Step 1: Determine the Required Gain K from the e_{ss} Specification

Steady-state error for a Type 1 system is $e_{ss} = \frac{1}{K_v}$

$$K_v = \lim_{s \to 0} s \left[\frac{K}{s(s+2)} \right] = \frac{K}{2}$$

$e_{ss}=1/K_v=0.05$ ➔ $K=40$

$$G(s) = \frac{40}{s(s+2)}$$

Example 2 Cont. . . .

lag compensator: Solution

Step 2: Find New Crossover with Safety

$PM_{uncomp}=18$

$\angle G(j\omega_{gcnew})=-180+(PM_{Desired})+(Safety)=-180+45+10=-125$

$\angle G(j\omega)=-90-\tan^{-1}(\omega/2)=-125$

$\omega_{gcnew}=1.40 rad/sec$

$$\frac{1}{\beta} * \left|G\left(j\omega_{gc-new}\right)\right| = 1 \Rightarrow \beta = \left|G\left(j\omega_{gc-new}\right)\right| = 11.7 \approx 12$$

Example 2 Cont. . . .

lag compensator: Solution

Step 3: Place Zero and Pole

Place the zero at $\omega_z = 1/T$

$= 0.1 * \omega_{gc\_new}$ =0.14 ➔T=7.14

- The pole is then at $\omega_p = 1/(\beta T)$

➔1/(12*7.14)=0.0117

$$G_c(s) = \frac{7.14s + 1}{85.7s + 1}$$

$$G_c(s)G(s) = \frac{7.14s + 1}{85.7s + 1} * \frac{40}{s(s + 2)}$$

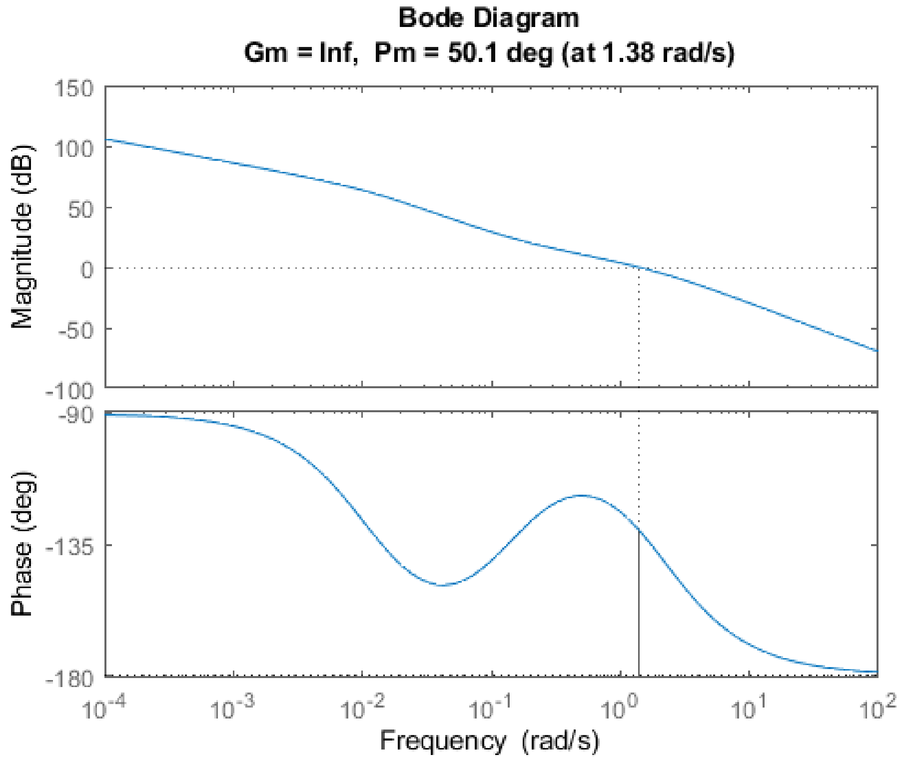

Fig 14: Compensated bode plot for example 2 [14]

Summary of lag compensators

- **Effects on Step response**: Same or similar overshoot, Slower settling time and Drastically reduced steady-state error.
- **Limitations**: Slows down the system response and Can make the system more sluggish.
- **We use it when**: steady-state error is the primary concern and the transient response is acceptable but needs better accuracy.

Lead-Lag Compensators

- Combines the benefits of both.
  - **Lead Part:** Speeds up response and improves PM.
  - **Lag Part**: Improves steady-state accuracy.

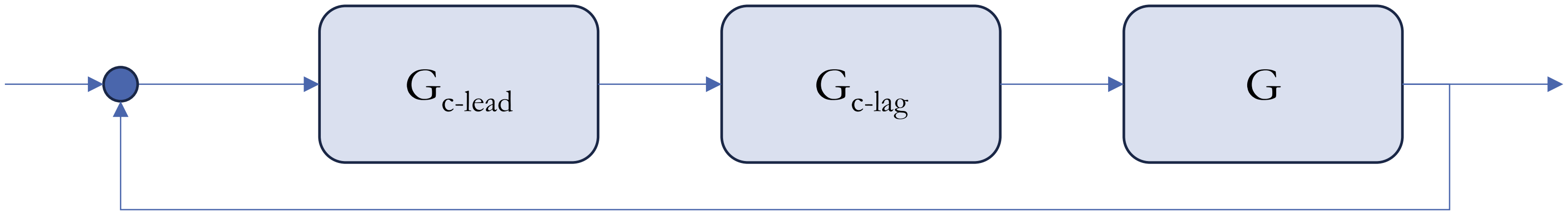

Fig 15: Closed-loop system with lead-lag compensator [15]

Comparison Table

Feature	Lead Compensator	Lag Compensator
Main Goal	Improve Stability/Speed	Improve Steady-State
Effect on BW	Increases	Decreases
Effect on PM	Increases	Slight Decrease (managed)
Noise	Amplifies HF Noise	Attenuates HF Noise

Summary & Key Takeaways

- PID is powerful but limited; compensators offer precise design.
- Lead: For Phase Margin and Speed. Use $\alpha > 1$.
- Lag: For Steady-State Error. Use $\beta < 1$.
- Design is done by shaping the Bode plot of the open-loop system.

References

[1] Chalachew Werku, 2025, Closed-loop system with compensator, Self-created
[2] Chalachew Werku, 2025, Bode plot, Magnitude vs Phase plot, Self-created
[3] Chalachew Werku, 2025, Gain margin vs Phase margin, Self-created
[4] Chalachew Werku, 2025, Frequency vs time response of a system, Self-created
[5] Chalachew Werku, 2025, Loop-shaping (bode plot), Self-created
[6] Chalachew Werku, 2025, Desired region in bode plot, Self-created
[7] Chalachew Werku, 2025, Pole and zero locations of compensators, Self-created
[8] Chalachew Werku, 2025, Standalone lead-compensator, Self-created
[9] Chalachew Werku, 2025, Key parameters on lead-compensator, Self-created
[10] Chalachew Werku, 2025, Uncompensated bode plot for example 1, Self-created
[11] Chalachew Werku, 2025, Compensated bode plot for example 1, Self-created
[12] Chalachew Werku, 2025, Standalone lag-compensator, Self-created
[13] Chalachew Werku, 2025, Key parameters on lag-compensator, Self-created
[14] Chalachew Werku, 2025, Compensated bode plot for example 2, Self-created
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