Week 12

Kinetics of Rigid Bodies – General Equations

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Contents

By the end of this lecture, you are able to:

1. Define and explain Kinetics of Rigid Bodies
2. Define and explain – General Equations of motion
3. Define and explain Kinetics of translational motion of Rigid Bodies
4. Define Equations of motion for rotational motion
5. Define Equations of motion for General plane motion motion

Understand the kinetics of rigid bodies

- The kinetics of rigid bodies deals with the relationships between external forces and moments acting on a body and the translational and rotational motions produced by these forces[1].
- While the kinematics of rigid bodies (studied earlier) was only concerned with describing motion ,such as velocity, acceleration, and angular velocity — kinetics focuses on why that motion occurs.[1].
- Previously, in the kinetics of particles, we treated the entire mass as concentrated at a single point. Here, however, the shape, size, and mass distribution of the body become important.
- For a rigid body:
  - Forces may act at different points
  - The body can both translate and rotate.
  - The mass moment of inertia plays a key role in rotational motion.

- Therefore, in rigid-body kinetics, we require three equations of motion:
  - Two force equations (for translation in x and y directions), and
  - One moment equation (for rotation about a point or the mass center).
- Kinetics of rigid bodies is analyzed using three major methods:
  - Force and Acceleration Method
  - Work and Energy Method
  - Impulse and Momentum Method
- Each method is applied to three types of motion:
  1. Pure Translation,
  2. Rotation about a Fixed Axis, and
  3. General Plane Motion (a combination of both).

General Equations of Motion in the System

- The fundamental principle underlying all kinetic analysis is Newton's Second Law of Motion, which states that:

$$\sum F = m\vec{a}$$

- For a rigid body, this must be extended to account for both the translational motion of the body's mass center and the rotational motion about that mass center.
- This leads to two coupled vector equations of motion:

$$\sum F = m\overrightarrow{a_G}$$

$$\sum M_G = I\alpha_G$$

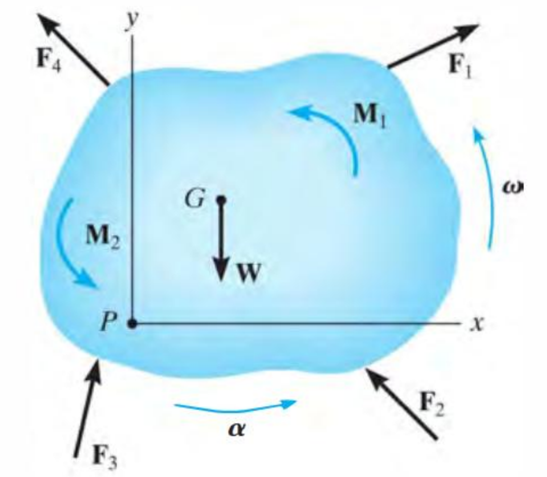


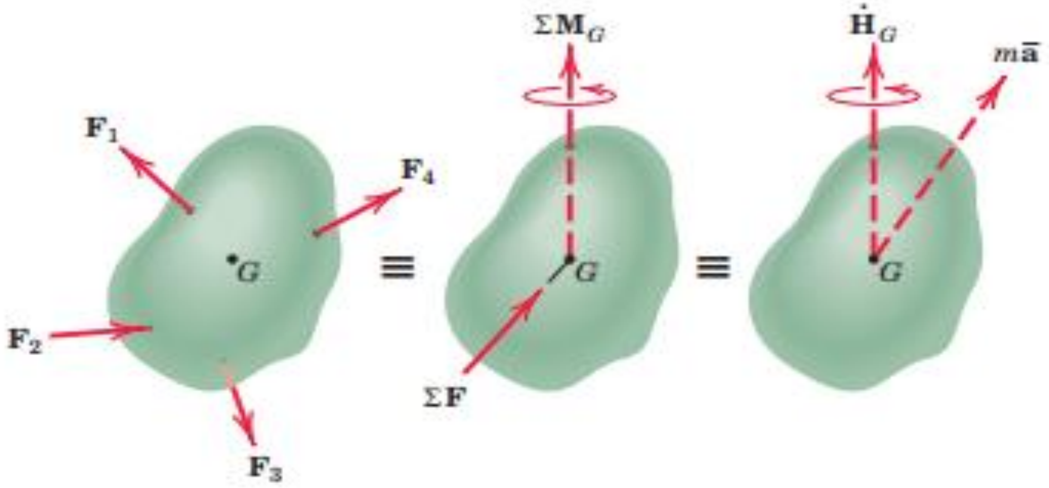
Figure 1. Equation of motion

where:

- m = total mass of the body
- a_G = linear acceleration of the mass center GGG
- $\sum M_G$ = sum of moments of external forces about the mass center
- I = mass moment of inertia about an axis through GGG
- α_G = angular acceleration of the body about that axis

Physical Meaning

- $\sum F = m\overrightarrow{a_G}$,is referred to as the translational equation of motion for the mass center of a rigid body. It states that the sum of all the external forces acting on the body is equal to the body's mass times the acceleration of its mass center G.
- It is a direct extension of the particle equation. It governs the motion of the body's center of mass as if all the mass were concentrated there.



a. FBD b. Force couple c. KD

Figure 2. Moment and force

- For motion of the body in the x-y plane, the translational equation of motion may be written in the form of two independent scalar equations, namely.

$$\sum F_x = ma_{(G)_x} \qquad \sum F_y = ma_{(G)_y}$$

- $\sum M_G = I\alpha_G$, is rotational equation of motion and states that the sum of the moments of all the external forces about the body's mass center G is equal to the product of the moment of inertia of the body about an axis passing through G and the body's angular acceleration

Mass Moment of Inertia (I)

- To describe rotation, we must introduce a new property the mass moment of inertia (I) — which measures a body’s resistance to angular acceleration about an axis. For example, the body's moment of inertia **about the z axis** in In integral for it can be expressed as:

$$I = \int r^2 dm$$

  - where r is the perpendicular distance from the axis of rotation to the differential element of mass dm.

- Analogous to mass in translational motion, the moment of inertia depends not only on total mass but also on how that mass is distributed relative to the axis.

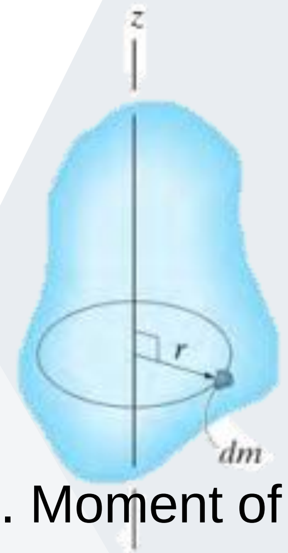


Figure 3. Moment of inertia

Parallel-axis theorem

- For a given body , A larger I means more resistance to angular acceleration. Units: $kg \cdot m^2$ (SI) or $slug \cdot ft^2$ (US Customary)
- If the moment of inertia of the body about an axis passing through the body's mass center is known, then the moment of inertia about any other parallel axis (let say point A) can be determined by using:

$$I = I_G + md^2$$

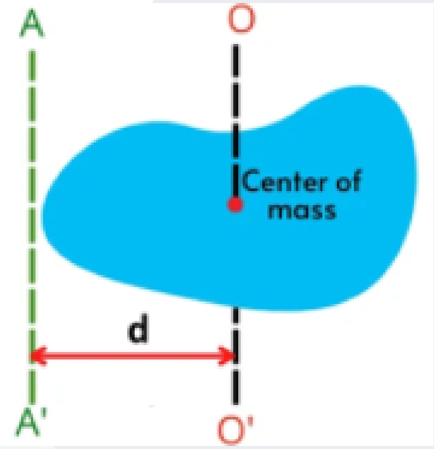


Figure 4. Parallel-axis

- Where:

I_G, is moment of inertia of the body trough axis passing through O-O′

m, mass of the body

d perpendicular distance between A-A’, and O-O′

- If the radius of gyration is known , moment of inertia can also obtained by:

$$I = mk^2 \qquad where\ k\ is\ \text{Radius of Gyration:}$$

Moment of Inertia of Composite Bodies

- Real objects are often made of multiple simple shapes: plates, rods, disks, spheres, etc.
- The total moment of inertia (I) is found by adding or subtracting the inertia of each component.
- Algebraic addition is used:
  - ✓Positive for solid parts
  - ✓Negative for holes or removed material
- Use integration for complex shapes or tables for standard forms (rods, disks, spheres).

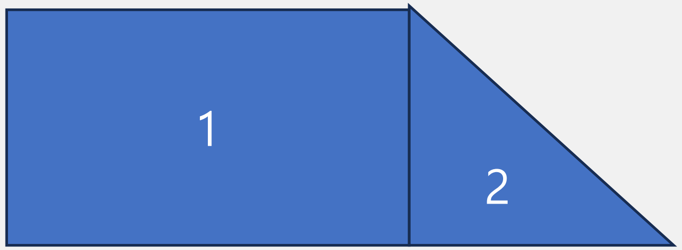


Figure 5. Solid parts

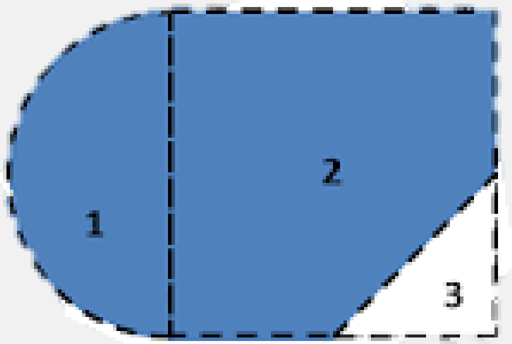


Figure 6. Removed material

General Application of the Equations of Motion

- The Rigid body's motion can be viewed as two simultaneous phenomena:

    1. **Translation of the mass center G**, as if the entire mass were concentrated at G.
    2. **Rotation about G**, caused by the moments of external forces.

Hence, the general motion of a rigid body can be represented as:

Motion = Translation of G + Rotation about G

- This decomposition forms the foundation for solving all planar kinetic problems.

- To summarize this analysis, three independent scalar equations can be written to describe the general plane motion of a symmetrical rigid body.

$$\sum F_x = ma_{(G)_x} \qquad \sum F_y = ma_{(G)_y} \qquad \sum M_G = I\alpha_G$$

- When applying these equations, one should always draw FBD in order to account for the terms involved in $\sum F_x$, $\sum F_y$, and $\sum m_G$, kinetic diagram for the body, accounts for the terms $ma_{(G)_x}$, $ma_{(G)_y}$ $and\ I\alpha_G$

Equations of Motion: Translation

- In translation, every line in the body remains parallel to its original position during motion.
- All particles of the body have identical linear velocities and accelerations.
- No rotation occurs about any axis , hence, angular velocity (ω) = 0 and angular acceleration (α) = 0.

- Translation can be of two types:

✓Rectilinear translation – all points move along straight, parallel paths.

✓Curvilinear translation – all points move along identical curved paths.

- Since angular acceleration (α), the moment equation eliminates rotational inertia. Then the complete set of equation for any translating body is:

Rectilinear Translation

$$\sum M_G = I\alpha_G = 0$$

$$\sum F_x = ma_{(G)_x} \quad or \quad \sum F_y = ma_{(G)_y}$$

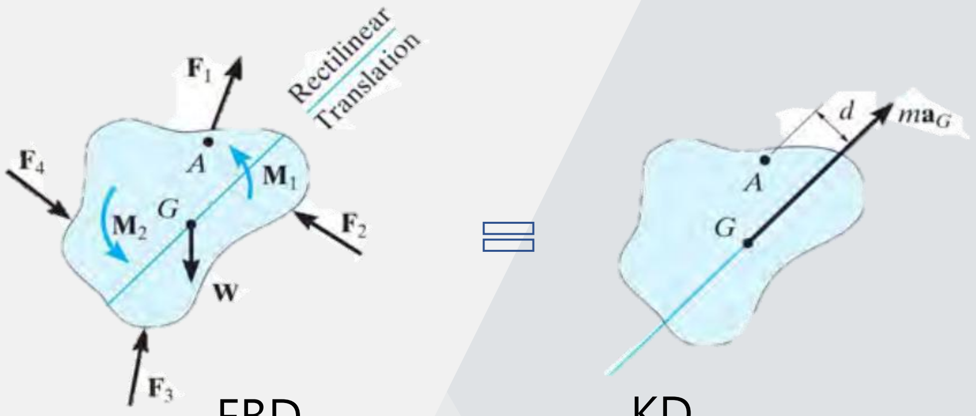


Figure 7. Rectilinear Translation

- It is also possible to sum moments about other points on or off the body, in which case the moment of ma_G must be taken into account. For example, if point A is chosen, which lies at a perpendicular distance d from the line of action of ma_G , the following moment equation applies:

$$\sum M_A - (ma_G)d = 0 \quad I.e \sum M_A = (ma_G)d$$

Here the sum of moments of the external forces and couple moments about A equals the moment of ma_G about A

Curvilinear Translation

$$\sum M_G = I\alpha_G = 0$$

$$\sum F_n = ma_{(G)_n} \qquad \sum F_t = ma_{(G)_t}$$

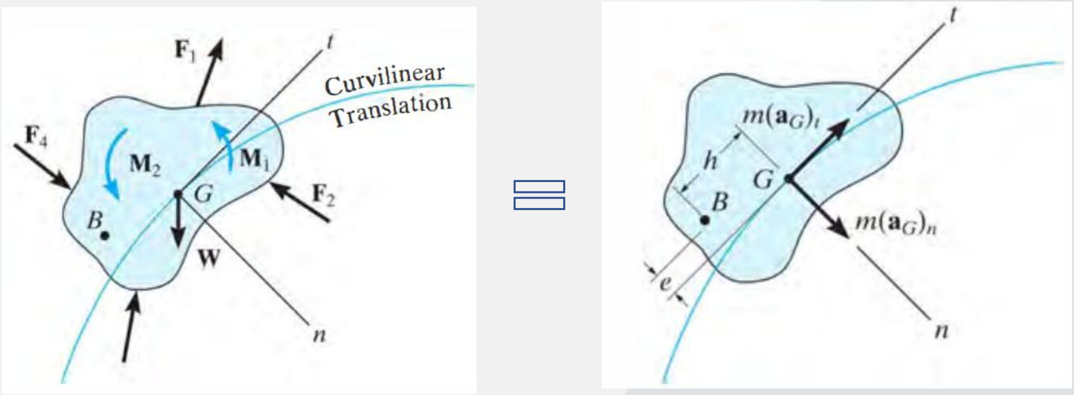


Figure 8. curvilinear Translation

If moments are summed about the arbitrary point B, Fig. 6, then it is necessary to account for the moments , of the two components $ma_{(G)_n}$ and $ma_{(G)_t}$ about this point.

From the kinetic diagram, h and e represent the perpendicular distances (or "moment arms") from B to the lines of action of the components. The required moment equation therefore becomes:

$$\sum M_B = e(ma_{(G)_t}) - h(ma_{(G)_n})$$

Equations of Motion
Rotation about a Fixed Axis

- For this motion, we saw that all points in the body describe circles about the rotation axis, and all lines of the body in the plane of motion have the same angular velocity and angular acceleration .
- Only angular quantities (θ, ω, α) are required to describe motion.
- The acceleration components of the mass center for circular motion are most easily expressed in n-t coordinates
- Therefore, forces acting on the body have tangential and normal (radial) components relative to the path of G.
- Consider the rigid body (or slab) shown in Fig.7. The center of mass (G) moves in a circle, and the acceleration of this point is best represented by its tangential and normal components.

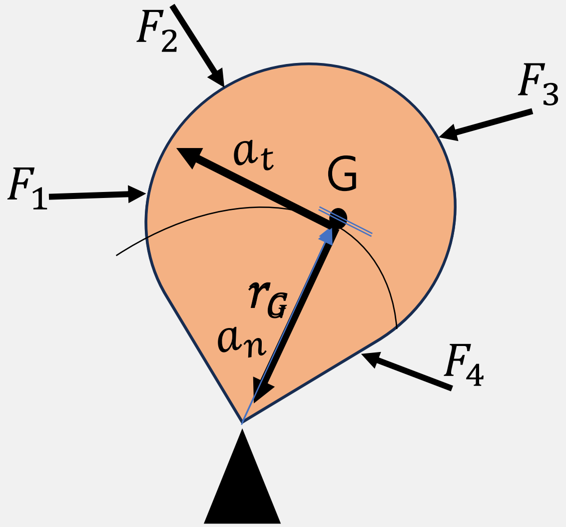


Figure 9. Rotation

- The center of mass (G) moves in a circle (if the axis doesn't pass through it).

Translational Equations for the Center of Mass of Motion

- Even though the body is rotating, its mass center experiences translational acceleration. Thus, Newton's second law still applies to G in normal–tangential components:.

$$\sum F_t = ma_{(G)_t} = m(r_G\ \alpha)$$

$$\sum F_n = ma_{(G)_n} = m(\frac{{V_G}^2}{r_G}) = \omega^2\ r_G$$

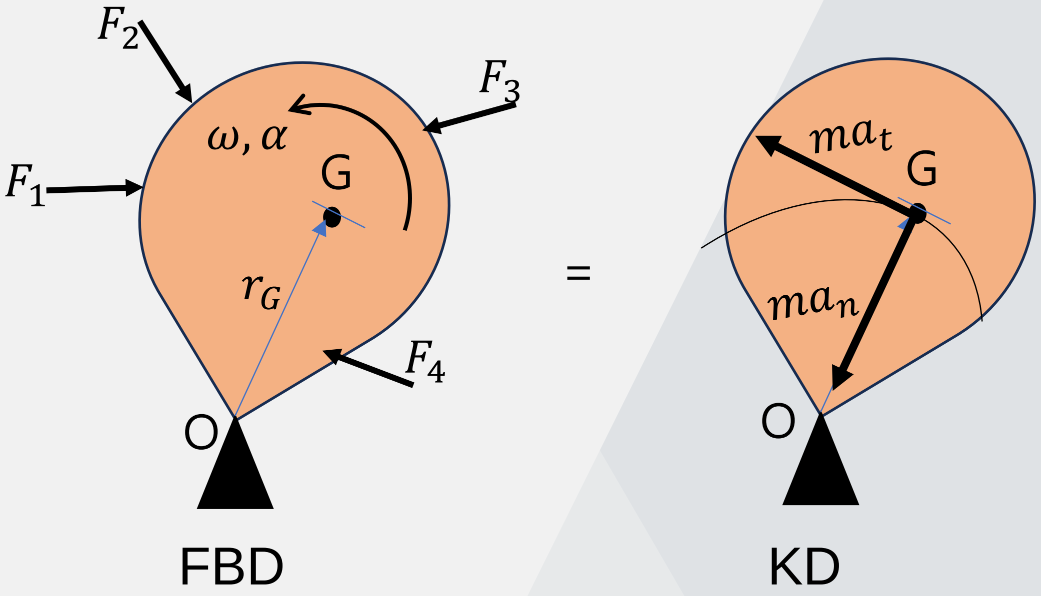


Figure 10. Rotation force representation

Where:

r_G = distance from the axis of rotation to G

$a_{(G)_t} = r_G\ \alpha$= tangential acceleration of G and act in a direction which is consistent with the body's angular acceleration α .

$a_{(G)_n}$=$\omega^2\ r_G$= normal (centripetal) acceleration of G and always directed from point G to O

Rotational (Moment) Equation About the Axis

- To account for rotational effects, take moments of all forces about the fixed axis (O):

$$\sum M_o = I_O\alpha$$

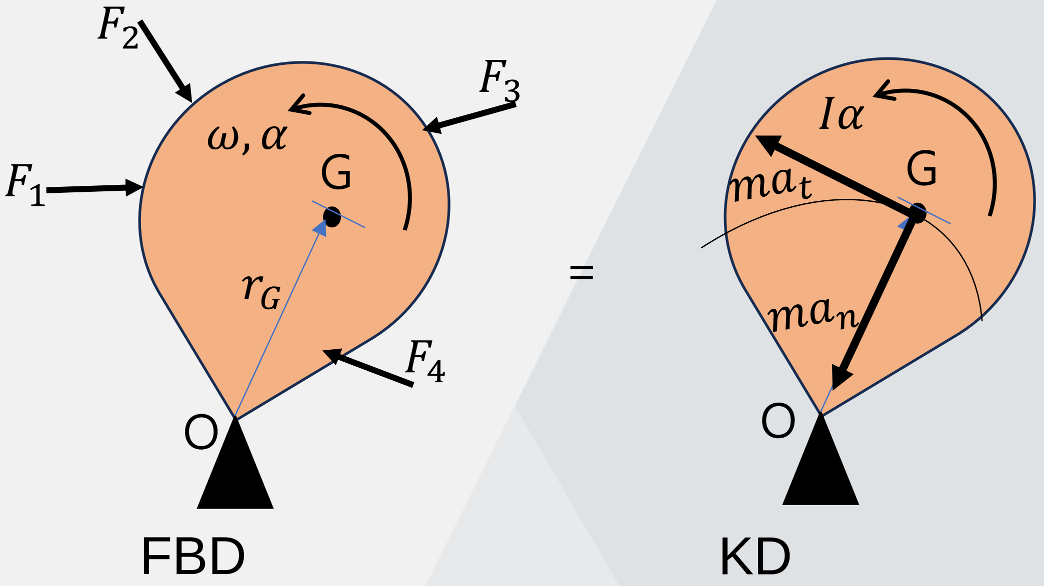


Figure 11. Rotation moment representation

Where:
I_O : mass moment of inertia about the fixed axis and From the parallel-axis theorem, $I_O = I_G + md^2$
α: angular acceleration

- This equation is the rotational counterpart to $\sum F_t = ma_{(G)_t}$.

Rotational (Moment) Equation About the Axis

- The equations of motion for rotational motion of rigid bodies which apply to the body can be written in the form

$$\sum F_t = ma_{(G)_t} = m(r_G\ \alpha)$$

$$\sum F_n = ma_{(G)_n} = m(\frac{{V_G}^2}{r_G}) = m\omega^2\ r_G$$

$$\sum M_o = I_O\alpha$$

- The tangential forces are responsible for changing the rotational speed (produce angular acceleration) and relates torque-producing forces to angular acceleration
- The normal forces act toward the center of rotation, maintaining the circular path of motion and Provides centripetal (inward) force for circular motion
- The moment equation ensures that the net torque equals the product of rotational inertia and angular acceleration and Relates net torque to angular acceleration about O.

Important point

- Rotation about a fixed axis involves both linear and angular motion of the mass center.
- Complete dynamic description requires, Tangential force equation, Normal force equation and Moment equation.
- In many problems, only $\sum M_o = I_O \alpha$ is used when

✓The axis is fixed and reactions at bearings are not required, or

✓We only need to find α or torque.

- But if reactions are unknown, we must also apply:

$$\sum F_t = ma_{(G)_t} = m(r_G\ \alpha)$$

$$\sum F_n = ma_{(G)_n} = m(\frac{{V_G}^2}{r_G}) = m\ \omega^2\ r_G$$

- The moment equation gives angular acceleration, while the force equations give support reactions or constraints.

Procedure for Analysis

Step 1: Free-Body Diagram (FBD)

- Establish the inertial x–y coordinate system.
- Specify the directions and senses of Tangential acceleration ,normal acceleration and angular acceleration α.
- Draw all external forces and moments acting on the body.

Step 2: Identify Key Quantities

- Determine or compute the moment of inertia,
  - ✓ If moments are summed about the body's mass center, G, then $\sum M_G = I_G\alpha$, since $ma_{(G)_t}$ and $ma_{(G)_n}$ create no moment about G.
  - ✓ If moments are summed about the pin support o on the axis of rotation O , then , $ma_{(G)_n}$ create no moment about O, then $\sum M_O = I_O\alpha \ and \ I_O = I_G + md^2$.
- If needed, sketch the kinetic diagram to visualize inertia effects.
- Identify all unknowns (e.g., reaction forces, α, applied moment).

Procedure for Analysis

Step 3: Apply the Equations of Motion

- Use the three rotational motion equations of motion and combine the equations for a complete solution.
- Use kinematics to relate angular variables if acceleration is unknown or variable.

$$\omega = \omega_O + \boldsymbol{\alpha_c t}$$

$$\theta = \theta_O + \omega_O t + \frac{1}{2}(\boldsymbol{\alpha}_c t^2)$$

$$\omega^2 = (\omega_O)^2 + 2\boldsymbol{\alpha_c}(\theta - \theta_O)$$

$$\alpha = \frac{d\omega}{dt}$$

$$\omega = \frac{d\theta}{dt}$$

Step 4: Solve and Interpret

- Solve for unknowns (α, reaction forces, or applied moment).
- If ΣM and ΣF equations yield negative α → actual rotation opposite to assumption.

Example

At the instant shown in Fig. 12 ,the 20-kg slender rod has an angular velocity of ω = 5 rad/s. Determine the angular acceleration and the horizontal and vertical components of reaction of the pin on the rod at this instant.

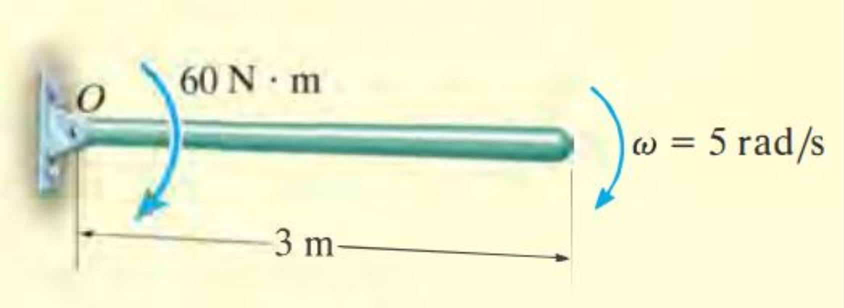
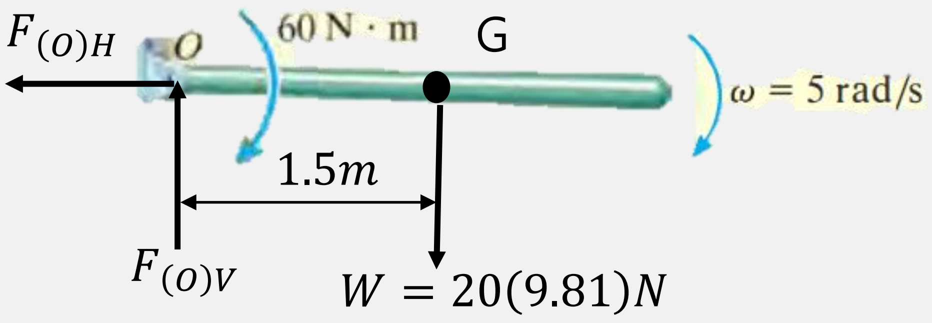


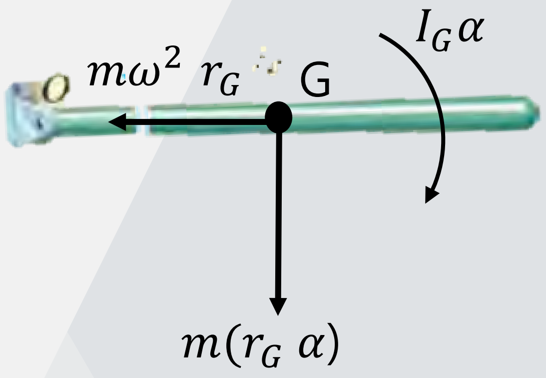
Figure 12. Example 1

SOLUTION



FBD

=



KD

Givens

M=60 N.m

ω = 5 rad/s

M= 20-kg

L=3m

Required

α = ?

$F_{(O)H}$ =?

$F_{(O)V}$ =?

$$\sum F_n = ma_{(G)_n} = m\omega^2\, r_G$$

$$F_{(O)H} = 20\,(5^2)1.5 = 750N$$

$$\sum M_G = I_G\alpha$$

$$F_{(O)V}(1.5) + 60 = [\tfrac{1}{2}(20)(3)^2] \qquad F_{(O)V} = 19.5N$$

$$\sum F_t = ma_{(G)_t} = m(r_G\, \alpha)$$

$$-F_{(O)V} + (20(9.81)) = 20(1.5)\ \alpha \qquad \alpha = 5.9\, rad/s^2$$

Equations of Motion
General Plane Motion

- A rigid body in general plane motion (GPM) experiences simultaneous translation and rotation in a single plane
- All points move in parallel planes; thus, the body's motion can be described within one plane (x–y).
- This is the most general planar case, combining:

    1. Translation of the center of mass (G)
    2. Rotation about the center of mass

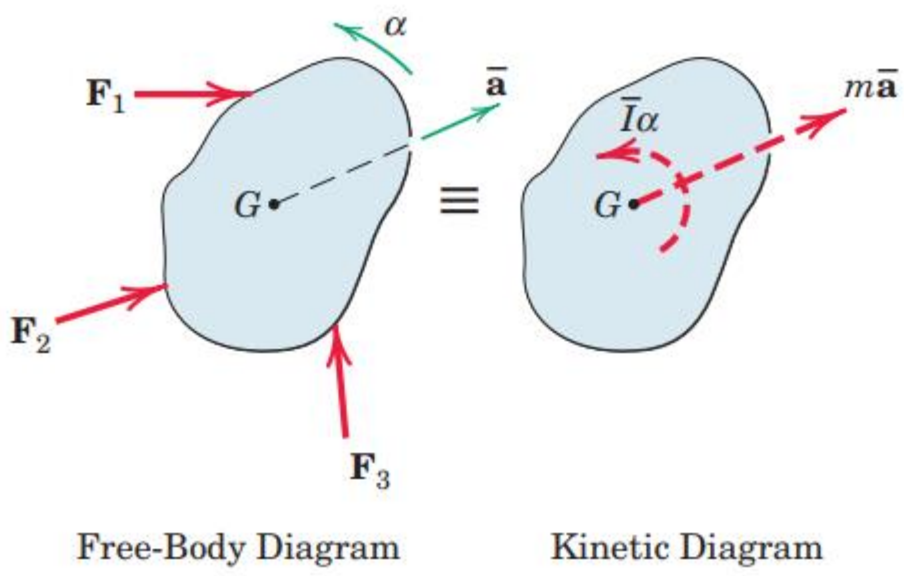


Figure 13. General plane motion

Equations of Motion For general plane motion

- To describe the motion dynamically (forces and acceleration), we apply Newton's second law for translation and rotation.

$$\sum F_x = ma_{(G)x}$$

$$\sum F_y = ma_y$$

$$\sum M_G = I_G\alpha$$

- Where

$\sum F_x$, and $\sum F_y$, cause linear acceleration of G.

$\sum M_G$ = causes angular acceleration about G.

I_G = mass moment of inertia about the center of mass.

α = angular acceleration.

Together, these three equations completely describe any rigid body moving in a plane.

Moment Equation About Any Point (Not G)

- Sometimes it is easier to take moments about another point (for example, a hinge, a support, or a contact point) [3].

$$\sum M_p = I_G\alpha + m(r_{G/p} \times a_G)$$

- The extra term $m(r_{G/p} \times a_G)$ accounts for the translational acceleration of the center of mass.
- If P is fixed or instantaneously at rest, this term becomes zero, giving:

$$\sum M_p = I_p\alpha$$

- where

$I_p = I_G + md^2$ (parallel-axis theorem).

Kinematics of General Plane Motion

- For any point P on the body:

$$v_{(p)} = v_{(G)} + \omega \times r_{p/G}$$

$$a_{(p)} = a_{(G)} + \alpha \times r_{p/G} + \omega \times (\omega \times r_{p/G})$$

- The velocity of any point = velocity of G + rotational velocity about G.
- The acceleration includes:

  Translational part: $a_{(G)}$
  Tangential part: $\alpha \times r_{p/G}$
  Normal (centripetal) part: $\omega \times (\omega \times r_{p/G})$

Problem-Solving Steps for General Plane Motion

- **Step 1** : Free-Body Diagram (FBD) and kinetic diagram
    - Establish the coordinate system (x–y, tangential–normal, etc.)
    - Shows all external forces and applied moments.
    - Shows **inertia effects:**

    ✓Inertia force maG (opposite to motion of G)

    ✓Inertia moment I_G α (opposite to rotation)

- **Step 2** : Apply the Equations of Motion

$$\sum F_x = ma_{(G)_x} \qquad \sum F_y = ma_y \qquad \sum M_G = I_G\alpha$$

- These are the three independent scalar equations.
- If moments are taken about another point P:

$$\sum M_p = I_G\alpha + m(r_{G/p} \times a_G)$$

Problem-Solving Steps for General Plane Motion

- **Step 3** : Kinematic Relationships
  - Use kinematics if the solution cannot be obtained directly from the equations of motion.
  - Write relationships between accelerations of key points.

$$a_{(p)} = a_{(G)} + \alpha \times r_{p/G} + \omega \times (\omega \times r_{p/G})$$

- **Step 4** : Solve and Interpret Results
  - Solve the system of equations for unknowns:
    - Forces (F, N)
    - Accelerations ($a_{(G)}$, α)
    - Moments or torques

Summary

- The Equations of Motion relate the external forces and moments acting on a rigid body to its linear and angular accelerations.
- They form the foundation of rigid-body dynamics for both simple and complex motions.
- Used to analyze: Translation, Rotation about a fixed axis, General plane motion

✓ Before applying the equations of motion, always draw a free-body diagram in order to identif y all the forces acting on the body

✓ A kinetic diagram may also be drawn in order to represent ma_G and $I_G\alpha$ graphically.

Summary

- Two Fundamental Equations of Motion are used and applied differently depending on the motion of the rigid body

Pure Translation:

$$\sum M_G = I\alpha_G = 0$$

Rectilinear

$$\sum F_x = ma_{(G)_x}$$

Curvilinear

$$\sum F_t = ma_{(G)_t} = m(r_G\ \alpha)$$

$$\sum F_n = ma_{(G)_n} = m(\frac{{V_G}^2}{r_G}) = \omega^2\ r_G$$

Rotation about Fixed Axis:

$$\sum M_o = I_O\alpha$$

$$\sum F_n = ma_{(G)_n}$$

$$\sum F_t = ma_{(G)_t}$$

General Plane Motion: Combination of translation (of G) + rotation (about G).

Summary

In This Lecture We Covered:

1. Introduction to Kinetics of Rigid body→ Scope and definition of kinetics
2. Definition Equation of Motion of Rigid bodies
3. Equation of Motion for Translational motion
4. Equation of Motion for Rotational motion
5. Equation of Motion for General plane motion

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