Week 6

Work–Energy Principle for Particles

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Contents

By the end of this lecture, you are able to:

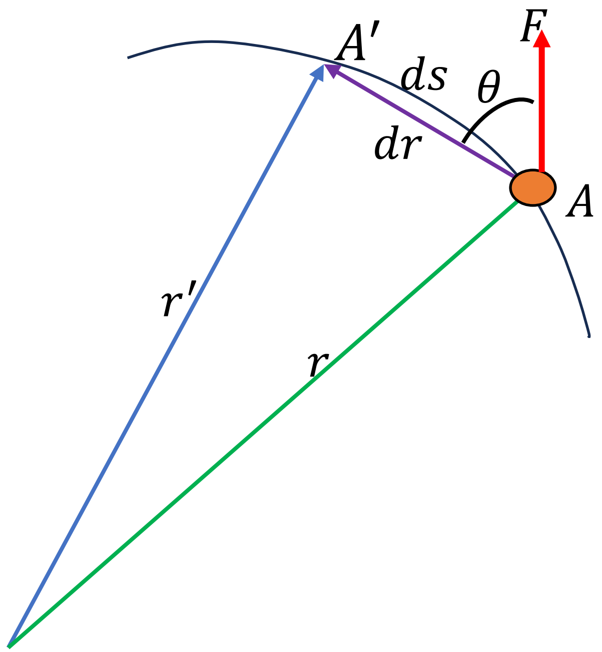
1. Understand the work energy principle
2. Understand the work done by different forces
3. Define and explain Kinetic energy
4. Explain work-energy equation
5. Understand potential energy and conservation of energy

Understand the work energy principle

- There are two general classes of problems in which the cumulative effects of unbalanced forces acting on a particle are of interest to us [1].
- These cases involve (1) integration of the forces with respect to the displacement of the particle and (2) integration of the forces with respect to the time they are applied.
- Integration with respect to displacement leads to the equations of work and energy,
- In today topic, we will analyze motion of a particle using the concepts of work and energy.[1].
- The resulting equation will be useful for solving problems that involve force, velocity, and displacement.
- Before we do this, however, we must first define the work of a force.

Definition of Work

- To develop the quantitative meaning of the term "work, Consider a particle that moves on path s from a point A to a neighboring point A' due to Force F . If r denotes the position vector corresponding to point A, we can denote the small vector joining A and A' by the differential dr which is the displacement [1].



- If the angle between the tails of dr and F is θ , Fig. 1, The work done by the force F during the displacement dr is defined as : $du = F.dr$

Figure 1. Work

The magnitude of this dot product can also be written as : $du = Fcos\theta\ ds$

where ds is the magnitude of dr

- **In general from** $du = F.dr$ or $du = Fcos\theta\ ds$, **Three particular case are drawn** [2]:

  - If $0^0 \leq \theta \leq 90^0$, then the force component and the displacement have the same sense so that the work is positive.
  - If $90^0 \leq \theta \leq 180^0$, these vectors will have opposite sense, and therefore the work is negative.
  - If the force is perpendicular to displacement, since cos 90° = 0, or if the force is applied at a fixed point, in which case the displacement is zero the work is zero.

- Work is a scalar quantity, so it has a magnitude and a sign but no direction
- Note that work is expressed in units obtained by multiplying units of length by units

of force.

Common Types of Work

- We can use the general $du = F.dr$ equations to derive formulas for the work done by a force in several common and important situations, as we now show.

Work of a constant Applied Force

- Consider a body on which moves from position 1 to position 2 under the action of the constant force P applied

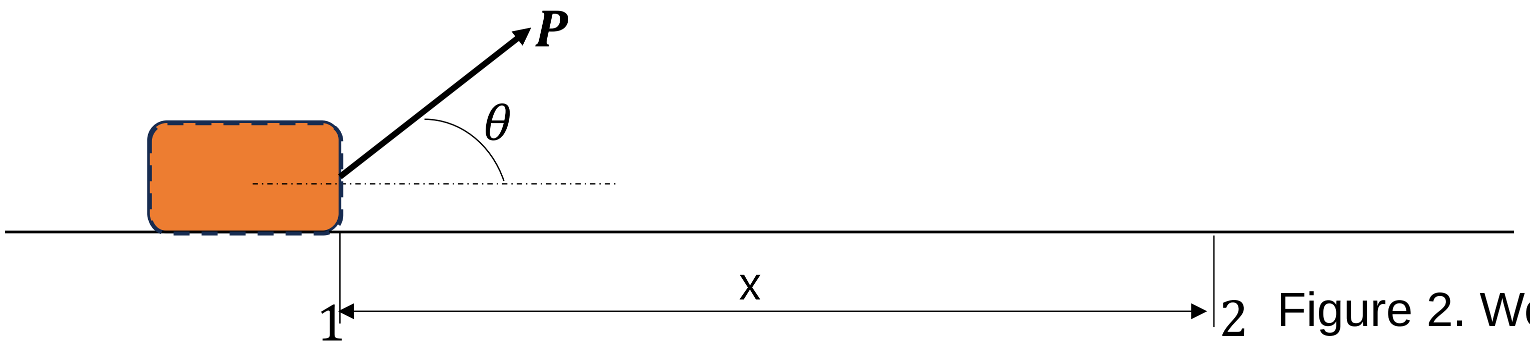


Figure 2. Work of applied force

- With the force P and the differential displacement dr written as vectors, the work done on the body by the force is

$$\int_1^2 du = \int_1^2 F.dr = \int_1^2 \big((P\cos\theta)i + (P\sin\theta)j\big).dx\, i$$

$$U_{1\rightarrow 2} = \int_1^2 ((P\cos\theta))dx$$

$$U_{1\rightarrow 2} = (P\cos\theta)(x_2 - x_1)$$

- All points move in parallel straight lines

Common Types of Work

Work of Force of Gravity

- Consider a body of weight W, which moves up along the path s shown in Fig. 3 from position y_1 to position y_2.

$$\int_1^2 du = \int_1^2 F.dr = \int_1^2 (-wj).(dx\, i + dyj)$$

$$U_{1\rightarrow 2} = \int_{y_1}^{y_2} (-(w)dy = -w(y_2 - y_1)$$

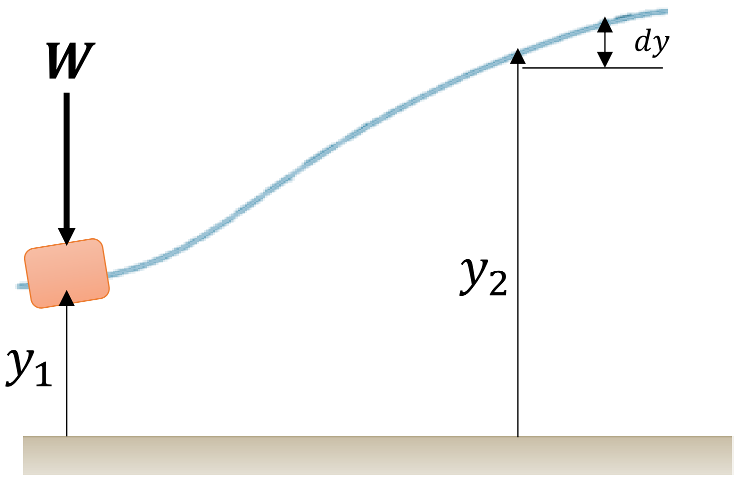


Figure 3. Work of force of gravity

- The work is positive when $\Delta y \leq 0$, that is, when the body moves down.
- When the body moves up (and $\Delta y \geq 0$, the force and displacement are in opposite directions, and the work is negative [2].

- Thus, the work is independent of the path and is equal to the magnitude of the particle's weight times its vertical displacement.

Common Types of Work

Work of a Spring Force

- Consider a body A attached to a fixed point B by a spring; we assume that the spring is undeformed when the body is at A_O
- The magnitude of the spring force F is proportional to the deflection x of the spring measured from the unstretched position A_O.

$$F_s = k\Delta x$$

where k is the spring constant expressed in N/m or kN/m and

$$x = L_{streached} - L_{unstreached}$$

- We can obtain the work of force F exerted by the spring during a finite displacement of the body from

$$\int_1^2 du = \int_1^2 F.dr = \int_{x_1}^{x_2} (-kx).dx$$

$$U_{1\rightarrow 2} = \frac{1}{2}(-k)(x_2{}^2 - x_1{}^2)$$

The work is negative since F_s acts in the opposite sense to ds but if the direction are the same consider it as positive

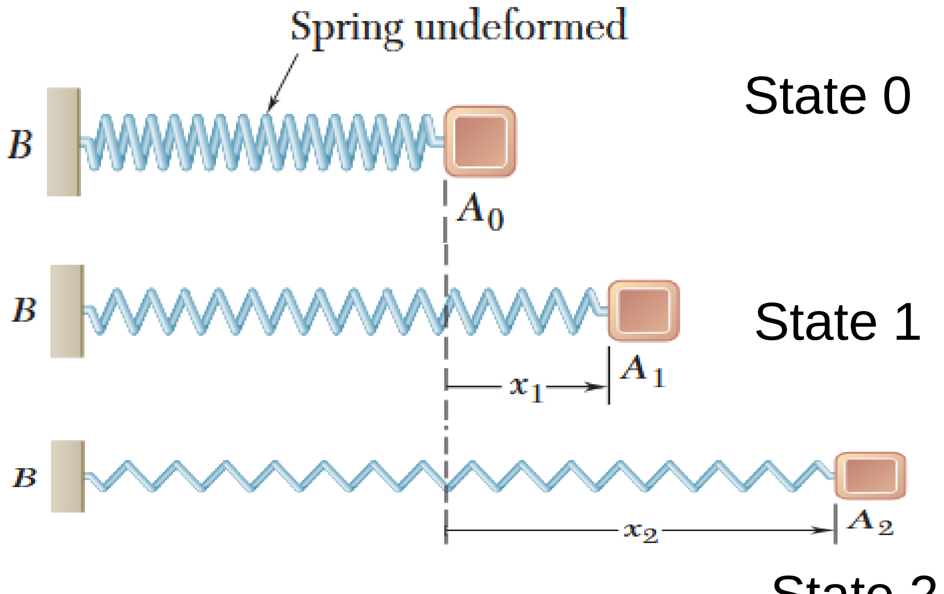


Figure 4. Work of force of spring

Common Types of Work

Work of a friction Force

- Consider a body on which moves from position 1 to position 2 under the action of the constant force P applied on a rough surface s.

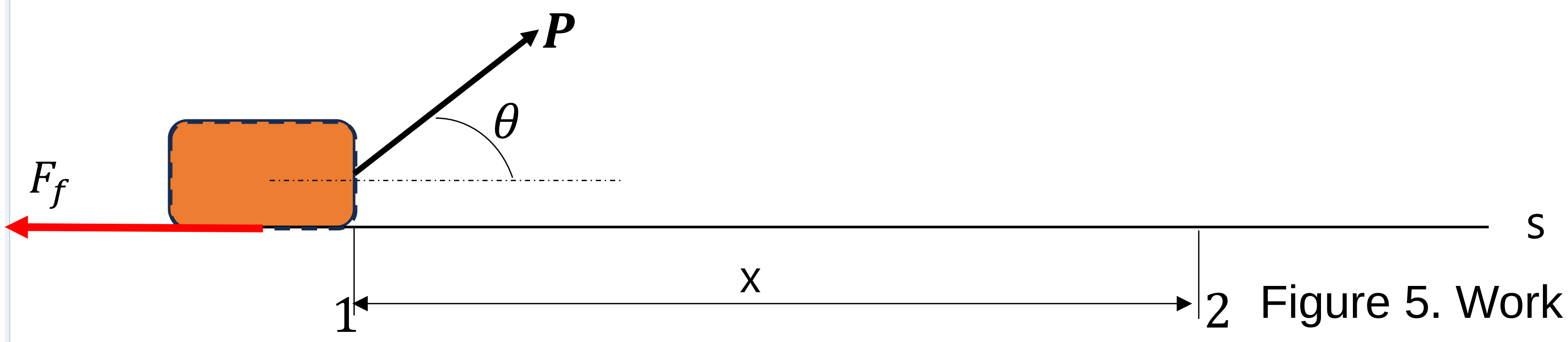


Figure 5. Work of Friction force

- With the force P, If a body slides a distance x over a surface under kinetic friction f_k,then the work done by friction is:

$$U_{1\rightarrow 2} = -F_f . x = -\mu_k F_N . X$$

- Where: μ_k coefficient of kinetic friction

  F_N = normal reaction force

- The negative sigh indicates friction acts opposite to motion

Kinetic energy

- It is the energy that an object possesses because of its motion [3].
- It depends on the mass of the object and the square of its velocity.
- Mathematically, The kinetic energy of the particle is defined as:

$$T = \frac{1}{2} m V^2$$

- Where: T= kinetic energy (in joules, J)

  m= mass of the object (in kilograms, kg)

  v= velocity of the object (in meters per second, m/s)

The faster an object moves or the heavier it is, the more kinetic energy it has.

Principle of Work and Energy

- Consider a particle of mass m acted upon by a force F and moving along a path that is either rectilinear or curved . Expressing Newton's second law in terms of the tangential components of the force and of th e acceleration , we have [4] :

$$\sum F_t = ma_t \qquad or \ \sum F_t = \mathrm{m}\, v\frac{dV}{ds}$$

$$\sum F_t \ ds = \mathrm{m}v \ dv$$

- Integrating from point 1, where s_1 and v_1 , to point 2, where s_2 and v_1 , we have:

$$\sum \int_{s_1}^{s_2} F_t . ds = \mathrm{m} \int_{v_1}^{v_2} (v). dv = \frac{1}{2} m({v_2}^2 - {v_1}^2)$$

$$\sum \int_{s_1}^{s_2} F_t . ds = \frac{1}{2} m({v_2}^2 - {v_1}^2)$$

- The left-hand side of represents the work of the force F exerted on the particle; and it is a scalar quantity. The right hand side equation indicates kinetic energy also a scalar quantity.
- This equation represents the principle of work and energy for the particle.

Principle of Work and Energy

- This equation states that when a particle moves from Point 1 to 2 under the action of a force F, the work of the force F is equal to the change in kinetic energy of the particle. This is known as the principle of wo rk and energy [2].

$$\sum U_{1\rightarrow 2} = \frac{1}{2} m({v_2}^2 - {v_1}^2)$$

- Rearranging the terms Work energy principle is often often expressed in the form

$$T_1 + \sum U_{1\rightarrow 2} = T_2$$

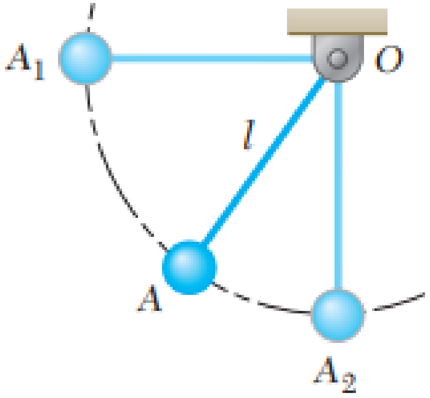
where: $T_2 = \frac{1}{2} m {v_2}^2$

$$T_1 = \frac{1}{2} m\, {v_1}^2$$

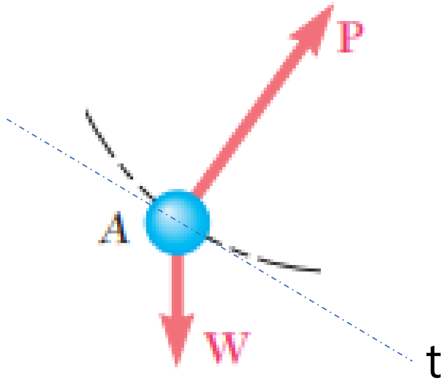
$\sum U_{1-2}$ is The sum of works done by applied force ,spring force ,wight and so on.

Applications of of Work and Energy

- The principle of work and energy greatly simplifies the solution of many problems involving forces, displ acements, and velocities [2].
- Consider, for example, the pendulum OA consisting of a bob A of weight W attached to a cord of length l
- The pendulum is released with no initial velocity from a horizontal position OA1 and allowed to swing in a vertical plane. We wish to determine the speed of the bob as it passes through A2, directly under O.



(a) A bob swings from an initial position A_1 to a final position A_2



(b) FBD of the bob at position A

Figure 6. Pendulum

- This example illustrates the following advantages of the method of work and energy:
  - In order to find the speed at A2, there is no need to determine the acceleration in an intermediate position A and to integrate the acceleration expression from A1 to A2.
  - Forces that do no work are eliminated from the solution of the problem.
  - All quantities involved are scalars and can be added directly, without using x and y components.
- Its disadvantage : can't find acceleration and normal forces, i.e. we need Newton's law.
  - To find the pendulum's cord tension at A_2, draw free-body and kinetic diagrams and apply Newton's second law in tangential and normal directions

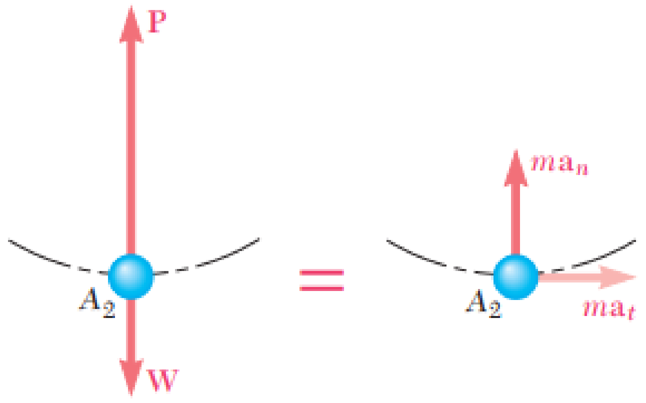


Figure 7. Determining the force on a pendulum

Summary of work energy equation

Important points

- This method eliminates the need to calculate the acceleration and enables you to relate the velocities of the particle at two points along its path of motion
- The governing equation: $T_1 + \sum U_{1 \rightarrow 2} = T_2$
- The kinetic energy (T) at the initial and final points is always positive
- The work can be done by wight ,spring ,applied and friction forces
- Work is positive when the force component is in the same sense of direction as its displacement, otherwise it is negative.
- A force does work when it moves through a displacement in the direction of the force.

1 Understanding & Solving problem

Procedure for Analysis

- Establish the inertial coordinate system and draw a free-body diagram of the particle in order to account for all the forces that do work on the particle as it moves along its path
- Compute the work of each of the external forces
- Calculate the kinetic energy at initial T_1 and Final point T_2
- Substitute the values for the total work done and the kinetic energies in to the general equation and find for the unknown

CONSERVATION OF ENERGY

- The principle of work and energy is useful for solving many different types of engineering problems [3].
- However, in many engineering applications, the total mechanical energy remains constant, although it may be transformed from one form into another.
- This is known as the principle of conservation of energy.
- To formulate this principle, we must first define a quantity known as potential energy.

Potential energy

- If the work of a force is independent of the path and depends only on initial and final positions on the path, then we can classify this force as a conservative force [1]. E.g. weight and the spring force
- For both cases the amount of work done the weight in not affected by the path as long as the mass and the vertical height kept the same.

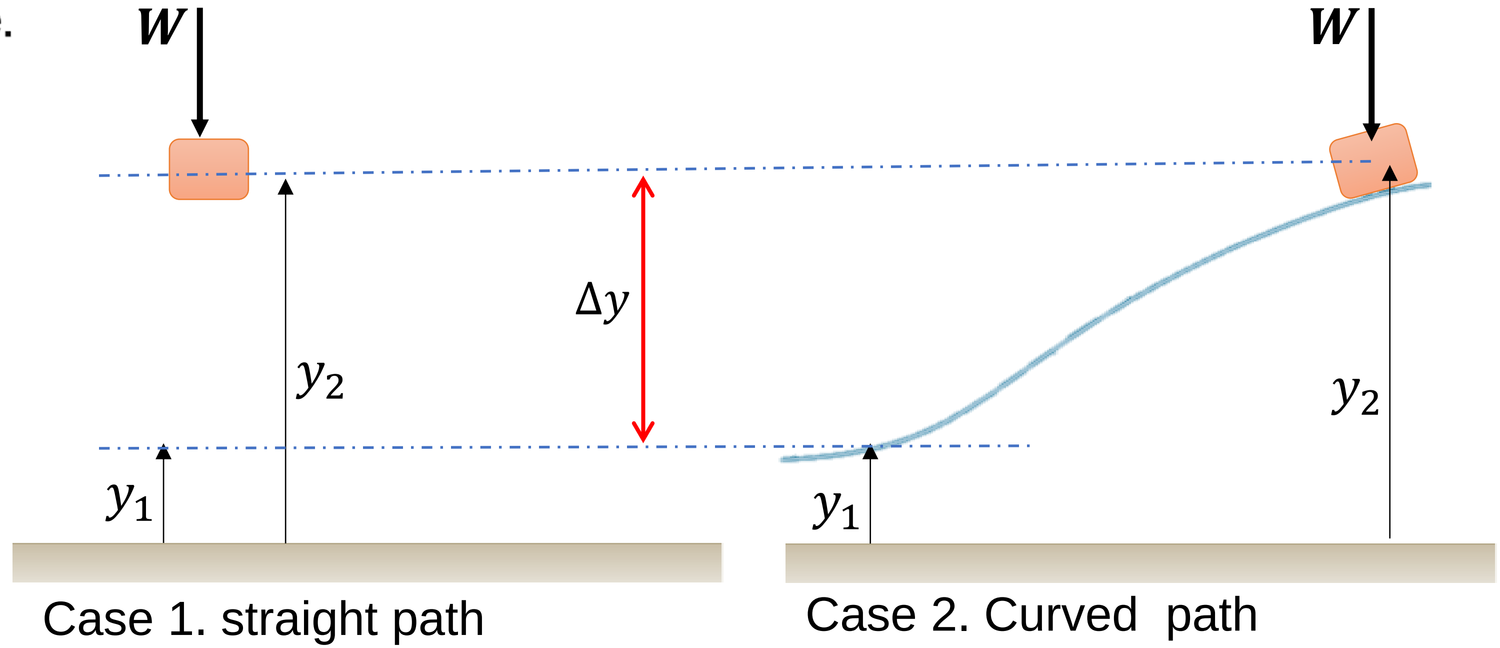


Figure 8. Force of gravity

- Work done by weight depends only on vertical displacement and Work done by spring depends only on elongation or compression.
- Thus, potential energy is a measure of the amount of work a conservative force will do when it moves from a given position to the datum.

Gravitational Potential Energy

- Consider a body of weight W, which moves along the path s shown in Fig. 7 ,between 3 different points (O, y_1 and y_2.

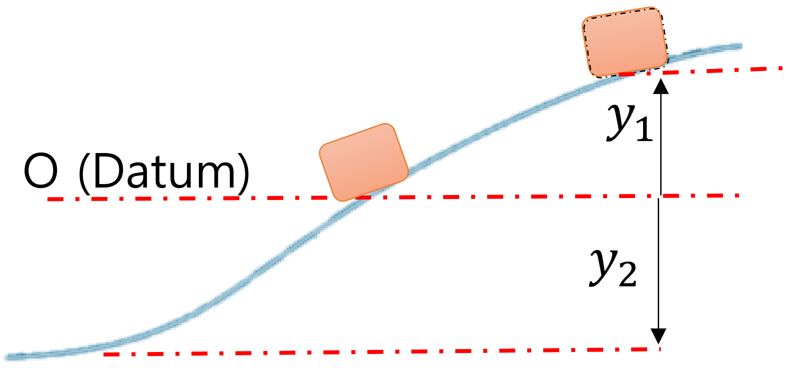


Figure 9. potential of gravity

- if we denoted gravitational potential energy as v_g, and mathematically expressed as: $v_g = wy$
- if we measure the gravitational potential energy of the body from arbitrarily selected datum O. Then gravitational potential energy at two different point is $(v_g)_1 = wy_1$ and $(v_g)_2 = -wy_2$

- W has the capacity of doing positive work when the particle is moved back down to the datum.
- Likewise, if the particle is located a distance y below the datum, Vg is negative since the weight does negative work
- Recall that the work done by the force of gravity W during this displacement is

$$U_{1\rightarrow 2} = -w(y_2 - y_1) = wy_1 - wy_2 = (v_g)_1 - (v_g)_2$$

Elastic Potential Energy (Spring Energy (V_e)

- When a spring is stretched or compressed a distance s from its unstretched position, it stores elastic potential energy.

$$v_e = \frac{1}{2} k s^2$$

where
v_e = elastic potential energy
k= spring stiffness (N/m)
s = elongation or compression (m)

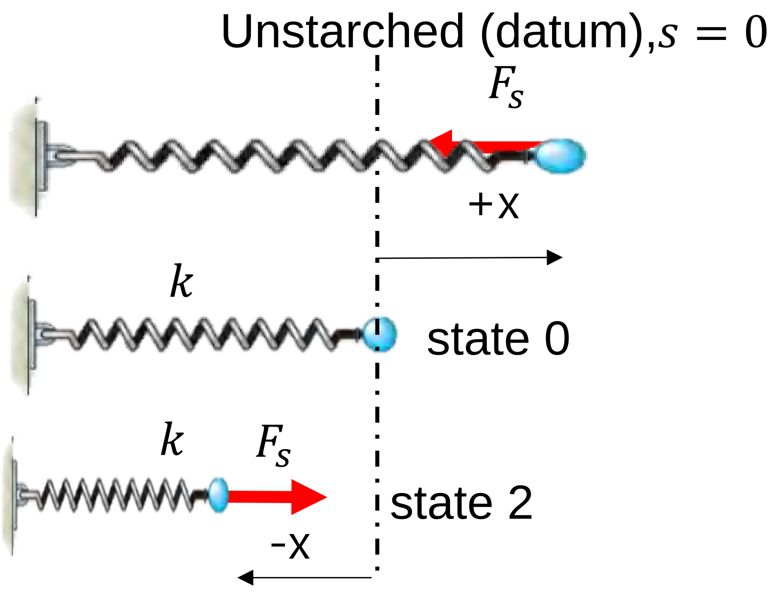


Figure 10. spring potential

- Ve is always positive, because the spring, when deformed, always has the potential to do positive work as it returns to its unstretched (datum) position.

- Recall that the work done by the force of Spring during this displacement is

$$U_{1\rightarrow 2} = \frac{1}{2}(-k)(x_2{}^2 - x_1{}^2) = (v_s)_1 - (v_s)_2$$

Potential Function

- In the general case, if a particle is subjected to both gravitational and elastic forces, the particle's potential energy can be expressed as a potential function, which is the algebraic sum.

$$V = v_g + v_e$$

- Where : V Potential energy function

  : v_g gravitational potential energy

  : v_e spring (elastic) potential energy

- The work done by a conservative force in moving the particle from one point to another point is measured by the difference of this function [2]., i.e

$$U_{1 \to 2} = V_1 - V_2$$

NB. Measurement of V depends on the location of the particle with respect to a datum.

CONSERVATION OF ENERGY

- When a particle is acted upon by a system of both conservative and nonconservative forces, the portion of the work done by the conservative forces can be written in terms of the difference in their potential energies using i.e. $U_{(1 \rightarrow 2)\ cons} = V_1 - V_2$

- As a result, the principle of work and energy can be written as

$$T_1 + \sum U_{(1 \longrightarrow 2) cons} + U_{(1 \longrightarrow 2) non\ cons} = T_2$$

- Here $\sum \boldsymbol{U}_{(\mathbf{1} \longrightarrow \mathbf{2}) \boldsymbol{non-cons}}$, represents the work of the nonconservative forces acting on the particle.
- If only conservative forces do work then we have:

$$T_1 + V_1 = T_2 + V_2$$

- This equation is referred to as the conservation of mechanical energy or simply the conservation of energy.

Summary on conservation of energy

Important points

- The conservation of energy equation can be used to solve problems involving velocity, displacement, and **conservative force systems**.
- It is easier to apply than the principle of work and energy because this equation requires specifying the particle's kinetic and potential energies at only two points along the path.
- The governing equation: $T_1 + V_1 = T_2 + V_2$
- The kinetic energy (T) at the initial and final points is always positive
- The gravitational protentional energy positive upward from the datum and negative downward from the datum also for a spring potential energy , is always positive.

1 Understanding & Solving problem

Procedure for Analysis

- Determine whether all the forces involved are conservative.

  If some of the forces are not conservative, for example, if friction is involve, you must use the general equation

- Draw two diagrams showing the particle located at its initial and final points along the path and choose a datum.
- Calculate the kinetic energy at initial T_1 and Final point T_2
- Compute the spring potential energy and gravitational potential energy at each end of the path.
- Substitute all your values to the general conservation of energy equation and solve for the known.

system of particles

- The principle of work and energy can be extended to include a system of particles isolated within an enclosed region of space as shown in Fig 8 [1].
- if we apply the principle of work and energy to this and each of the other particles in the system, then s ince work and energy are scalar quantities, the equations can be summed algebraically, which gives

$$\sum T_1 + \sum U_{1-2} = \sum T_2$$

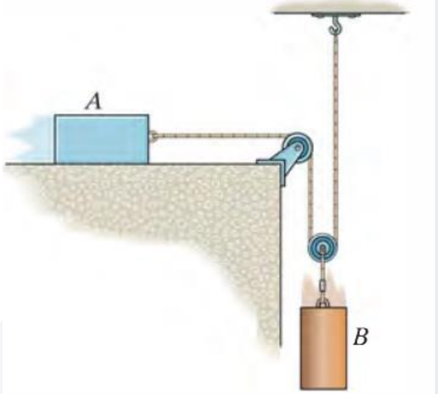


Figure 11. dependent motion

- If a system of particles is subjected only to conservative forces, then a n equation similar to the equation, this can be written for the particles.

$$\sum T_1 + \sum V_1 = \sum T_2 + \sum V_2$$

Summary on continuous motion

Work –energy equation

- useful for solving problems that involve force, velocity, and displacement
- Equation $$T_1 + \sum U_{1-2} = T_2$$
- Applied in all cases.

Conservation of energy

- A conservative force does work that is independent of its path. Two examples are the weight of a particle and the spring force.
- The work done by a conservative force depends upon its position relative to a datum.
- When this work is referenced from a datum, it is called potential energy..
- Equation $$T_1 + V_1 = T_2 + V_2$$
- Applied in certain cases.

Problem 1

1. If the motor exerts a constant force of 300 N on the cable, determine the speed of the 20-kg crate when it travels s = 10 m up the plane, starting from rest. The coefficient of kinetic friction between the crate and the plane is μ_k = 0.3.

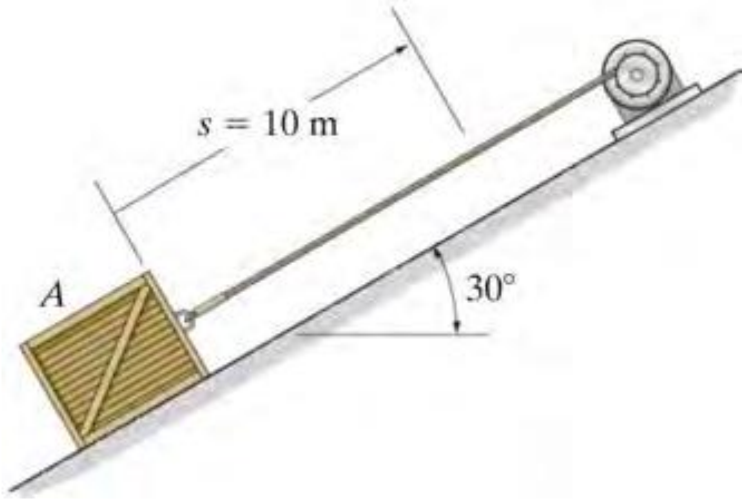


Figure 12. problem 1

Given

- $M=20kg$
- P=T= 300N m/s
- $\Delta S = 10m$
- $V_1 = 0(rest)$
- $\vartheta = 30^0$
- μ_k = 0.3

Required

$V_2 = ?$

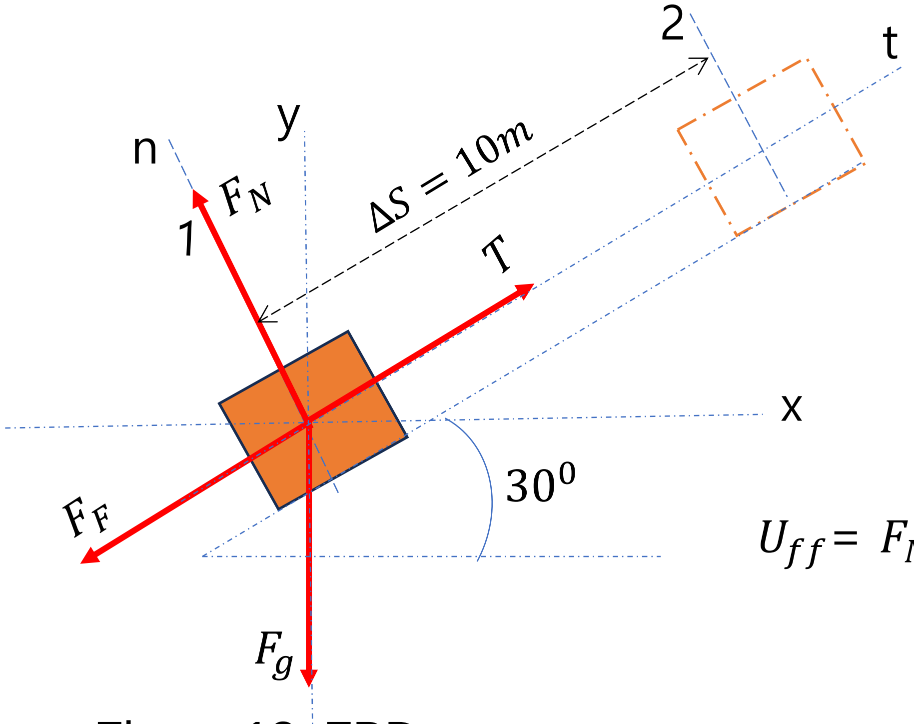


Figure 13. FBD

$$T_1 + \sum U_{1\rightarrow 2} = T_2$$

$$T_1 = \frac{1}{2} m\, {v_1}^2 = 0$$

$$T_2 = \frac{1}{2} m {v_2}^2 = 10{v_2}^2$$

$$\sum U_{1\rightarrow 2} = U_P - U_g - U_{ff}$$

$$U_P = T\Delta S = 3000J$$

$$U_g = (mg\sin\theta)\Delta S = 981J$$

$$U_{ff} = F_N.\mu_k \Delta S = (mg\cos\theta)(\mu_k)\Delta S = 509.7J$$

$$T_1 + \sum U_{1\rightarrow 2} = T_2$$

$$0 + 3000J - 981J - 509.7J = 10{v_2}^2$$

$$v_2 = 12.3\ m/s$$

Activity

1. The 1 0-lb block A travels to the right at V_A= 2 ft/s at the instant shown. If the coefficient of kinetic friction is 0.2 between the surface and A, determine the velocity of A when it has moved 4 ft. Block B has a weight of 20 lb [2]. .

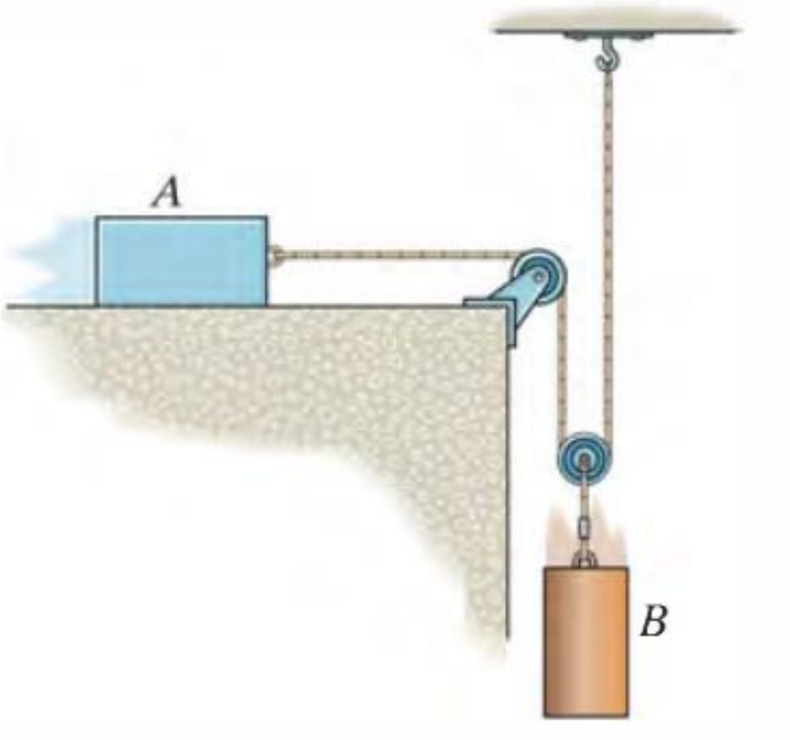


Figure 14. Problem

Activity

2. The 2-kg collar is released from rest at A and travels along the smooth vertical guide. Determine the speed of the collar when it reaches position B. Also, find the normal force exerted on the collar at this position. The spring has an unstretched length of 200 mm..

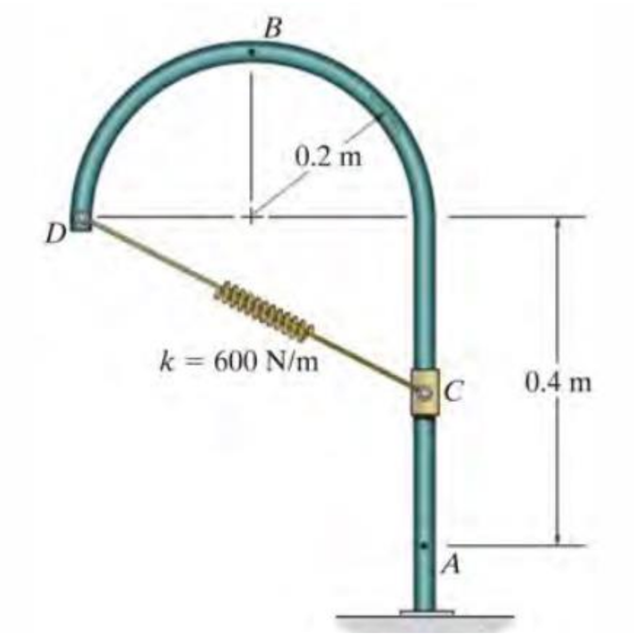


Figure 15. Problem 2

Source: Engineering Mechanics: Dynamics, Hibbeler, Rusel M.,Prentice Hall, 10th ed., 2003,page 300

Summary

In This Lecture We Covered:

1. Introduction to Kinetics of Particles → work energy method
2. Work and its types
3. Kinetic energy
4. Work energy equation
5. conservation of energy

References

[1] Dynamics, Hibbeler, Rusel M.,Prentice Hall,10th ed., 2003

[2] Engineering Mechanics - Dynamics, Meriam J.L.,John Wiley & Sons, 9th ed., 2020.,

[3] Vector Mechanics for Engineers: Dynamics, Johnston, E. R., & Clausen, W. E., McGraw-Hill, 11th ed., 2015

[4] Cengel, Yunus, and John Cimbala. *Ebook: mechanics fundamentals and applications (si units)*. Mc Graw Hill, 2013.