Week 4

Chapter 2
Relative and Constrained Motion of Particles

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Contents

By the end of this lecture, you are able to:

1. Motion of multiple Particles
2. Constrained Motion of Particles
3. Relative motion with moving frames
4. Solve problem based on the concepts

4.1 Understand Motion of multiple Particles

- In the previous section, we studied the motion of a single particle or a body idealized as a particle. In this section, we extend our focus to the motion of several particles, or two or more bodies idealized as particles, studied at the same time.
- Here, we will analyze their relative motion and understand how they influence each other.
- The motion of several particles in engineering mechanics is studied by analyzing either independent motion, dependent (constrained) motion.
- Independent motion : Each particle moves freely, without connection (e.g., two planes on different path)
- Dependent (constrained) motion : The motion of one particle is directly tied to another by a rope, pulley, or rigid link (e.g., two blocks connected by a rope).

- As shown in Figure 1, the motion of the two planes is independent as they move freely. In contrast, in Figure 2, the movement of block A affects the motion of block B because they are connected by a rope [1].

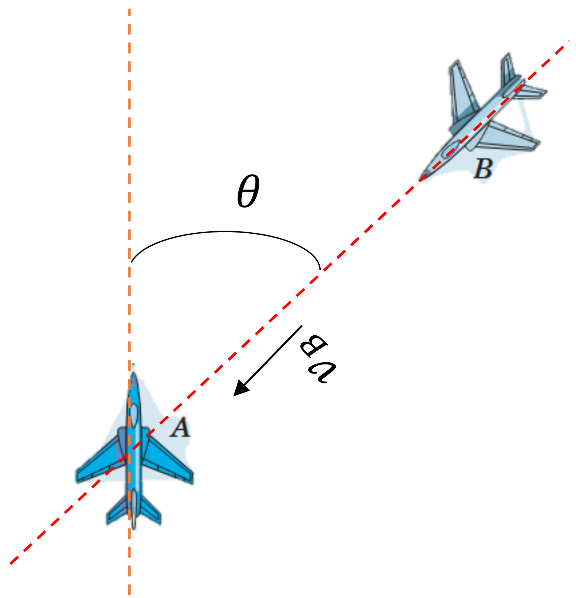


Figure 1. Independent motion

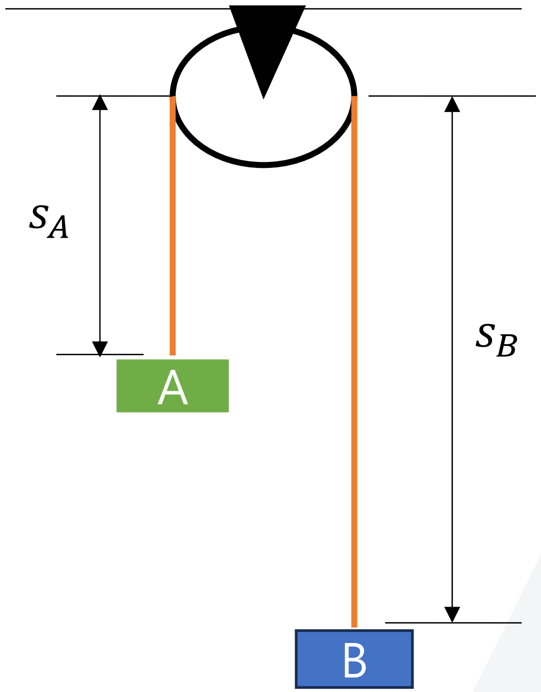


Figure 2. Dependent (constrained) motion

- Constraint equations and relative motion principles are key tools for solving problems involving interconnected particles.

Relative – independent motion

- In the preceding sections on kinematics, displacements, velocities and accelerations have all been absolute. That is, they have been measured from a stationary reference [1].
- It is not always possible or convenient to use a fixed set of axes for the observation of motion.
- There are many engineering problems for which the analysis of motion is simplified by using measurements made with respect to a moving reference system.
- It is often useful to consider the motion of one point relative to another point that has a different motion.
- In this section translating inertial frames of reference will be considered for the analysis.
- an inertial reference frame is a non-accelerating frame [1].

Position

- Consider particles A(boat) and B (duck), which move along the arbitrary paths shown in Fig. 3. The absolute position of each particle, r_A and r_B is measured from the common origin o of the fixed x, y, z reference frame. The relative-position of B w.r.t A denoted by vector $r_{B/A}$ [2].

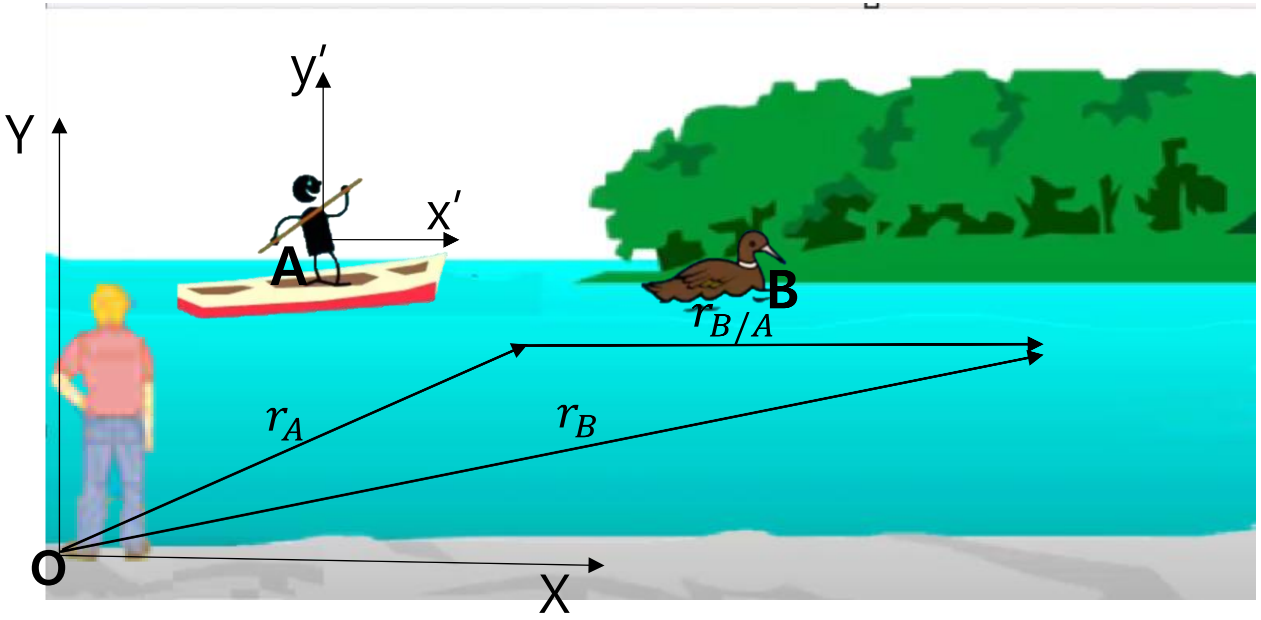


Figure 3. The position vector

- The origin of a second frame of reference x', y', z ' is attached to and moves with particle and only permitted to translate relative to the fixed frame

- The three vectors shown in Fig. 3. can be related by the equation

$$\boldsymbol{r_B} = \boldsymbol{r_A} + \boldsymbol{r_{B/A}}$$

Velocity

- An equation that relates the velocities of the particles is determined by taking the time derivative of the above equation; i.e.,

$$V = \frac{dr}{dt}$$

$$\frac{d\boldsymbol{r}_B}{dt} = \frac{d\boldsymbol{r}_A}{dt} + \frac{d\boldsymbol{r}_{B/A}}{dt}$$

$$\boldsymbol{V}_B = \boldsymbol{V}_A + \boldsymbol{V}_{B/A} \quad \text{(m/s)}$$

- Here $\boldsymbol{V}_B$ and $\boldsymbol{V}_A$ refer to absolute velocities, since they are observed from the fixed frame; whereas the relative velocity $\boldsymbol{V}_{B/A}$ is observed from the translating frame [2].
- N.B. since the x', y', z ' axes translate, the components of $\boldsymbol{r}_{B/A}$ will not change direction and therefore the time derivative of these components will only have to account for the change in their magnitudes.

Acceleration

Relative cont'd….

- The time derivative of velocity yields a similar vector relation between the absolute and relative accelerations of particles A and B [2].

$$a = \frac{dv}{dt} \qquad \boldsymbol{a_B} = \boldsymbol{a_A} + \boldsymbol{a_{B/A}} \quad \left({}^{m}/_{s^2}\right)$$

Here $\boldsymbol{a_{B/A}}$ is the acceleration of B as seen by the observer located at A and translating with the x ', y', Z' reference frame.

- When the particle moves on curved path the acceleration components the particle consists of both tangential and normal components of acceleration

- For example if Particle B is changing its speed on curved path, the above equation can be rewritten as :

$$\boldsymbol{a_{(B)t}} + \boldsymbol{a_{(B)n}} = \boldsymbol{a_A} + \boldsymbol{a_{B/A}} \quad \left({}^{m}/_{s^2}\right)$$

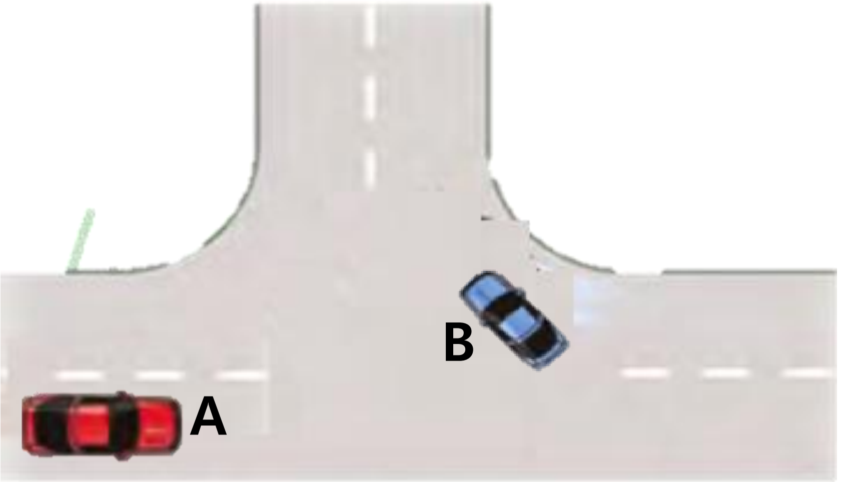


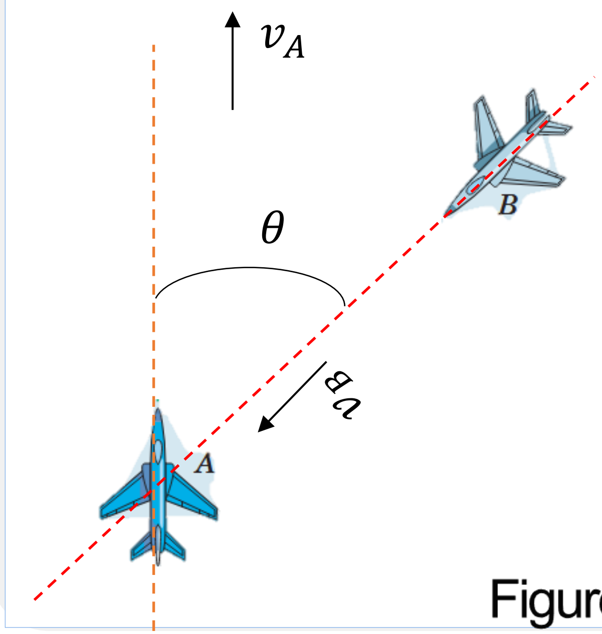
Figure 4. Car on straight and curved path

SOLVING PROBLEMS of relative motion

- Since the relative motion equations are vector equations, problems involving them may be solved in one of two ways.

1. Vector Analysis

- For instance, the velocity vectors and acceleration vector could be written as Cartesian vectors and the resulting scalar equations solved for up to two unknowns [1].
- We will express every given values of velocity and acceleration vector in I ,j, and k components.



$$V_B = V_A + V_{B/A}$$

$$-v_B \sin\theta \; i - v_B \cos\theta \; j = V_A j + V_{B/A}$$

$$a_B = a_A + a_{B/A}$$

$$-a_B \sin\theta \; i - a_B \cos\theta \; j = a_A j + V_{B/A}$$

Figure 5. Relative motions of planes

Relative cont'd....

1. Graphical Method

- Since vector addition or subtraction forms a triangle, sine and cosine laws can be applied to solve for relative or absolute velocities and accelerations [1].
- Draw vectors V_B and V_A from a common point. Apply the laws of sines and cosines to determine $V_{B/A}$.
- As review, their formulations are provided below.

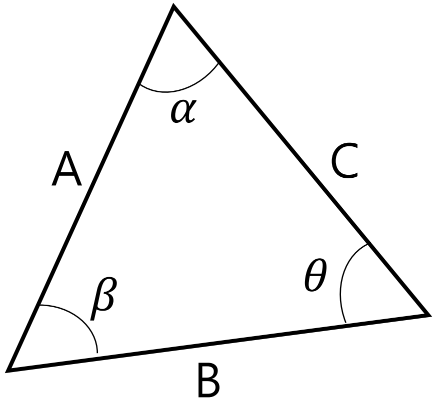


Figure 6. Triangle

Law of Sines:

$$\frac{A}{Sin\theta} = \frac{B}{sin\alpha} = \frac{C}{sin\beta}$$

Law of Cosines:

C= $\sqrt{A^2 + B^2 - 2AB\cos\beta}$

B= $\sqrt{A^2 + C^2 - 2AC\cos\alpha}$

A= $\sqrt{C^2 + B^2 - 2BC\cos\theta}$

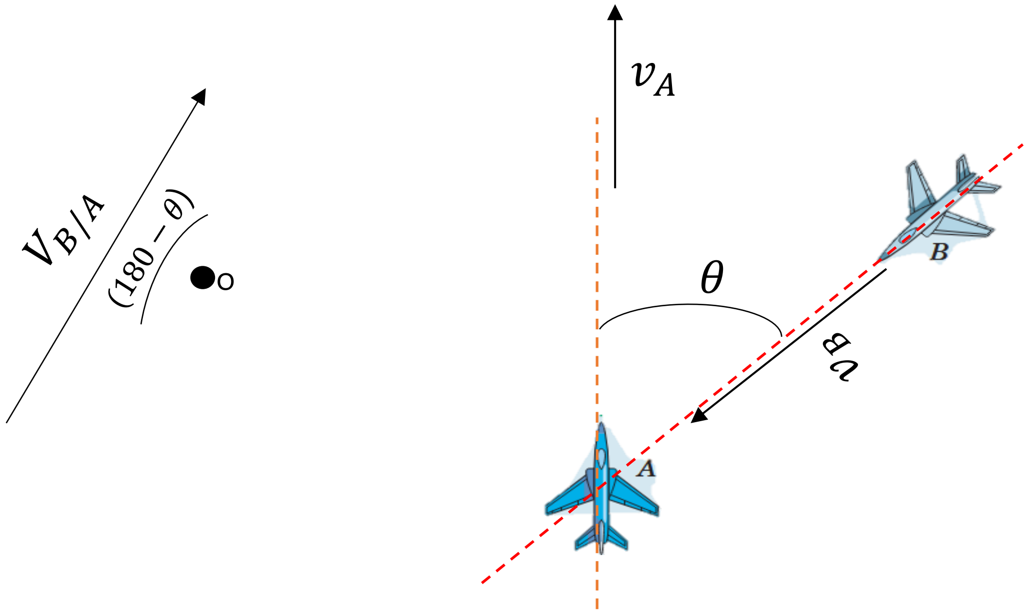


Figure 7. Velocity polygon

- Alternatively, if V_B and V_A are drawn to the correct scale, the relative velocity $V_{B/A}$ can be obtained directly from the velocity polygon, without the need to apply the sine or cosine laws.

Understanding & Solving problem

Procedure for Analysis

- It is first necessary to specify the particle origin for the translating x', y', z' and fixed axis. Usually this point has a known velocity or Acceleration .
- Since vector addition forms a triangle, there can be at most two unknowns, represented by the magnitudes and/ or directions of the vector quantities..
- These unknowns can be solved for either graphically, using trigonometry (law of sines, law of cosines), or by resolving each of the three vectors into rectangular or Cartesian components, thereby generating a set of scalar equations..

Dependent /constrained motion analysis

- In some types of problems the motion of one particle will depend on the corresponding motion of another particle [3].
- This dependency commonly occurs if the particles, here represented by blocks, are interconnected by inextensible cords which are wrapped around pulleys.
- For example, the position of block B in Fig. 8 depends upon the position of block A.
- When this is the case, only the magnitudes of the velocity and acceleration of the particles will change,

not their line of direction

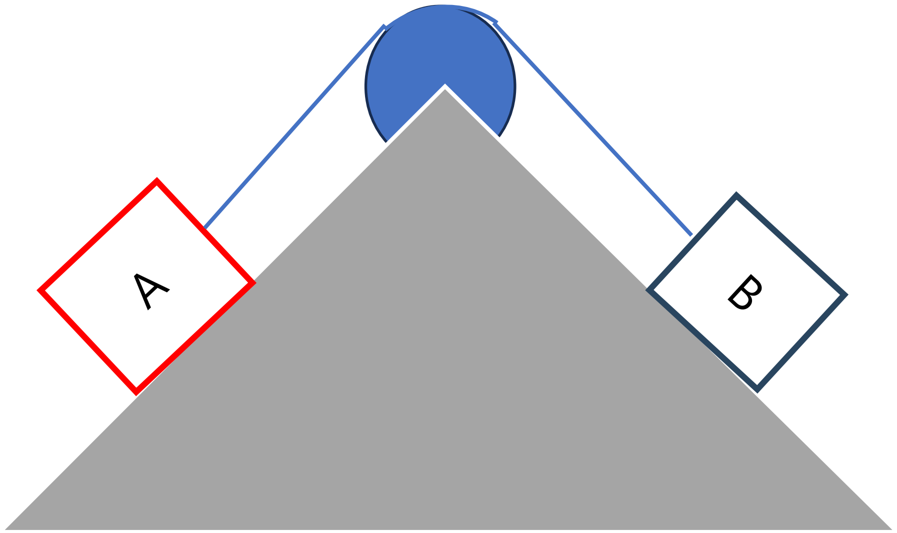


Figure 8. Dependent motion

motion expression –case 1

- We can show this mathematically by first specifying the location of the blocks using position coordinates S_A and S_B [3].

- Note that each of the coordinate axes is:

  1. measured from a fixed point or fixed datum line.

  2. measured along each inclined plane in the direction of motion of each block A and B.

  3. Has a positive sense from C to A and D to B

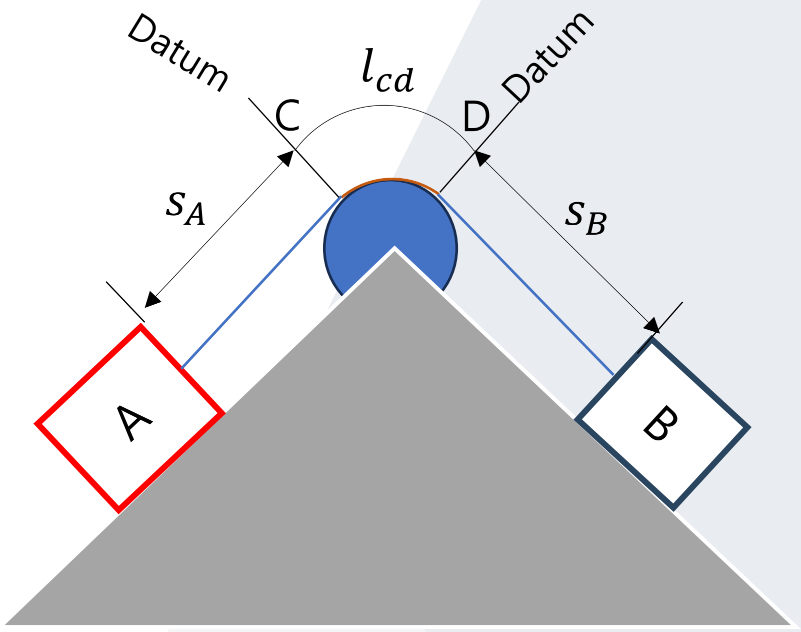


Figure 9. constrained motion

- If the total cord length is L_T, the two position coordinates are related by the equation

$$S_A + S_B + L_{CD} = L_T$$

Here L_{CD} is the length of the cord passing over arc CD

- Taking the time derivative of this expression, realizing that ICD and IT remain constant, while S_A and S_B measure the segments of the cord that change in length [1]. We have :

$$\frac{d(S_A)}{dt} + \frac{d(S_B)}{dt} + \frac{d(L_{CD})}{dt} = \frac{d(L_T)}{dt} \quad \text{or} \quad \frac{d(S_A)}{dt} + \frac{d(S_B)}{dt} = 0$$

$$V_A = -V_B$$

- In a similar manner, time differentiation of the velocities yields the relation between the accelerations, i.e.

$$a_A = -a_B$$

- The negative sign indicates that when block A has a velocity downward, i.e., in the direction of positive S_A, it causes a corresponding upward velocity of block B; i.e., B moves in the negative S_B direction

motion expression –case 2

- In this case, We used to datums to specify the horizonal position of block A is specified by SA, and the vertical position of block B is defined by SB [1].
- We followed the same procedure as previous one to specify the positions .

Then the position coordinates can be related by the equation

$$S_A + 2S_B + h = L_T$$

- Since, During the motion the length of the red colored segments of the cord, L_T and h are constant during the motion, the two time derivatives yield.

$$2V_B = -V_A \qquad 2a_B = -a_A$$

Hence, when B moves downward (+SB), A moves to the left (-SA) with twice the motion

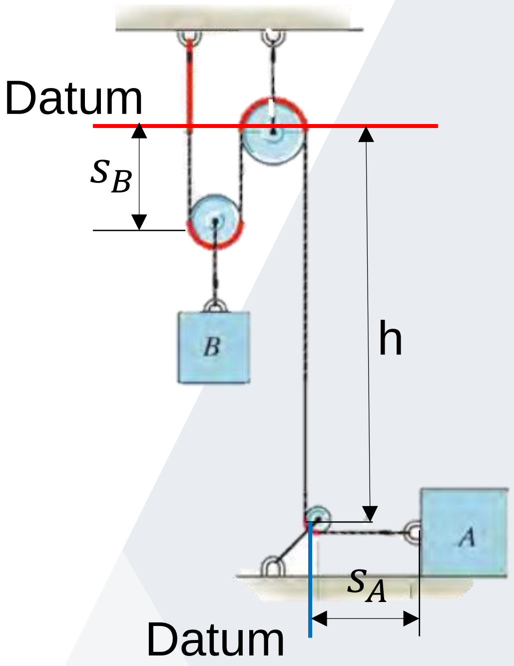


Figure 10. Case 2

The pervious example can also be worked by defining the position of block B from the center of the bottom pulley (a fixed point), Fig. 11. In this case:

$$S_A + h + 2(h - S_B) = L_T$$

Time differentiation yields.

$$2V_B = V_A \qquad 2a_B = a_A$$

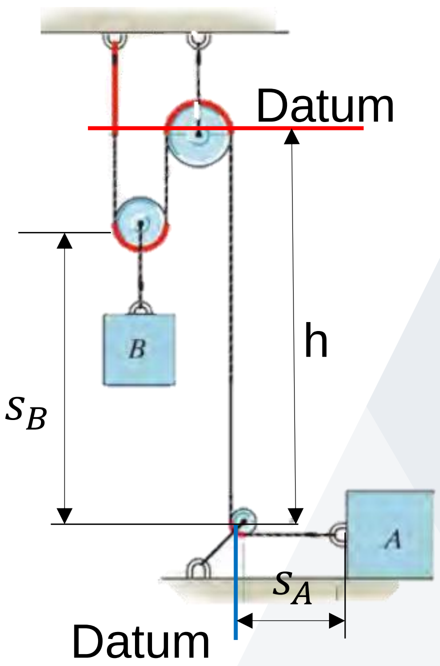


Figure 11. Case 2 alternative

motion expression –case 3

Here we have another case were we drive the kinematics relationship of the blocks using Pythagoras equation [3].

From the pulley geometry the total length of the rope L_T will be:

$$L_T = 2(h - y_A) + l \qquad \text{Where } l = \sqrt{h^2 + {x_A}^2}$$

$$L_T = 2(h - y_B) + \sqrt{h^2 + {x_A}^2}$$

Differentiation with time yields:

$$0 = (-2\dot{y}_B) + \frac{x_A\,\dot{x}_A}{\sqrt{h^2 + {x_A}^2}}$$

Substituting $\dot{x}_A = V_A$ and $\dot{y}_A = V_B$ gives:

$$(V_B) = \frac{1}{2}\frac{x_A V_A}{\sqrt{h^2 + {x_A}^2}}$$

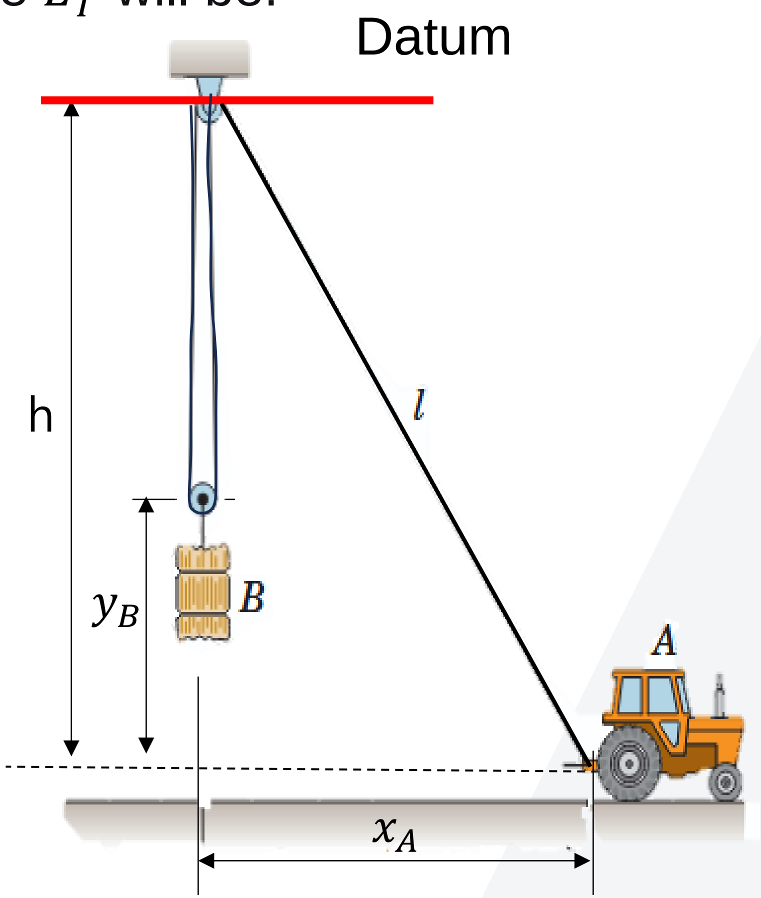


Figure 12. Case 3

1 Understanding & Solving problem

Procedure for Analysis

- Establish each position coordinate with an origin located at a fixed point or datum. It is not necessary that the origin be the same for each of the coordinates; however, it is important that each coordinate axis selected be directed along the path of motion of the particle.

- Using geometry or trigonometry, relate the position coordinates to the total length of the cord, L_T, or to that portion of cord, L, which excludes the segments that do not change length as the particles move-such as arc segments wrapped over pulleys.

- If a problem involves a system of two or more cords wrapped around pulleys, then the position of a point on one cord must be related to the position of a point on another cord using the above procedure. Separate equations are written for a fixed length of each cord of the system and the positions of the two particles are then related by these equations.

1 Understanding & Solving problem

Procedure for Analysis

Time Derivatives

- Two successive time derivatives of the position-coordinate equations yield the required velocity and acceleration equations which relate the motions of the particles.

- The signs of the terms in these equations will be consistent with those that specify the positive and negative sense of the position coordinates.

System of particles motion analysis

Relative motion

- If two particles A and B undergo independent motions, then these motions can be related to their relative motion using a translating set of axes attached to one of the particles (A).

$$V_B = V_A + V_{B/A}$$

$$a_B = a_A + a_{B/A}$$

- For planar motion, each vector equation produces two scalar equations, one in the x, and the other in the y direction.
- For solution, the vectors can be expressed in Cartesian form, or Graphical methods are used .

Constrained / dependent motion

- The dependent motion of blocks that are suspended from pulleys and cables can be related by the geometry of the system.
- First we establishing datum for position coordinates,
- Using geometry and/or trigonometry, the coordinates are then related to the cable length to get position equation.
- The first time derivative of this equation gives a relationship between the velocities of the blocks, and a second gives their accelerations.

Problem 1

Two planes A and B are traveling with the constant velocities shown. Determine the magnitude and direction of the velocity of plane B relative to plane A [1].

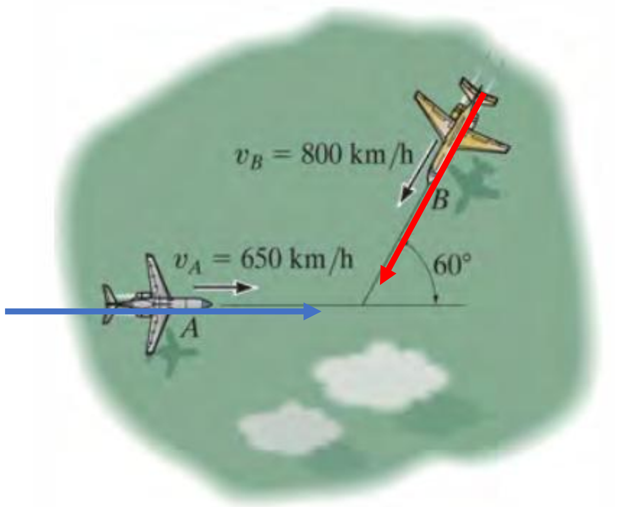


Figure 13. Problem 1

Given

$V_A = 650KM/h$

$V_B = 800Km/h$

$\theta = (60^0)$

Required

a) $V_{B/A} = ?$

b) $\theta_{B/A} = ?$

Solution

Using vector analysis

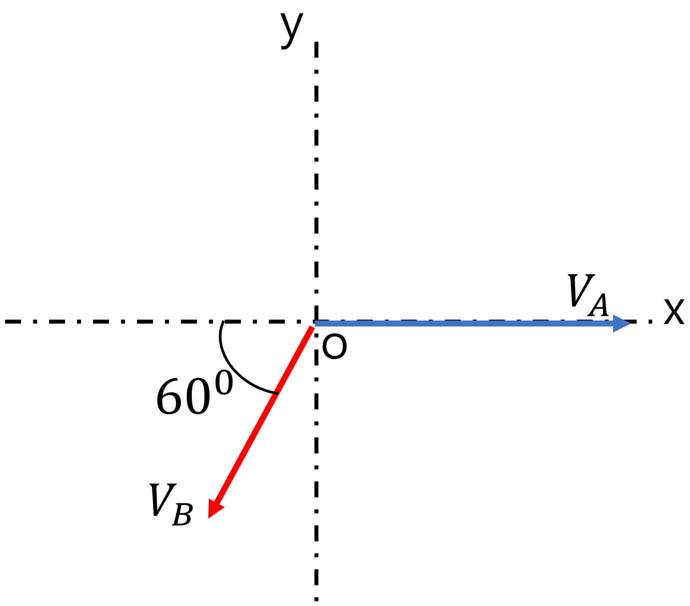


Figure 14. vector diagram

The $V_{B/A}$ obtained from the relation

$$V_B = V_A + V_{B/A}$$

$$-V_B \cos\theta i - V_B \sin\theta j = V_A i + V_{B/A}$$

$$-800\cos 60\, i - 800 \sin 60 j = 650i + V_{B/A}$$

$$V_{B/A} = -1050i - 692.8j$$

$$|V_{B/A}| = \sqrt{(-1450)^2 + (-800)^2} = 1257.93km/h$$

$$\theta_{B/A} = \tan^{-1}(\frac{692.8}{1050}) = 33.4^0$$

ans

Problem 2

The boats A and B travel with constant speeds of V_A = = 15 m/s and V_B = 10 m/s when they leave the pier at o at the same time. Determine the velocity of B relative to A [1].

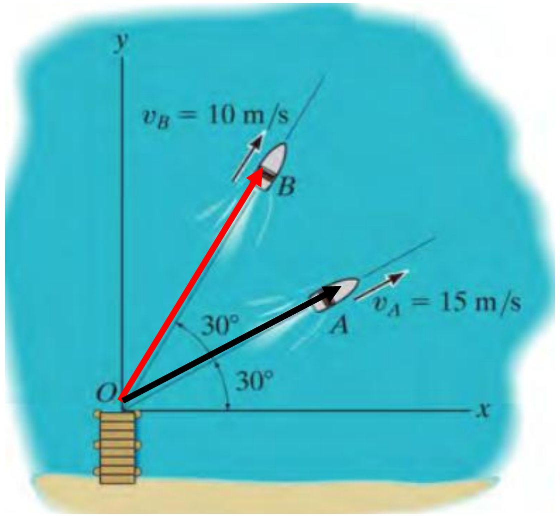


Figure 15. problem 2

Solution

Using graphical approach

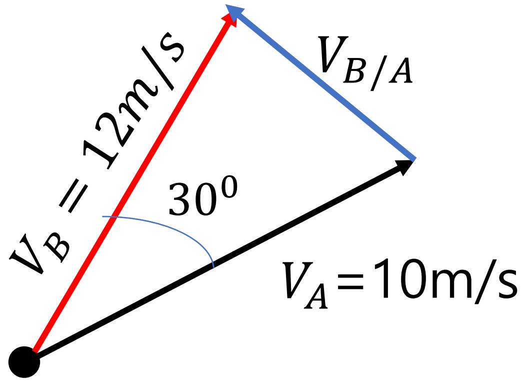


Figure 16. velocity diagram

Given

- $V_A = 15m/s$
- $V_B = 10m/s$
- $s_c = -6m$

Required

$a)\ V_{(B/A)} = ?$

Using trigonometric relation

$$V_{B/A} = \sqrt{V_A{}^2 + V_B{}^2 - 2V_AV_B\cos\theta}$$

$V_{B/A} = 8.07m/s$ ans

Problem 3

- At the instant shown, the car at A is traveling at 10 m/s around the curve while increasing its speed at 5 m/s^2. The car at B is traveling at 18.5 m/s along the straightaway and increasing its speed at 2 m/s^2 Determine the relative velocity and relative acceleration of A with respect to B at this instant [1].

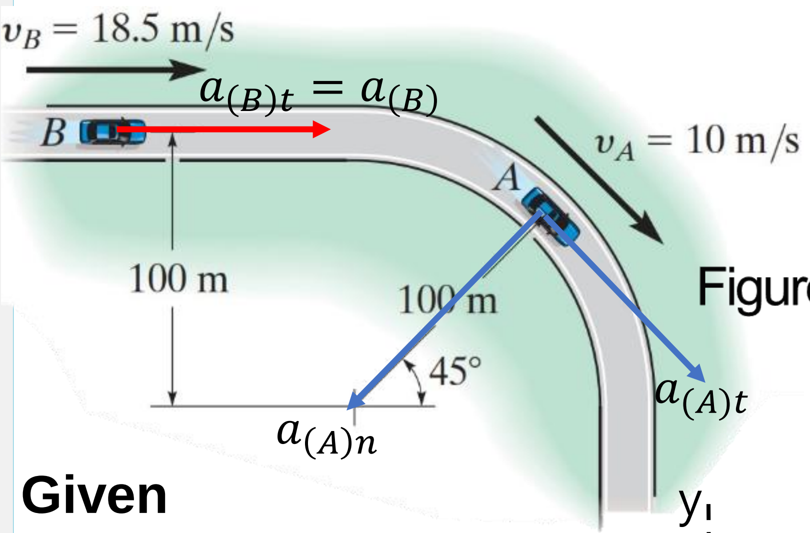


Figure 17. Problem 3

Solution

Given

$v_A = 10\ m/s$

$v_B = 18.5\ m/s$

$a_{(A)t}$ = 5 m/s^2

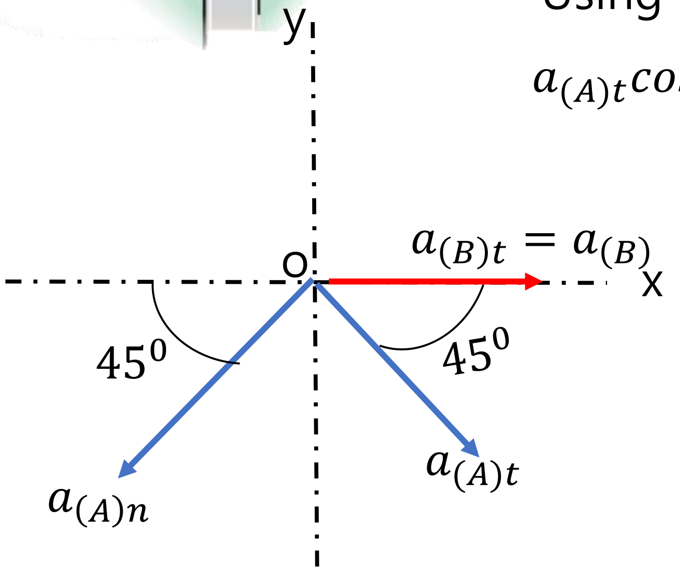
a_{tB} = 2 m/s^2

$\rho = 100m$

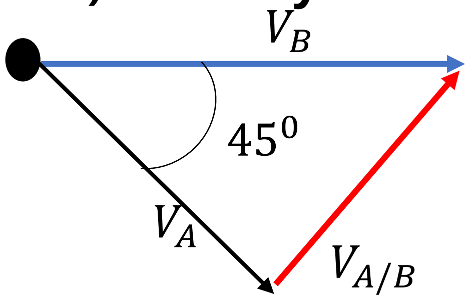
$\theta = 45^0$

Required

- a) $v_{A/B}$=?
- $b)\ a_{A/B}$ =?



a) velocity A relative to B



Using trigonometric relation

$$V_{A/B} = \sqrt{{V_A}^2 + {V_B}^2 - 2V_AV_Bcos\theta}$$

=13.4 m/s

ans

b) Acceleration of A relative to B

Using vector approach $a_A = a_B + a_{A/B}$ or $a_{(A)t} + a_{(A)n} = a_{(B)t} + a_{(A/B)}$

$$a_{(A)t}cos45i - a_{(A)t}sin45\,j - a_{(A)n}cos45i - a_{(A)n}sin45j = a_{(B)}i + a_{(A/B)}$$

$$a_{(A)n} = \frac{{V_B}^2}{r} = \frac{18.5^2}{100} = 1\ {}^{m}/_{s^2}$$

$$a_{(A/B)} = [0.828i - 4.244j]\ \text{m/}s^2$$

$$a_{A/B} = \sqrt{0.828^2 + (-4.244)^2}$$

ans

Problem 4

If block A is moving downward with a speed of 4 ft/s while C is moving up at 2 ft/s, determine the speed of block B [1].

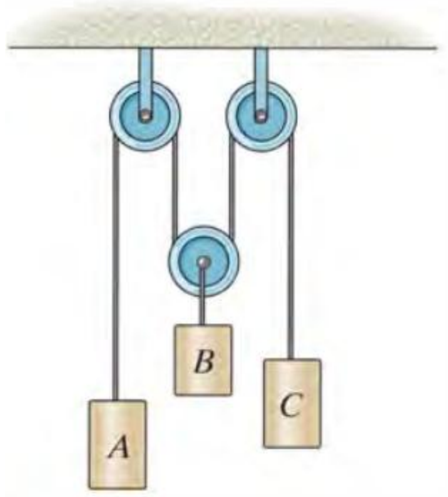


Figure 17. Problem 4

Given

$\downarrow V_A = 4\ ft/s$

$\uparrow V_c = 2\ ft/s$

Required

a) $V_B = ?$

Solution

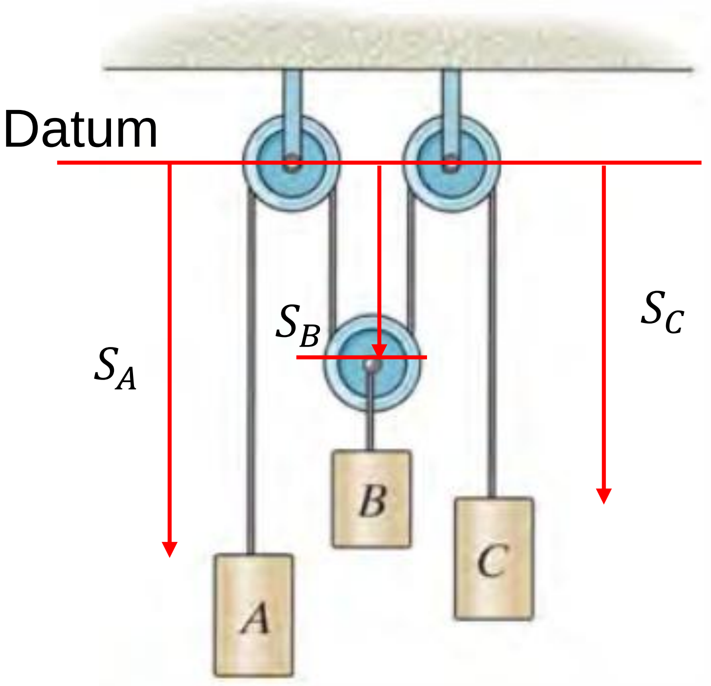


Figure 18. Positions

Position: $L_T = S_A + 2S_B + S_C$

Velocity

$0 = V_A + 2V_B + V_C$

$0 = -4 + 2V_B + 2$

$V_B = \frac{2}{2} = 1 ft/s$

Problem 5

Determine the speed of B if A is moving downwards with a speed of V_A = 4 m/s at the instant shown [1].

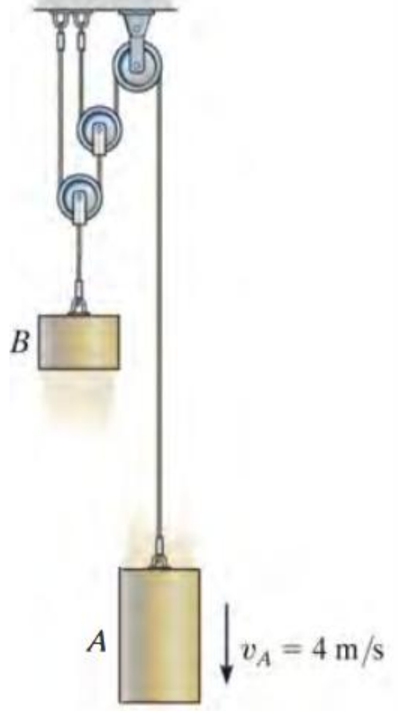


Figure 19. Problem 5

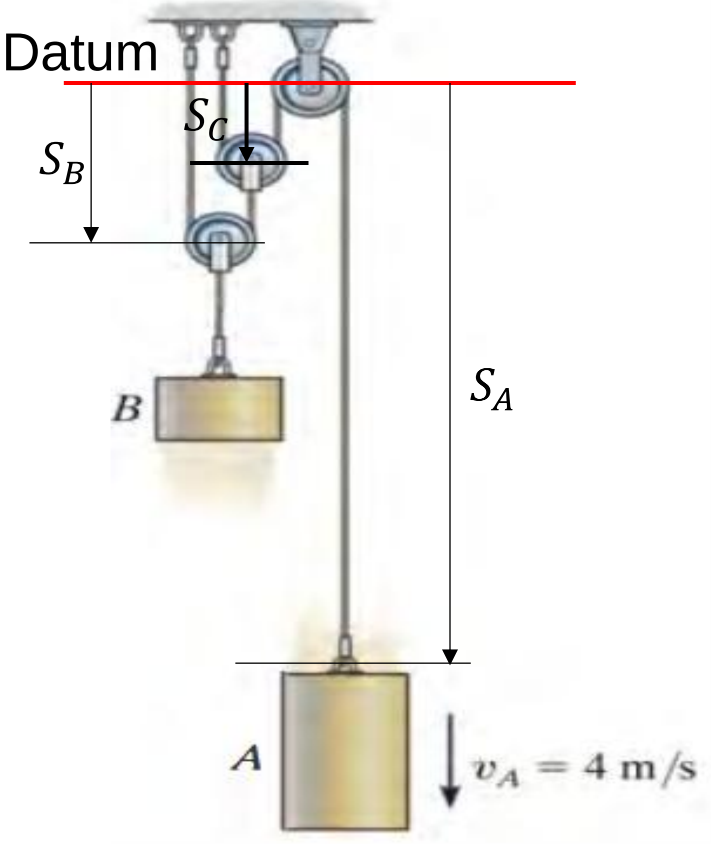
Given

$\downarrow V_A = 4\ m/s$

Required

$V_B = ?$

Solution



Position

$$L_{T_1} = S_A + 2S_c$$

$$L_{T_2} = S_B + (S_B - S_c)$$

Velocity

$$-V_A = 2V_c$$

$$V_C = -2V_B$$

$$V_B = 1\text{m/s}$$

Activities

1) The cable at B is pulled downwards at 4 ft/s, and the speed is decreasing at 2 $^{ft}/_{s^2}$. Determine the velocity and acceleration of block A at this instant [1].

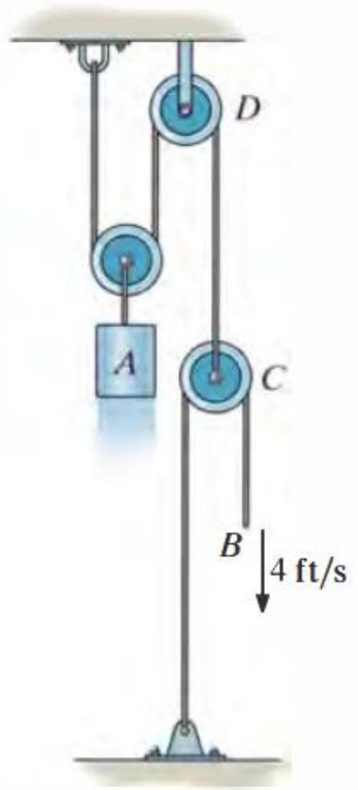


Figure 20. Activity 1

Activities

2) At the instant shown, cars A and B are traveling at the speeds shown. If B is accelerating at 1200 $^{Km}/_{h^2}$ while A maintains a constant speed, determine the velocity and acceleration of A with respect to B [1].

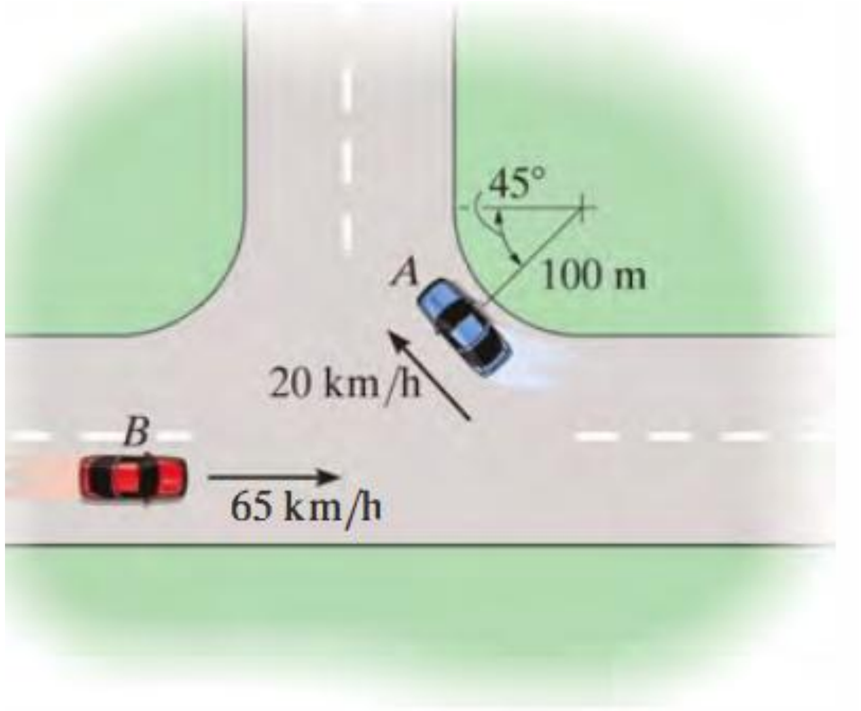


Figure 21. Activity 2

Activity

3. The man pulls the boy up to the tree limb C by walking backward. If he starts from rest when $x_A = 0$ And moves backward with a constant acceleration a_A = 0.2 $^{m}/_{s^2}$, determine the speed of the boy at the instant y_B = 4 m. Neglect the size of the limb. When x_A = 0, y_B = 8 m , so that A and B are coincident, i.e., the rope is 16 m long [1].

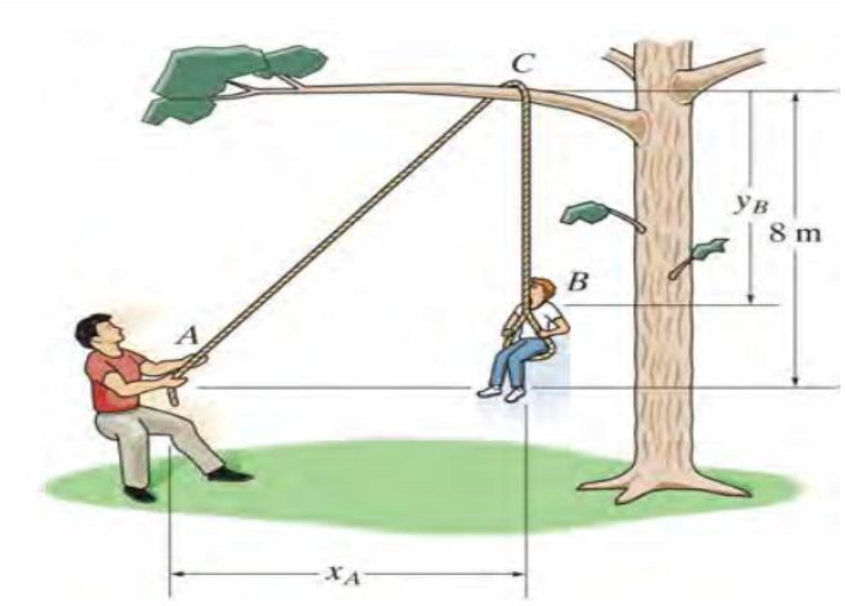


Figure 21. Activity 3

Summary

In This Lecture We Covered:

1. Introduction to Kinematics of system Particles
2. Independent motion –relative motion of Particles→ rectilinear or curvilinear
3. Dependent Motion of particles → rectilinear
4. Steps to Solve problems → description
5. Solved Problems

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