Week 10

Fixed Axis Rotation and Absolute Motion

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Contents

By the end of this lecture, you are able to:

1. Understand Fixed Axis Rotation
2. Define and explain Angular motion
3. Define angular motion parameters such us angular velocity, angular acceleration
4. Understand Absolute Motion analysis method for Rigid bodies

Fixed Axis Rotation of rigid bodies

- The rotation of a rigid body is described by its angular motion [1].
- The particles forming the rigid body move in parallel planes along circles centered on the same fixed axis.
- any point P located in the body travels along a circular path.
- The particles located on the axis of rotation have zero velocity and zero acceleration
- Particles or points on the body have the same angular motion and different linear motion

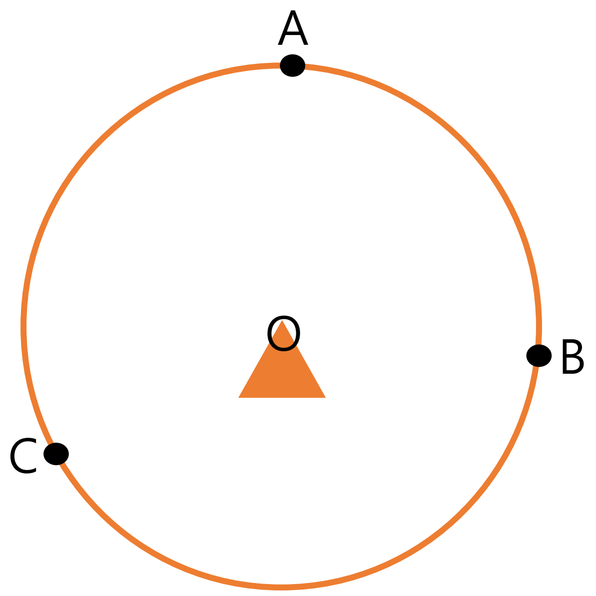

Figure 1. Rotation about fixed axis

- To study this motion, it is first necessary to discuss the angular motion, angular position, angular velocit y and angular acceleration of the body about the axis.
- Consider the rod shown below, which has two points, A and B, experiencing angular motion or rotation about its own fixed axis passing through point O.
- The angular motion of the rod can be represented by its angular velocity, angular acceleration, and angular position

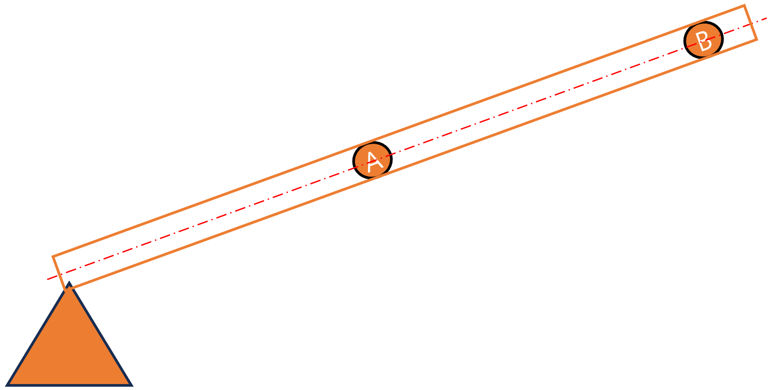

Figure 2. Rod

Angular position

- At the instant shown, the angular position of the rod or any point on the rod defined by the angle θ measured from a fixed reference line [1]..

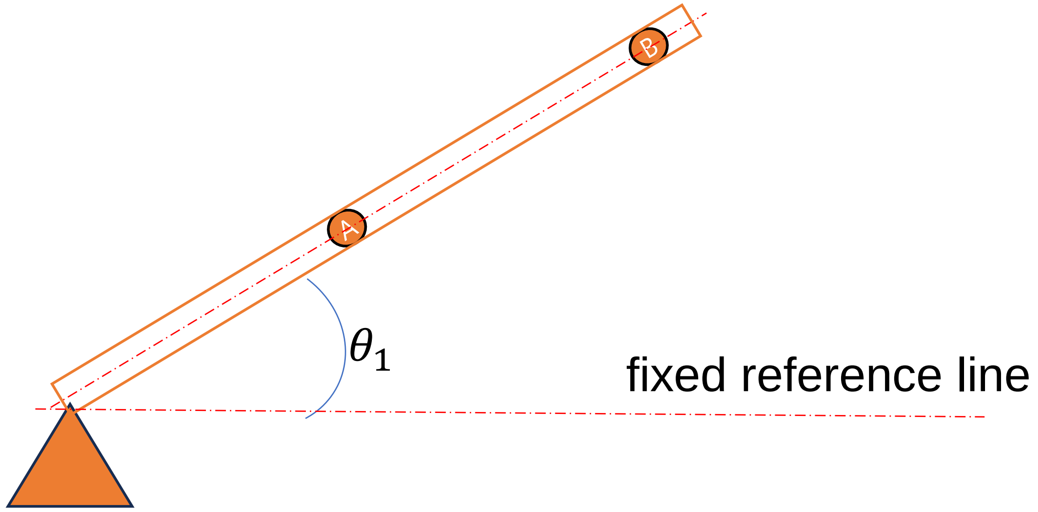

Figure 3. position of Rod

Angular displacement

- The change in the angular position, which can be measured as a differential $d\theta$, is called the angular displacement.
- This vector has a magnitude of $d\theta$, measured in degrees, radians, or revolutions, where 1 rev = 2π rad.
- The direction is determined by the right-hand rule [1]..

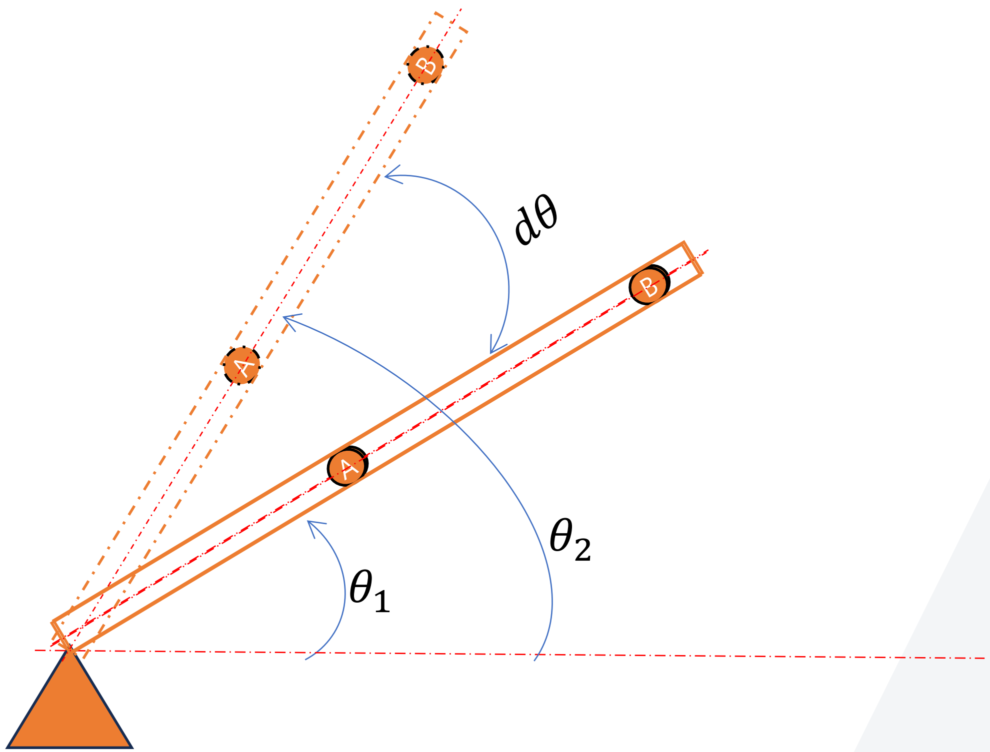

Figure 4. angular displacement of the Rod

Angular Velocity.

- The time rate of change in the angular position is called the angular velocity ω (omega). ;
- Since $\mathrm{d}\theta$ occurs during an instant of time dt, then;

$$\omega = \frac{d\theta}{dt}$$

- This vector has a magnitude which is often measured in rad/s.
- it is expressed here in scalar form since its direction is also along the axis of rotation
- we can refer to the sense of rotation as clockwise or counterclockwise.

Angular Acceleration

- The angular acceleration α (alpha) measures the time rate of change of the angular velocity.;
- The magnitude of this vector is

$$\alpha = \frac{d\omega}{dt}$$

- it is also possible to express α as function of position θ as:

$$\alpha = \frac{d^2\theta}{d^2t}$$

- The line of action of a is the same as that for ω , however, its sense of direction depends on whether ω is increasing or decreasing.
- If is ω decreasing, then α is called an angular deceleration and therefore has a sense of direction which is opposite to ω.

- By eliminating dt from angular velocity equation and angular acceleration equation, we obtain a differential relation between the angular acceleration, angular velocity, and angular displacement, namely,

$$\alpha d\theta = \omega d\omega$$

- The similarity between the differential relations for angular motion and those developed for rectilinear motion of a particle ($v = ds/dt$, $a = dv/dt$, and $a\,ds = v\,dv$) should be apparent [1]..

Equation for Constant Angular Acceleration

If the angular acceleration of the body is constant, $\alpha = \alpha_c$, then, when integrated the previous equation, yield a set of formulas which relate the body's angular velocity, angular position, and time [1].

angular Velocity as a Function of Time.	• Integrate $\alpha = \frac{d\omega}{dt}$ $\int_{\omega_O}^{\omega} d\omega = \int_0^t \alpha_c dt$	$\omega = \omega_O + \boldsymbol{\alpha_c t}$
Angular Position as a Function of Time.	• Integrate $\alpha = \frac{d\omega}{dt} = \omega_O + \alpha_c t$ $\int_{\theta_O}^{\theta} d\theta = \int_0^t (\ \omega \qquad)dt$	$\theta = \theta_O + \omega_O t + \frac{1}{2}(\boldsymbol{\alpha}_c t^2)$
Angular Velocity as a Function of angular Position .	• Integrate $\alpha d\theta = \omega d\omega$, $\int_{\theta_O}^{\theta} \alpha d\theta = \int_{\omega_O}^{\omega} \omega d\omega$	$\omega^2 = (\omega_O)^2 + 2\boldsymbol{\alpha_c}(\theta - \theta_O)$

- Here θ_0 and ω_0 are the initial values of the body's angular position and angular velocity, respectively.

Motion of Point A

- As the rigid body in rotates, point A and B ravels along a circular path of radius r with center at point O [2].

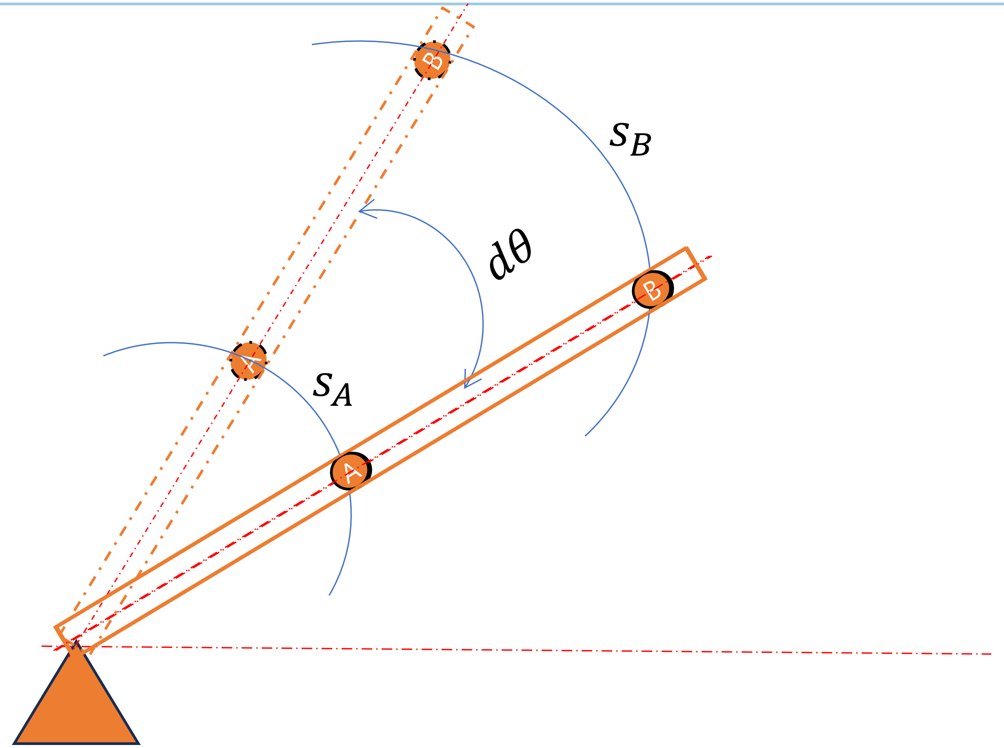

Figure 5. motion of particle or points on body

- This path followed by both points are s_A and s_B
- The distance traveled by point A is less than distance traveled by point B having the same angular change $d\theta$.

Position and Displacement.

- The position of the particles is defined by the position vector r, which extends from O to The particles

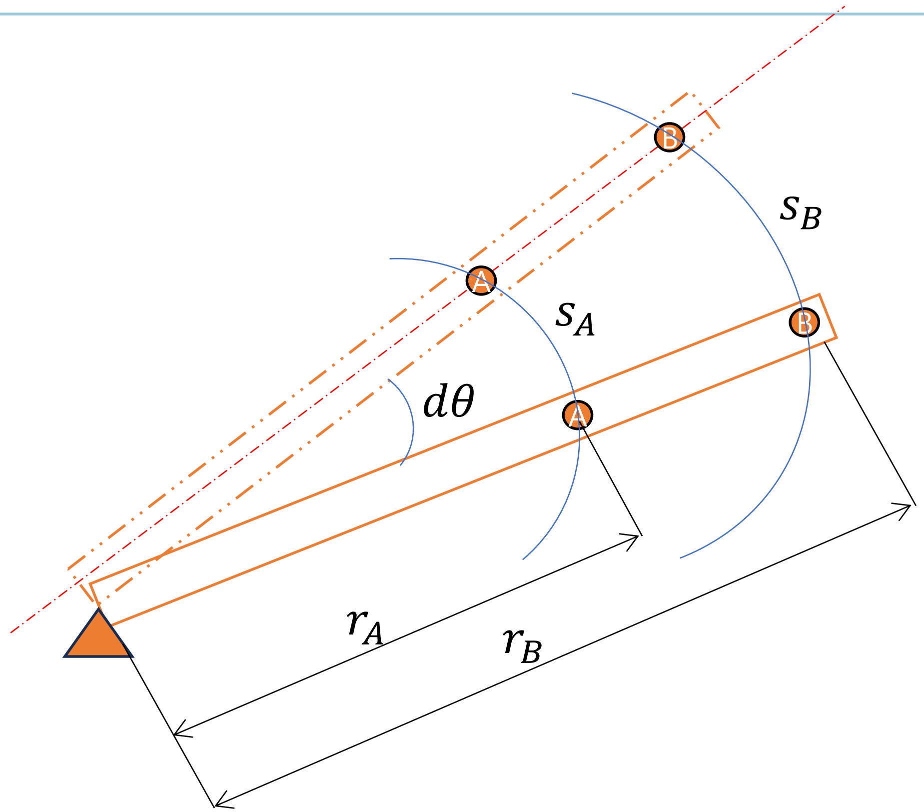

Figure 6. position of particle or points on body

- If the body rotates $d\theta$ then the points will displace ds = $\mathrm{r}d\theta$.

Velocity of points

- As shown in Figure 7, the direction of v is tangent to the circular path.
- The velocity of P has a magnitude which can be found by dividing $rd\theta$ by dt so that:

$$\boldsymbol{v} = \frac{\boldsymbol{ds}}{\boldsymbol{dt}} = \frac{\boldsymbol{d}(r\boldsymbol{d}\theta)}{\boldsymbol{dt}}$$

$$v = \boldsymbol{\omega r}$$

where ω is the angular speed at which the link is roatting

- Both the magnitude and direction of v can also be accounted for by using the cross product of ω and r . Here, We have:

$$v = \boldsymbol{\omega} \times \boldsymbol{r}$$

- For point A and point B it can be re written as:

$$V_A = \boldsymbol{\omega r_A} \qquad V_B = \boldsymbol{\omega r_B}$$

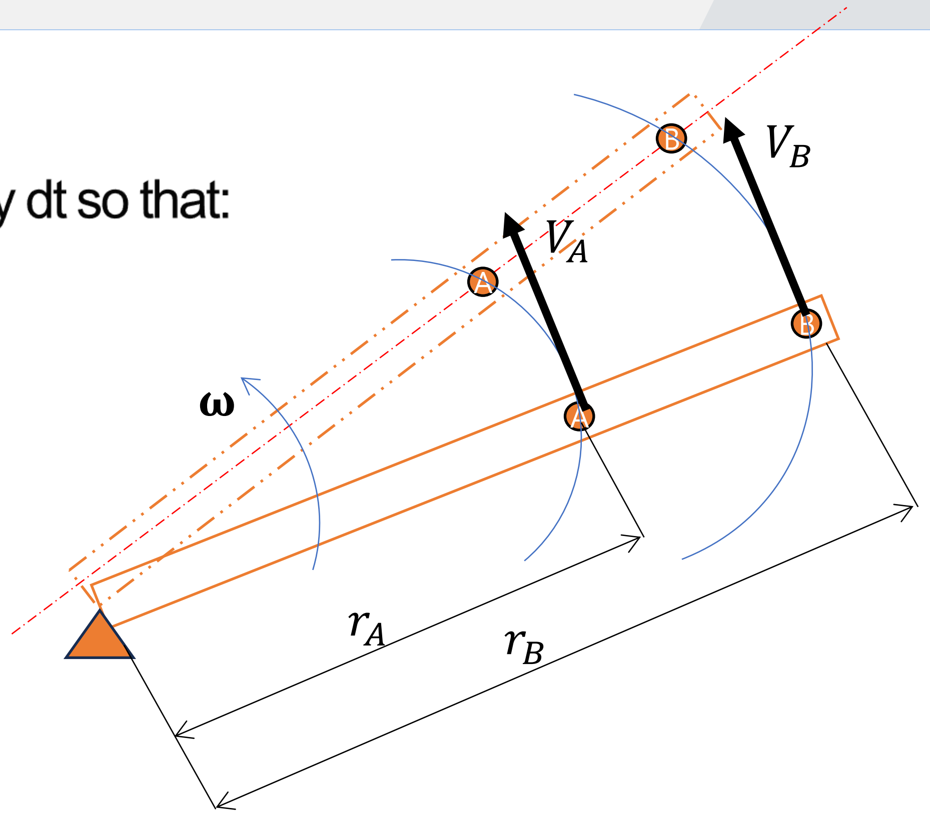

Figure 7. velocity of points

Acceleration of points

- The acceleration of the points can be expressed in terms of its normal and tangential components [3]. :

$$a = a_t + a_n \qquad since \qquad a_t = \frac{\boldsymbol{dv}}{\boldsymbol{dt}} = \boldsymbol{\alpha}\,\boldsymbol{r} \qquad and \qquad a_n = \boldsymbol{v^2/r} = \boldsymbol{\omega^2 r}$$

- where $\rho = r, v = w.r\ and\ \alpha = \frac{d\omega}{dt}$, we have

- Like the velocity, the acceleration of point P can be expressed in terms of the vector cross product.

$$a = \frac{\boldsymbol{dv}}{\boldsymbol{dt}} = \frac{\boldsymbol{d(\omega r)}}{\boldsymbol{dt}} = \frac{\boldsymbol{d\omega}}{\boldsymbol{dt}} \times \mathrm{r} + \frac{\boldsymbol{dr}}{\boldsymbol{dt}} \times \boldsymbol{\omega}$$

- Recalling that $\alpha = \frac{d\omega}{dt}$, and using, $\frac{dr}{dt} = v = \omega \times \omega \times r$

$$a = (\alpha \times r) + (\omega \times \omega \times r)$$

- we can express an in a much simpler form as $a_n = \omega \times \omega \times r = -w^2 r$, hence the above equation can be identified by its two components as:

$$a = a_t + a_n$$

$$a = (\alpha \times r) - (\omega^2 r)$$

- The tangential component of acceleration, represents the time rate of change in the velocity's magnitude. If the speed of the points (A and B) is increasing, then at acts in the same direction as v; if the speed is decreasing, at acts in the opposite direction of v; and finally, if the speed is constant, at is zero [3]..
- The normal component of acceleration represents the time rate of change in the velocity's direction. The direction of an is always toward O, the center of the circular path.

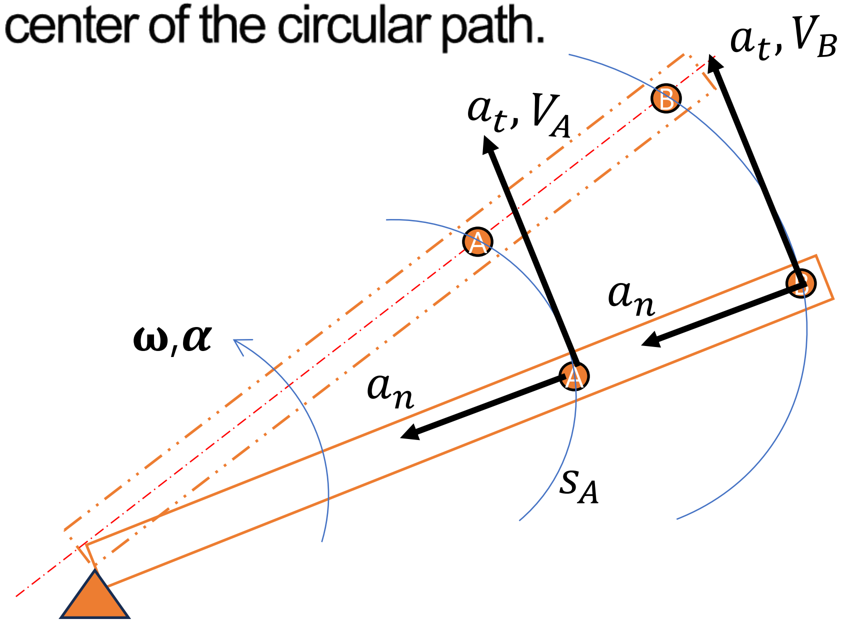

Figure 8. Acceleration of points on body

- Since a_t and a_n are perpendicular to one another, if needed the magnitude of acceleration can be determined from the Pythagorean theorem; namely:

$$a = \sqrt{a_t + a_n}$$

Summary on Rotational motion

- Points located on a body that rotates about a fixed axis follow circular paths
- All the points on a rotating body move with the same angular velocity and acceleration but different linear velocity and acceleration

$$\omega = \frac{d\theta}{dt} \qquad \alpha = \frac{d\omega}{dt} \qquad \alpha d\theta = \omega d\omega$$

- If the body's angular acceleration is constant, then the following equations can be used:

$$\omega = \omega_O + \boldsymbol{\alpha_c} t$$

$$\theta = \theta_O + \omega_O t + \frac{1}{2}(\boldsymbol{\alpha}_c t^2)$$

$$\omega^2 = (\omega_O)^2 + 2\boldsymbol{\alpha_c}(\theta - \theta_O)$$

time dependent

- In most cases the velocity of P and its two components of acceleration can be determined from the scalar equations

$$v = \boldsymbol{\omega r}$$

$$a_n = \boldsymbol{v}^2/\boldsymbol{r} = \boldsymbol{\omega}^2\boldsymbol{r} \qquad a_t = \boldsymbol{\alpha\, r}$$

If the geometry of the problem is difficult to visualize, the following vector equations should be used:

$$v = \boldsymbol{\omega} \times \boldsymbol{r} \qquad a_t = (\alpha \times r) \qquad a_n = \omega \times \omega \times r = -w^2 r$$

General plane motion analysis

Absolute Motion Analysis

- We now develop the approach of absolute-motion analysis to describe the plane kinematics of rigid bodies [1]..
- In this approach, we make use of the geometric relations which define the configuration of the body involved and then proceed to take the time derivatives of the defining geometric relations to obtain velocities and accelerations.
- Here ,The motion can be completely specified by knowing both the angular rotation of a line fixed in the body and the motion of a point on the body.
- One way to relate these motions is to use a rectilinear position coordinate s to locate the point along its path and an angular position coordinate θ to specify the orientation of the line.

The horizontal motion of the block can be rel ated to the angular motion of the rod using;

$$x = \frac{y}{tan\theta}$$

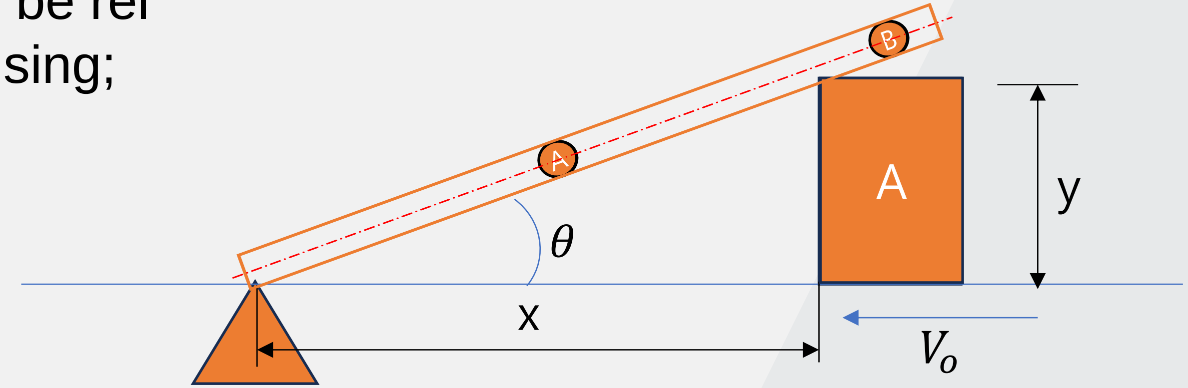

Figure 9. Absolute motion of rod

Procedures

- The velocity and acceleration of a point P undergoing rectilinear motion can be related to the angular velocity and angular acceleration of a line contained within a body [4].

Step 1: From the given and required data, identify the point that undergoes pure translation and determine how the body rotates.

Step 2: Using trigonometric and geometric relationships, relate the translational motion of the chosen point s to the rotation of the body. From the body's dimensions, express the linear displacement **s** as a function of the angular position θ:

$$s = f(\theta)$$

- When forming this relation:
  - ✓ Choose one that involves **no more than two changing va riables** to simplify differentiation.
  - ✓ Ensure that one of these variables has a known value.

Procedures

Step 3: Differentiate $s = f(\theta)$ with respect to time to obtain the relationship between **linear velocity (v)** a nd **angular velocity (ω)**.

Step 4: Differentiate again with respect to time to find the relationship between **linear acceleration (a)** and **angular acceleration (α)**.

- To better understand or see these steps, we will also consider the following example.

Sample problem 1. The block moves to the left with a constant velocity v_o. Determine the angular velocity and angular acceleration of the bar as a function of θ.

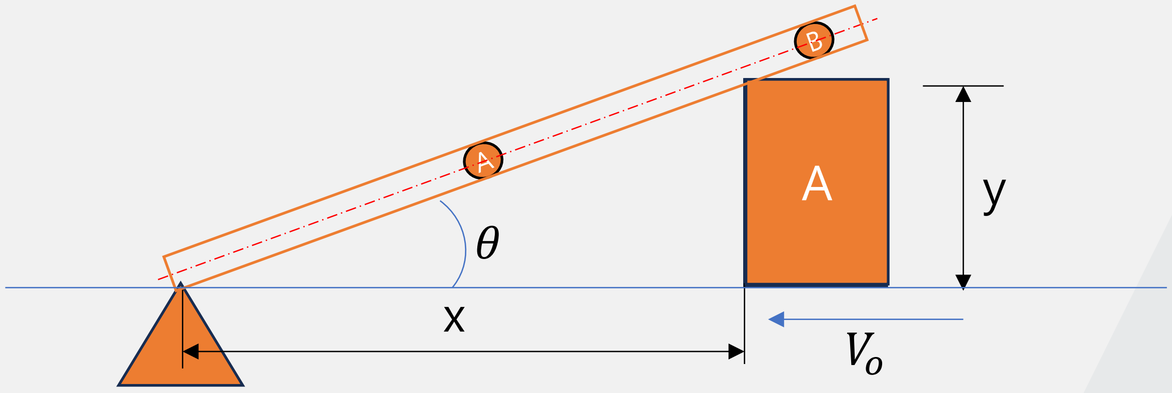

Figure 10. sample problem

Procedures

Step 1 Identify translation and rotation

- The block translates horizontally (left) with known constant speed v_o.
- The bar rotates about the fixed pivot O.
- The vertical coordinate of the contact point is the block height a (constant).

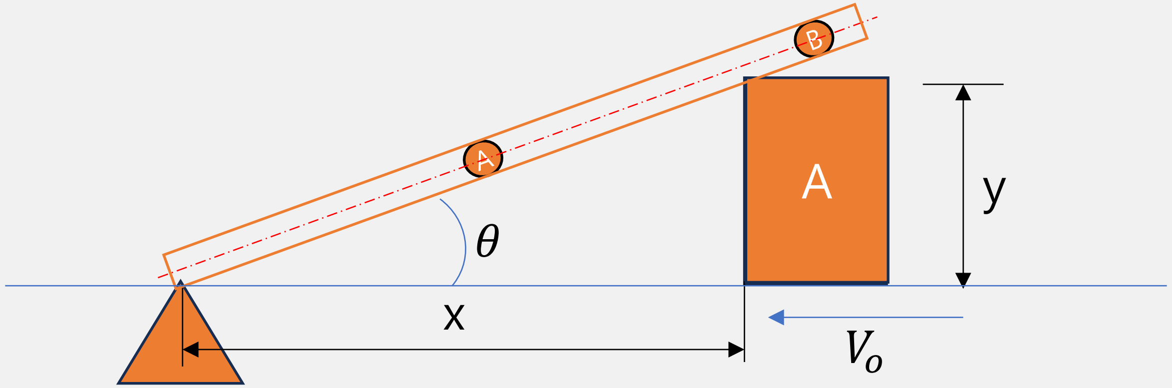

Figure 11. step 1

Procedures

Step 2 Relate the translational displacement to the rotation (geometry / trig)

- Based on the given and required data, we need to relate the block's linear motion x with the rod's angular motion. Several geometric relations are possible, but we'll choose the simplest one for easier analysis.

The possible alternatives are as follows:

$$x = a\cot\theta$$

$$a = x\tan\theta$$

$$\sin\theta = \frac{a}{\sqrt{a^2 + x^2}}$$

$$\cos\theta = \frac{x}{\sqrt{a^2 + x^2}}$$

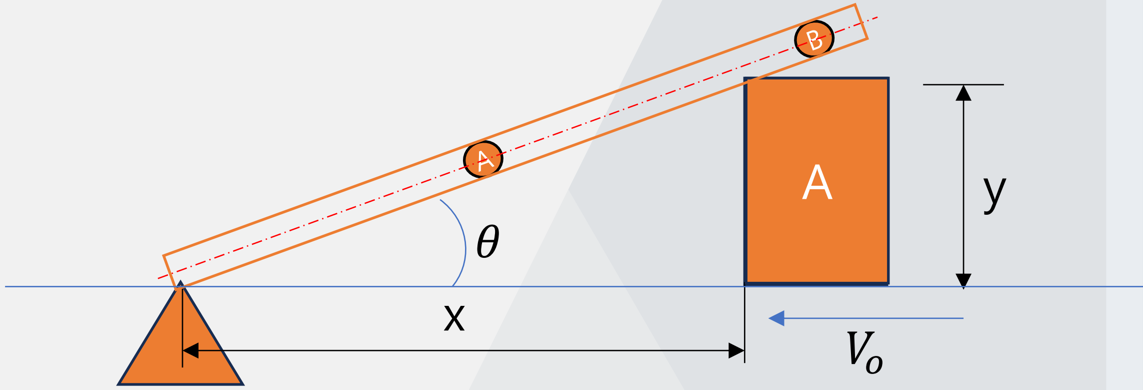

Figure 12. step 2

- Among these, we choose $x = a\cot\theta$ since it directly relates the known horizontal motion x to the angular position θ with only two changing parameters, making differentiation simpler.

Procedures

Step 3 Differentiate once: relation between linear velocity and angular velocity

- using chain rule, Differentiate $x = a\cot\theta$ with respect to time t:

$$\dot{x} = a\frac{d}{dt}(\cot\theta)$$
$$\dot{x} = a(-csc^2\theta)\dot{\theta}$$

- Sign convention: take x positive to the right; the block moves left with speed v_o, so $\dot{x} = v_o$. And we also know $\dot{\theta} = \omega$ Substitute:

$$-v_o = a(-csc^2\theta)\omega$$

- So the angular velocity of the bar as a function of θ is:

$$\omega(\theta) = v_o\frac{sin^2\theta}{a}$$

Procedures

Step 4 — Differentiate again: relation between linear acceleration and angular acceleration

- using chain rule, Differentiate $\omega(\theta) = v_o \frac{sin^2\theta}{a}$ with respect to time t:

$$\dot{\omega} = \frac{v_o}{a}\frac{d}{dt}(sin^2\theta)$$

$$\dot{\omega} = \frac{v_o}{a}(2sin\theta cos\theta\dot{\theta})$$

Where $\dot{\theta} = \omega = v_o \frac{sin^2\theta}{a}$

$(\dot{\omega} = \alpha)\ is\ angular\ acceleration$

- Substituting

$$\alpha = \frac{v_o}{a}(2sin\theta cos\theta)\ v_o \frac{sin^2\theta}{a}$$

- You can rewrite using $\sin 2\theta = 2sin\theta cos\theta$ if preferred:

$$\alpha(\theta) = \frac{{v_o}^2}{a^2}(sin2\theta)\ sin^2\theta$$

Main points

- The absolute-motion approach to rigid-body kinematics is quite straightforward, provided the configuration lends itself to a geometric description which is not overly complex [2]..
- By direct application of the time-differential equations $\mathrm{v} = \frac{ds}{dt}$, $\mathrm{a} = \frac{dv}{dt}$, $\omega = \frac{d\theta}{dt}$, and $\alpha = \frac{d\omega}{dt}$, the motion of the point and the angular motion of the line can then be related.
- This procedure is similar to that used to solve dependent motion problems involving pulleys.
- In this earlier treatment, the geometric relations were quite simple, and no angular quantities had to be considered.
- Here In rigid-body motion, geometric relations involve both linear and angular variables, so their relations of time derivatives include both linear and angular velocities and accelerations..

Summary

Rotation about a Fixed Axis

- For this type of motion, all of the particles move along circular paths
- all line segments in the body undergo the same angular motion.

Variable angular acceleration

$$\omega = \frac{d\theta}{dt} \qquad \alpha = \frac{d\omega}{dt} \qquad \alpha d\theta = \omega d\omega$$

Constant angular acceleration

$$\omega = \omega_O + \boldsymbol{\alpha}_c t \qquad \theta = \theta_O + \omega_O t + \frac{1}{2}(\boldsymbol{\alpha}_c t^2)$$

$$\omega^2 = (\omega_O)^2 + 2\boldsymbol{\alpha}_c(\theta - \theta_O)$$

- Once the angular motion of the body is known, then the Rotation about a fixed axis acceleration velocity of any particle a distance r from the axis can be obtained

Motion of point on the body

$$v = \boldsymbol{\omega r} \qquad a_n = \boldsymbol{v^2/r} = \boldsymbol{\omega^2 r} \qquad a_t = \boldsymbol{\alpha\, r}$$

Absolute motion analysis

- If the motion of a point on a body or the angular motion of a line is known, then it may be possible to relate this motion to that of another point or line using an absolute motion analysis.
- To do so, First, we wrote an equation which describes the general geometric configuration of a given problem in terms of knowns and unknowns. Then we differentiated this equation with respect to time to obtain velocities and accelerations, both linear and angular.

Problem 1

The hook is attached to a cord which is wound around the drum. If it moves from rest with an acceleration of $20\,{}^{ft}/_{s^2}$, determine the angular acceleration of the drum and its angular velocity after the drum has completed 10 rev. How many more revolutions will the drum turn after it has first completed 10 rev and the hook continues to move downward for 4 s ? [1].

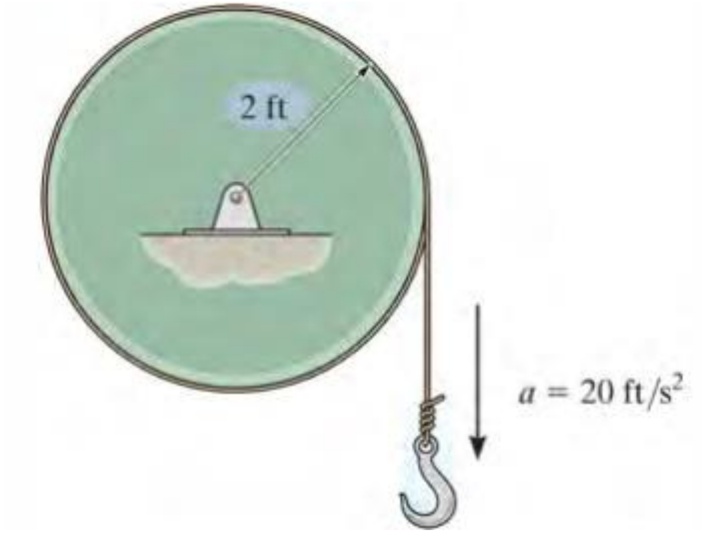

Figure 13. problem 1

Given

- $a_t = 20\,{}^{ft}/_{s^2}$
- N= 10rev
- t= $4s$

Required

- a) $\alpha_{(@10rev)}$
- $b)\ \omega_{(@10rev)}$ =?
- $c)\ N_{(@t=4s)}$ =?

Solution

Angular Motion: The angular acceleration of the drum can be determine by applying:

$$a_t = \alpha r \qquad 20 = \alpha 2 \qquad \alpha = 10\,{}^{rad}/_{s^2}$$

The angular velocity of the drum can be determine by applying:

$$\omega^2 = (\omega_O)^2 + 2\boldsymbol{\alpha_c}(\theta - \theta_O)$$

$$where\ (\theta - \theta_O) = 10\text{rev} \times \frac{2\pi}{1rev} = 2\pi\ and\ \ \omega_O = 0$$

$$\omega = 35.45 rad/s$$

The angular displacement of the drum 4 s after it has complet ed 10 rev can be determined by applying:

$$\theta = \theta_O + \omega_O t + \frac{1}{2}(\boldsymbol{\alpha}_c t^2)$$

$$(\Delta\theta) = \omega_O t + \frac{1}{2}(\boldsymbol{\alpha}_c t^2)$$

$$\Delta\theta = (35.45)4 + 0.5(10)(4^2) = 221.79\ \text{rad}$$

$$= 221.79\ \text{rad} \times \frac{1rev}{2\pi} = 35.3\ rev$$

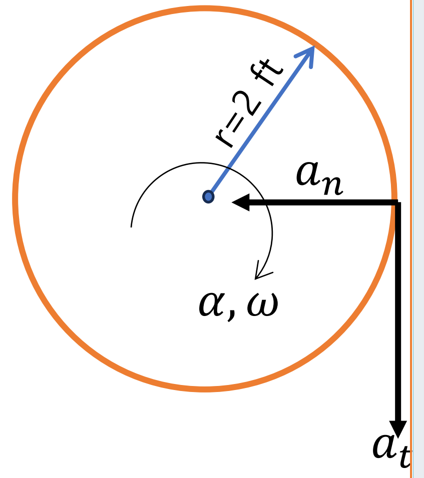

Figure 14. Drum

Problem 2

- The bridge girder G of a bascule bridge is raised and lowered using the drive mechanism shown. If the hydraulic cylinder AB shortens at a constant rate of 0.15m/s, determine the angular velocity of the bridge girder at the instant $\theta = 60^0$ [1].

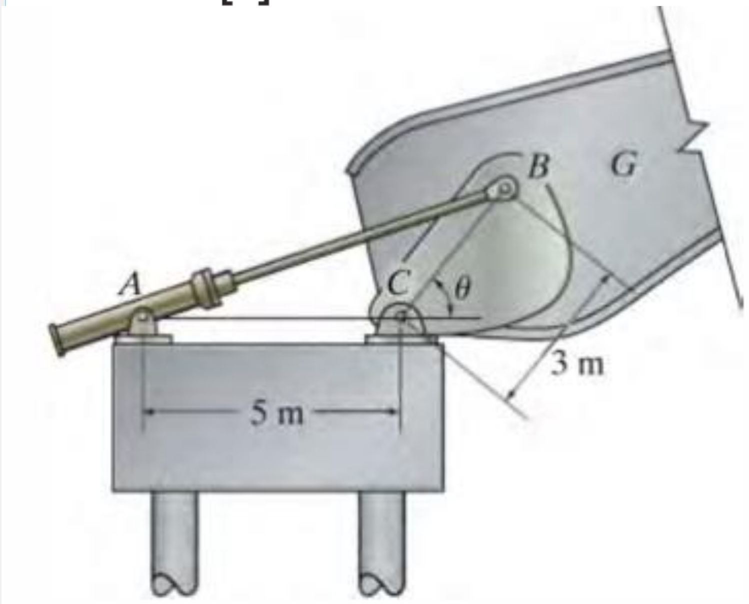

Figure 15. problem 2

Given

- $V_{AB} = -0.15\, {}^{m}/_{s}$
- $\theta = 60^0$

Required

$\omega_{(@\theta=60^0)} = ?$

Solution

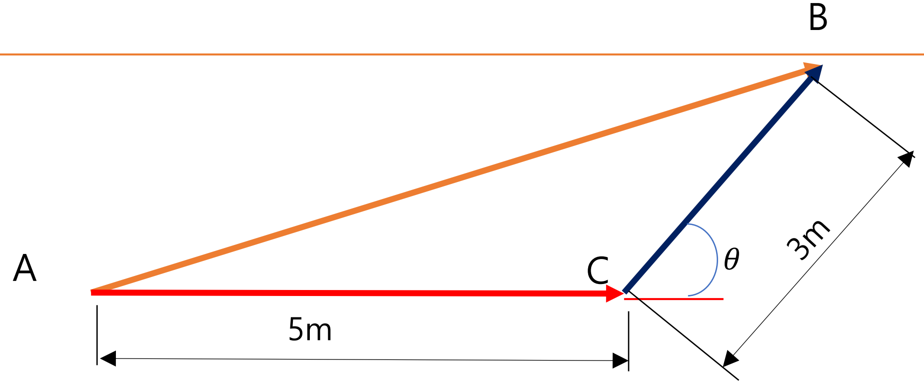

Figure 16. Geometry

Position Coordinates: Applying the law of cosines to the geometry :

$$AB^2 = 3^2 + 5^2 - 2(3)(5)\cos(180 - \theta) \qquad \cos(180-\theta) = -cos\theta$$

$$AB^2 = 34 + 30cos(\theta)$$

Time Derivatives: Taking the time derivative,

$$2(AB)(\dot{AB}) = 30(-\sin\theta)\,\dot{\theta}$$

When : $\theta = (60^0)$, AB=$\sqrt{34 - 30\cos(60)}$=4.35m

$$\dot{AB} = V_{AB} = -0.15\, {}^{m}/_{s}$$

it is direct towards the negative sense of AB

$$\dot{\theta} = \omega = 0.0808 rad/s$$

Activity 1

1.The bucket is hoisted by the rope that wraps around a drum wheel. If the angular displacement of the wheel is $\theta = (0.5t^3 + 15t)$, where t is in seconds, determine the velocity and acceleration of the bucket when t = 3 s [1].

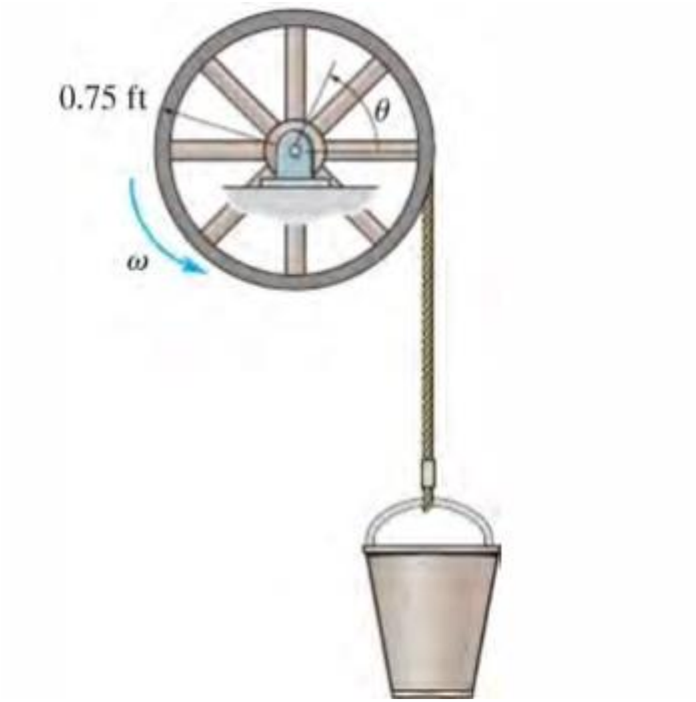

Figure 17. Activity 1

Source: Engineering Mechanics: Dynamics, Hibbeler, Rusel M.,Prentice Hall, 10th ed., 2003,page 338

Activity 2

2. The operation of reverse gear in an automotive transmission is shown. If the engine turns shaft A at $\omega_A = 40 rad/s$, determine the angular velocity of the drive shaft, ω_B The radius of each gear is listed in t he figure [1].

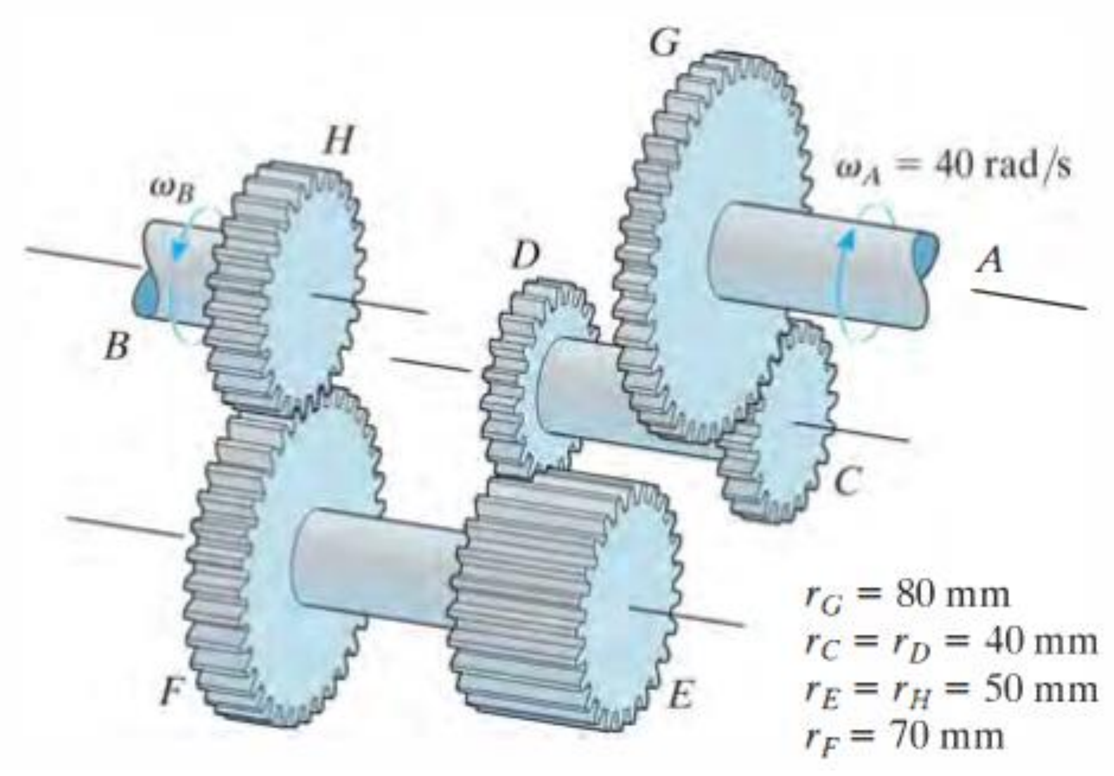

Figure 18. Activity 2

Source: Engineering Mechanics: Dynamics, Hibbeler, Rusel M.,Prentice Hall, 10th ed., 2003 page 339

Activity 3

3. Determine the velocity and acceleration of the plate at the instant $\theta = 30^0$, if at this instant the circular cam is rotating about the fixed point 0 with an angular velocity ω = 4 rad/s and an angular acceleration $\alpha = 2\,{}^{rad}/_{s^2}$ [1].

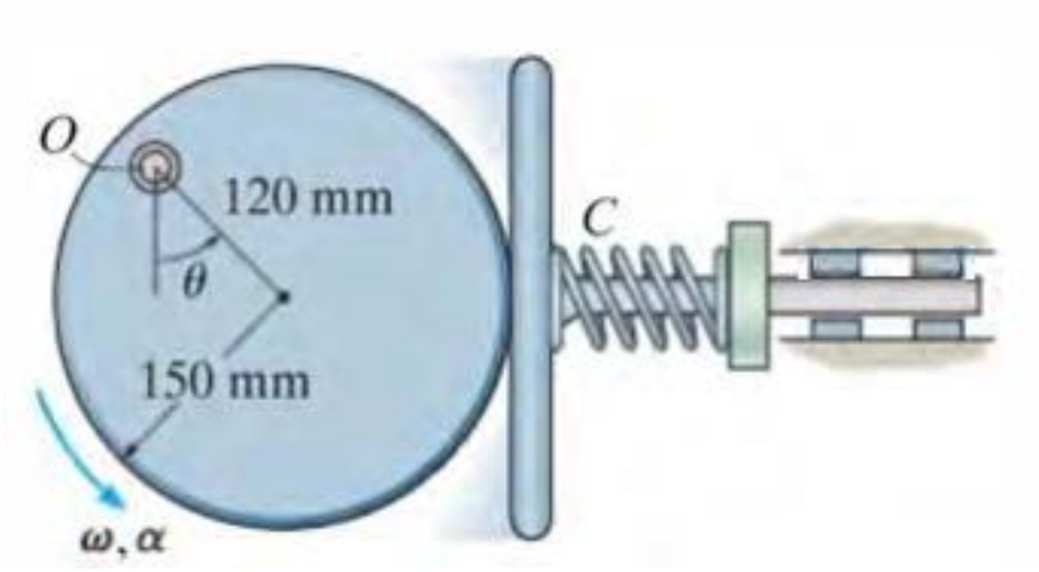

Figure 19. Activity 3

Source: Engineering Mechanics: Dynamics, Hibbeler, Rusel M.,Prentice Hall, 10th ed., 2003,page 351

Activity 4

3. For the instant represented when y=160mm the piston rod of the hydraulic cylinder C imparts a vertical motion to the pin B consisting of $\dot{y}$ = 400mm/s and $\ddot{y} = -100mm/s$. For this instant determine the angular velocity and the angular acceleration of link OA. Members OA and AB make equal angles with the horizontal at this instant [1].

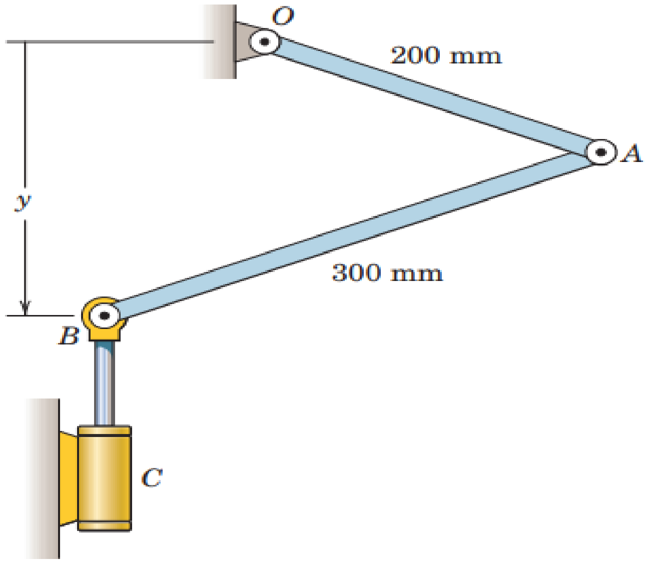

Figure 20. Activity 4

Source: Engineering Mechanics - Dynamics, Meriam J.L.,John Wiley & Sons, 9th ed., 2020, page 345

Summary

In This Lecture We Covered:

1. Introduction to rotation about fixed axis
2. Angular motion → constant and variable angular acceleration
3. Motion of points on rotating body
4. General plane motion → Absolute motion analysis
5. Problems solving on rotation and general plane motion

References

[1] Dynamics, Hibbeler, Rusel M.,Prentice Hall,10th ed., 2003

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