Week 13

Work–Energy Method for Rigid Bodies

Lecturer : Daniel Hambissa Datti

Contents

By the end of this lecture, you are able to:

1. Understand the work energy principle for Rigid Bodies
2. Understand the various ways a force and couple do work.
3. Define and explain Kinetic energy
4. Explain work-energy equation for Rigid Bodies
5. Understand conservation of energy for for Rigid Bodies

Understand the work energy principle for Rigid Bodies

- In the study of kinetics of particles, we learned that the work done by external forces acting on a particle equals the change in its kinetic energy [1].

$$T_1 + \sum U_{1 \to 2} = T_2$$

- This principle provided a powerful way to determine velocities or displacements without directly computing acceleration and time.

- A rigid body consists of a large number of particles whose distances remain fixed relative to each other. When the body moves, each particle moves such that the body may Translate, Rotate about a fixed axis, or Undergo general plane motion [1].

scope Cont'd…

- To extend the particle work–energy principle, we must include both the translational motion of the body's mass center and the rotational motion about that center.
- When we extend this concept to rigid bodies, we must also include rotation, Thus, the total kinetic energy of a rigid body has two parts:
    - Translational kinetic energy of its mass center
    - Rotational kinetic energy about its mass center
- But first it will be necessary to develop a means of obtaining the body's work and kinetic energy when the body is subjected to translation, rotation about a fixed axis, or general plane motion

Definition of Work for Rigid Bodies

- When a force F acts on a point of a body and the point moves through a small displacement dr, the differential work is [2]:

$$du = F.dr = F\cos\theta \, ds$$

- where α is the angle between the force direction and displacement.
- If the force varies with position, integrate:

$$U_{1\rightarrow 2} = \int_1^2 ((F\cos\theta))ds$$

- Several types of forces are often encountered in planar kinetics problems involving a rigid body.
- The work of each of these forces has been presented in next slides and listed as a summary.

Common Types of Work

Work of a constant Applied Force

- If an external force F_c acts on a body, Fig. 1, and maintains a constant magnitude F_c and constant direction θ, while the body undergoes a translation s, then the above equation can be integrated, so that the work becomes:

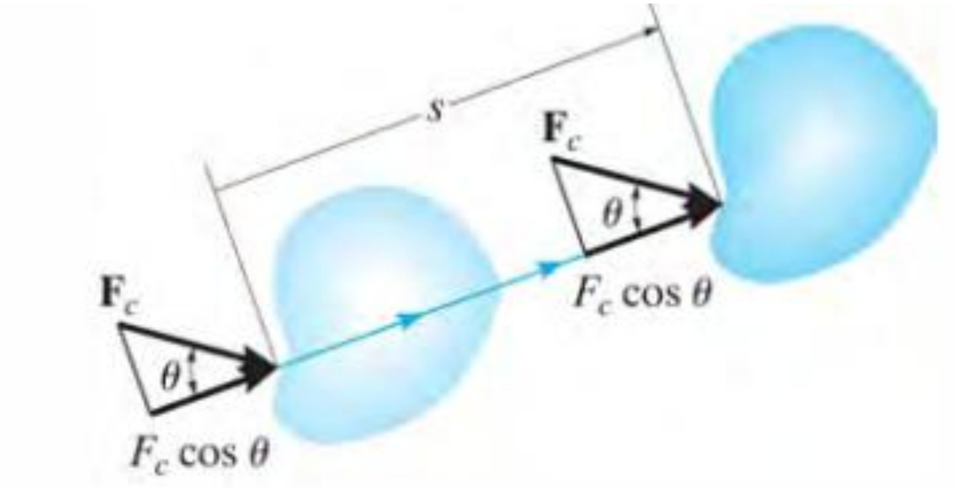


Figure 1. work of constant applied force

$$U_{F_c} = \int ((F_c cos\ \theta))ds$$

$$U_{F_c} = (F_c cos\ \theta)s$$

Common Types of Work

Work of Force of Wight

- The weight of a body does work only when the body's center of mass G undergoes a vertical displacement Δy. If this displacement is upward, Fig. 2, the work is negative, since the weight is opposite to the displacement.

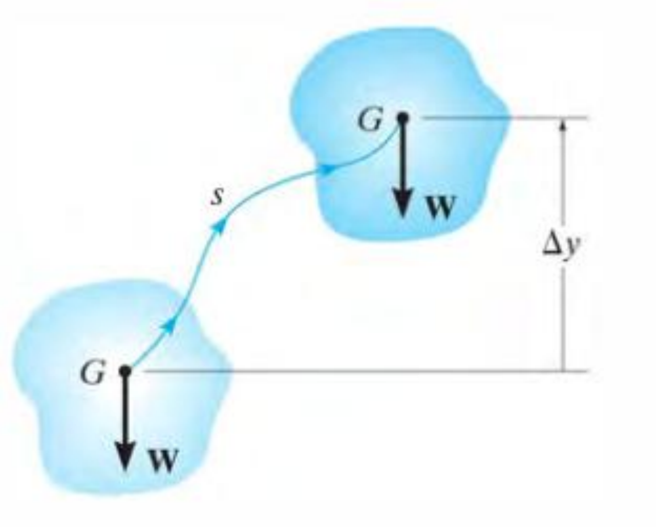


Figure 2. Work of force of gravity

$$U_w = -w\Delta y$$

Likewise, if the displacement is downward(Δy) the work becomes positive. In both cases t he elevation change is considered to be small so that W, which is caused by gravitation, is constant.

Common Types of Work

Work of a Spring Force

- If a linear elastic spring is attached to a body, the spring force Fs = ks acting on the body does work when the spring either stretches or compresses from s_1 to a further position s_2 .
- In both cases the work will be negative since the displacement of the body is in the opposite direction to the force, Fig. 3.The work is

$$U_s = \frac{1}{2}(-k)(s_2{}^2 - s_1{}^2)$$

Where: $s_2 > s_1$

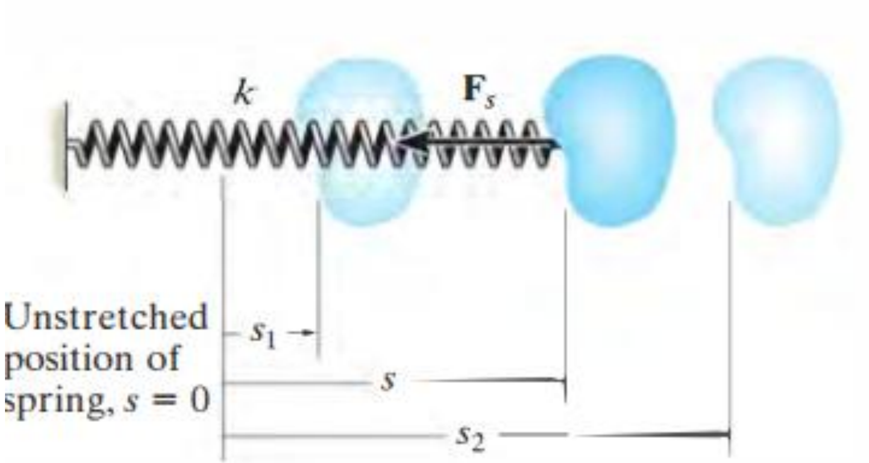


Figure 3. Work of force of spring

Forces That Do No Work

- There are some external forces that do no work or do not change its kinetic energy when the body is displaced.
- These forces act either at fixed points on the body, or they have a direction perpendicular to their displacement.

  ✓ Reactions at frictionless pins or rollers: point of contact has no motion along reaction line.

  ✓ Normal reaction on smooth surfaces: displacement is perpendicular to reaction.

  ✓ Weight of a body moving horizontally: no change in elevation → no vertical displacement.

  ✓ Friction in rolling without slipping: contact point has zero velocity; hence, no displacement.

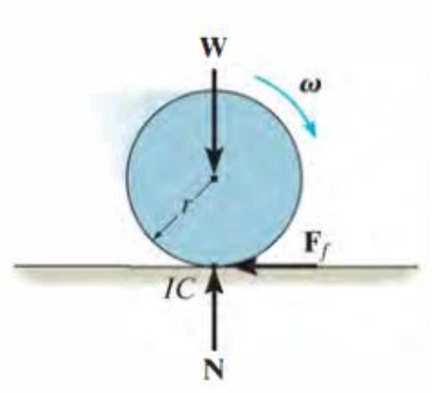


Figure 4. zero work forces

The Work of a Couple Moment

- We frequently need to evaluate the work done by a couple M which acts on a rigid body during its motion
- A couple consists of two equal and opposite forces separated by a distance.
- Its moment is $M = F \times d$
- If a couple acts on a rigid body causing rotation through an angle θ, then the work done by a couple M whose plane is parallel to the plane of motion is

$$U = \int_{\theta_1}^{\theta_2} M d\theta$$

When the moment M of the couple is constant:

$$U = M(\theta_2 - \theta_1)$$

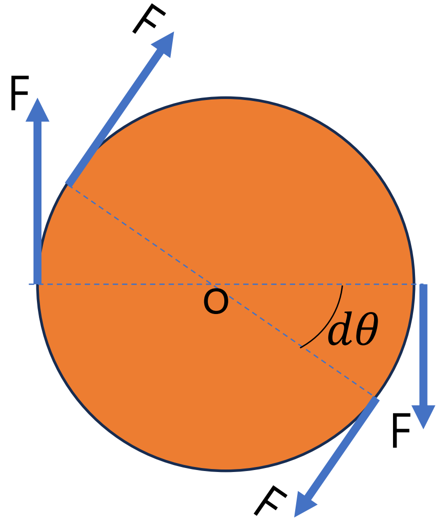


Figure 5. Couple

✓ Positive work: moment and rotation same sense.

✓ Negative work: moment resists motion.

Example

- The bar shown in Fig. 6 has a mass of 10 kg and is subjected to a couple moment of M = 50 N · m and a force of P = 80 N, which is always applied perpendicular to the end of the bar. Also, the spring has an unstretched length of 0.5 m and remains in the vertical position due to the roller guide at B. Determine the total work done by all the forces acting on the bar when it has rotated downward from $\theta = 0^0$ to $\theta = 90^0$.

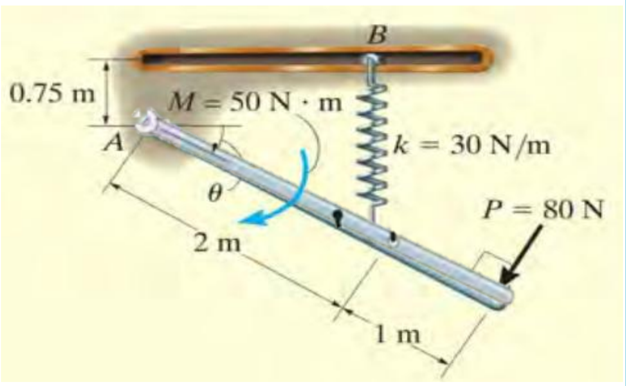
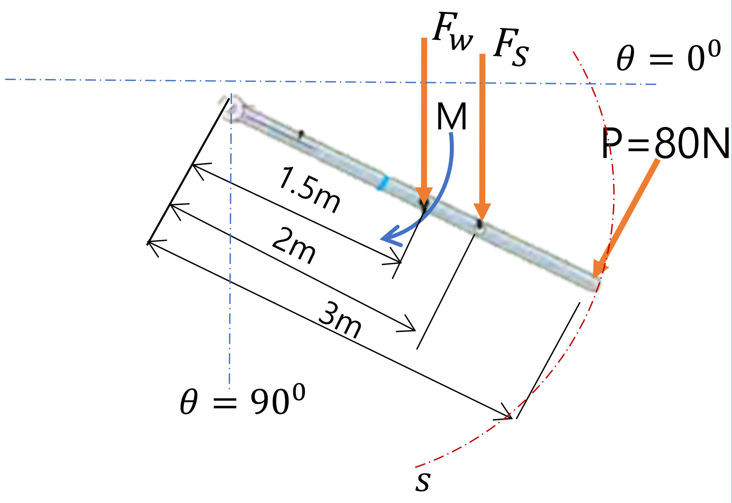


Figure 6. Example

SOLUTION

Weight W. Since the weight 10(9.81) N = 98.1 N is displaced downwar d 1.5 m, the work is

$$U_W = 9.81(1.5) = 147.2N$$



Spring Force Fs. When $\theta = 0^0$ the spring is stretched (0.75 m - 0.5 m) = 0.25 m and when $\theta = 90^0$, the stretch is (2 m + 0.75 m)- 0.5 m= 2.25 m. Thus,

$$U_s = \frac{1}{2}(-30)(2.25^2 - 0.25^2) = -75N$$

Example

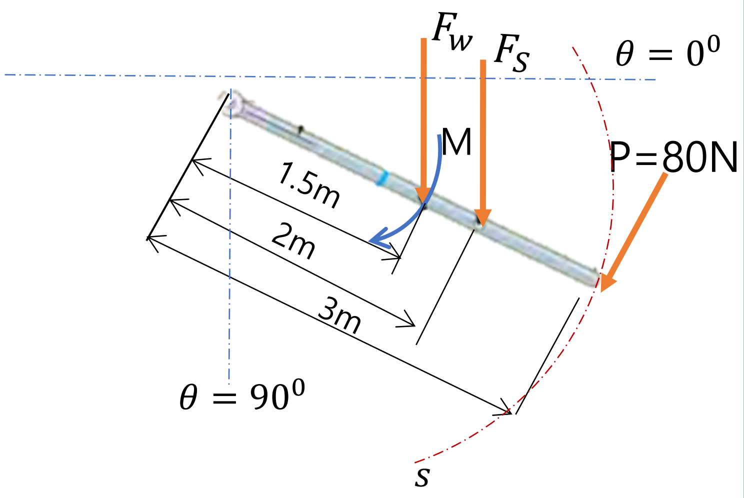
Solution *Cont'd…*

Force P. As the bar moves downward, the force is displaced through a distance of, S=($\pi/2$) (3 m) = 4.712 m.

$$U_P = 80(4.712) = 377.2N$$

Couple Moment M. The couple moment rotates through an angle of $\theta = \pi/2\ rad$. Hence:

$$U_M = 50\ (\pi/2) = 78.5N$$



Pin Reactions. Forces A x and Ay do no work since they are not displaced.

Total Work. The work of all the forces when the bar is displaced is thus:

U =147.2 J + 78.5 J - 75.0 J + 377.0 J=528 J

Kinetic Energy of rigid bodies

- We now use the familiar expression for the kinetic energy of a particle to develop expressions for the kinetic energy of a rigid body for each of the three classes of rigid-body plane motion [3].

(a) Translation

- The translating rigid body of Fig. 7 has a mass m and all of its particles have a common velocity v.

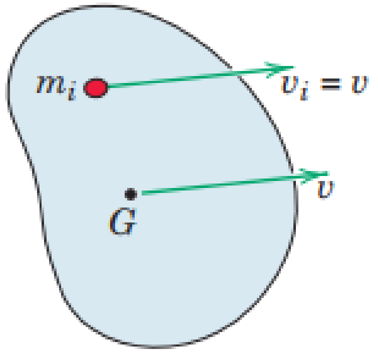


Figure 7. Translation

- The kinetic energy of any particle of mass m_i of the body is $T_i = \frac{1}{2} m_i V^2$,so for the entire body:

$$T = \sum\left(\frac{1}{2} m_i V^2\right) = V^2 \sum\left(\frac{1}{2} m_i\right) \quad \text{or} \quad T = \frac{1}{2} m\, V^2$$

- This expression holds for both rectilinear and curvilinear translation

Kinetic Energy of rigid bodies

(b) Fixed-axis rotation

- The rigid body in Fig. 8 rotates with an angular velocity v about the fixed axis through O.

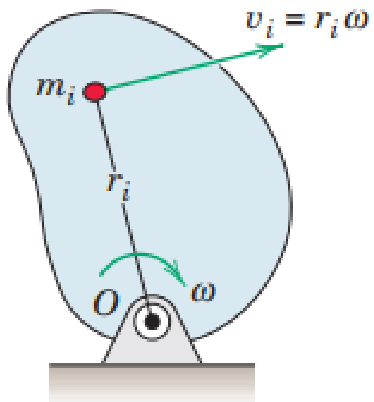


Figure 8. Rotation

- The kinetic energy of a representative particle of mass m_i of the body is $T_i = \frac{1}{2}m_i(r_i.\omega)^2$,so for the entire body

$$T = \sum(\frac{1}{2}m_i(r_i.\omega)^2) = \frac{1}{2}\ \omega^2 \sum(m_i {r_i}^2)$$

- But the moment of inertia of the body about O is $I_0 = \sum(m_i {r_i}^2)$ so $T = \frac{1}{2} I_0 \omega^2$

- Note the similarity in the forms of the kinetic energy expressions for translation and rotation..

- You should verify that the dimensions of the two expressions are identical.

Kinetic Energy of rigid bodies

General Plane Motion.

- The rigid body in Fig. 6/12c executes plane motion where, at the instant considered, the velocity of its mass center G is V_G and its angular velocity is ω.
- The velocity V_i of a representative particle of mass m_i may be expressed in terms of the mass-center velocity V_G.
- Therefore, the kinetic energy is

$$T = \frac{1}{2} m\, {V_G}^2 + \frac{1}{2} I_G \omega^2$$

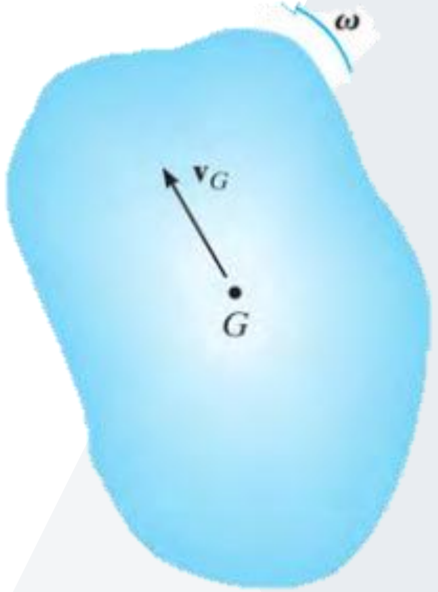


Figure 9. General plane motion

- Here:

✓The first term → translational kinetic energy of the mass center,

✓The second term → rotational kinetic energy about the mass center.

Alternative Expression Using Instantaneous Center

- The kinetic energy of plane motion may also be expressed in terms of the rotational velocity about the instantaneous center C of zero velocity
- If the instantaneous center C (the point with zero velocity) is known, we may write the kinetic energy of a rigid body in plane motion as

$$T = \frac{1}{2} I_{Ic} \omega^2$$

- where Ic is the moment of inertia of the body about its instantaneous center.

- This is often simpler for rolling or linkage problems.

System of Connected Rigid Bodies

- Because energy is a scalar quantity, the total kinetic energy for a system of connected rigid bodies is the sum of the kinetic energies of all its moving parts.
- Depending on the type of motion, the kinetic energy of each body is found by applying

$$T_{Total} = \sum \left(\frac{1}{2} m \, {V_G}^2 + \frac{1}{2} I_G \omega^2\right)$$

Principle of Work and Energy

- By applying the principle of work and energy developed to each of the particles of a rigid body and adding the results algebraically, since energy is a scalar, the principle of work and energy for a rigid body becomes[4] :

$$T_1 + \sum U_{1 \rightarrow 2} = T_2$$

- This equation states that the body's initial translational and rotational kinetic energy (T_1), plus the work done by all the external forces and couple moments acting on the body as the body moves from its initial to its final Position ($\sum U_{1 \rightarrow 2}$), is equal to the body's final translational and rotational kinetic energy.

- Note that the work of the body's internal forces does not have to be considered, because
  - ✓ They occur in equal but opposite collinear pairs, so that when the body moves, the work of one force cancels that of its counterpart.
  - ✓ since the body is rigid, no relative movement between these forces occurs, so that no internal work is done.

- Note that the work of the body's internal forces does not have to be considered, because
  - ✓ They occur in equal but opposite collinear pairs, so that when the body moves, the work of one force cancels that of its counterpart.
  - ✓ since the body is rigid, no relative movement between these forces occurs, so that no internal work is done.
- When several rigid bodies are pin-connected, linked by inextensible cables, or meshed with gears, they can be treated as one system in the work–energy equation.
- The internal forces at pins, cables, or gear teeth are equal and opposite on the connected members.
- These internal forces do no net work because their displacements are either zero or opposite in direction.
- Therefore, in the system work–energy equation, only external forces and couples are included :internal forces are eliminated.

Summary of work energy equation

Important points

- The Work–Energy Method in rigid-body kinetics provides an elegant and powerful way to analyze motion related to angular velocity ,linear velocity, force and couples
- The governing equation: $T_1 + \sum U_{1 \rightarrow 2} = T_2$
- The kinetic energy (T) The kinetic energy of a rigid body that undergoes planar motion can be referenced to its mass center. It includes a scalar sum of its translational and rotational kinetic energies.
- Work can be produced by both forces and couples. A force performs work when its point of application moves through a displacement ds in the same direction as the force.
- However, for a cylinder or any circular body rolling without slipping, the normal and frictiona forces do no work — the normal force acts at a point with no displacement, and the frictiona force acts on successive contact points that are momentarily at rest..

Understanding & Solving problem

Procedure for Analysis

- **Step 1:** Free–body / active-force diagram
  - Draw the inertial coordinate system and draw a free-body diagram of the particle in order to account for all the forces that do work on the particle as it moves along its path
  - Draw a neat figure of the system showing initial and final positions (states 1 and 2).
  - Mark all geometry needed for kinematics (lengths, radii, angles).
  - show only external forces and couples that may do work (weights, applied forces, springs, driving torques).

1 Understanding & Solving problem

Procedure for Analysis

- **Step 2;** Kinematic diagram (velocities & relations)
- Identify type of motion for each body (translation, fixed-axis rotation, or general plane motion)
- Mark velocities V_G and angular speeds ωω.
- If useful, locate the instantaneous center C for plane motion: $T = \frac{1}{2} I_{IC}\omega^2$ can simplify.

1Understanding & Solving problem

Procedure for Analysis

- **Step 3;** Kinematic diagram (velocities & relations)
  - Identify type of motion for each body (translation, fixed-axis rotation, or general plane motion)
  - Mark velocities V_G and angular speeds ωω.
  - If useful, locate the instantaneous center C for plane motion: $T = \frac{1}{2} I_{Ic}\omega^2$ can simplify.
- **Step 4;** Determine the work done $\sum U_{1-2}$
  - For each external force/couple that moves, compute its work between states using :

$$U_{1\rightarrow 2} = \int_1^2 ((F\cos\theta))ds$$

$$U = M(\theta_2 - \theta_1)$$

1 Understanding & Solving problem

Procedure for Analysis

- **Step 5;** Evaluate kinetic energy at states 1 and 2
  - For each body compute: $T = \frac{1}{2} m \, {V_G}^2 + \frac{1}{2} I_G \omega^2$
  - For rotating about a fixed axis use, $\mathrm{T} = \frac{1}{2} I_o \omega^2$
  - Sum kinetic energies for all bodies if system analysis.
- **Step 6;** Apply the work–energy equation and solve for the unknown
  - Use $T_1 + \sum U_{1 \rightarrow 2} = T_2$
- Solve algebraically for the unknown (e.g., final velocity v, ωω, angle, or work).
- If needed, use Newton equations after finding motion from energy to compute reaction forces or torques (because energy method does not give reactions).

Conservation of Energy for Rigid Bodies

- The principle of conservation of energy states that if only conservative forces (forces that store or release energy, such as gravity and springs) act on a rigid body, then the total mechanical energy of the body remains constant.

- The conservation of energy equation can be used to solve problems involving velocity, displacement, and **conservative force systems**.

$$T_1 + V_1 = T_2 + V_2$$

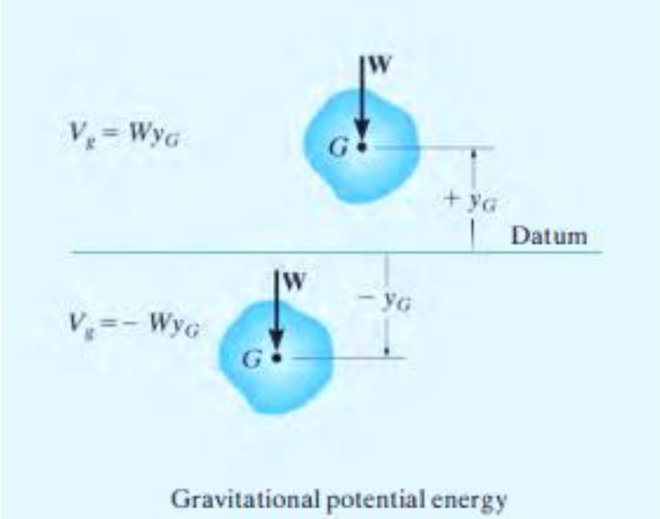
where

T = kinetic energy $= \frac{1}{2} m\, V_G{}^2 + \frac{1}{2} I_G \omega^2$

V = potential energy = $\mathrm{mg}h_g + \frac{1}{2}\mathrm{k}\; S^2$

Subscripts 1 and 2 represent two different positions of the body

- The potential energy is the sum of the body's gravitational and elastic potential energies.
- The gravitational potential energy will be positive if the body's center of gravity is located above a datum.
- If it is below the datum, then it will be negative.
- The elastic potential energy is always positive, regardless if the spring is stretched or compressed
- The kinetic energy (T) at the initial and final points is always positive



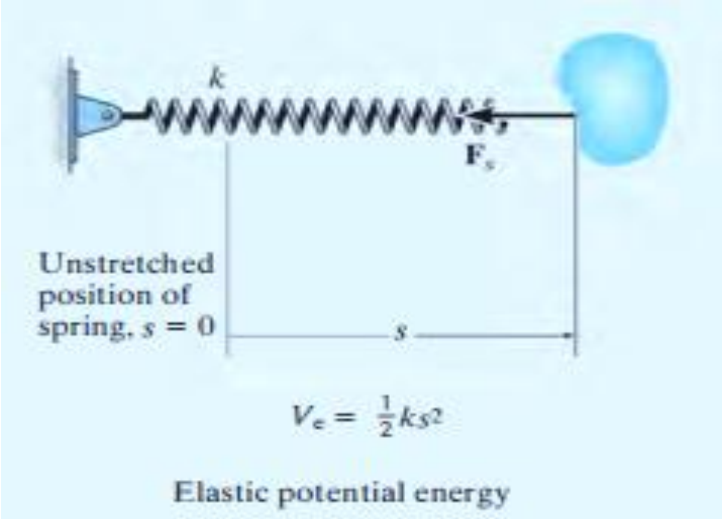


Figure 10. Potential energies

1 Understanding & Solving problem

Procedure for Analysis

- Determine whether all the forces involved are conservative.

  If some of the forces are not conservative, for example, if friction is involve, you must use the general equation

- Draw two diagrams showing the particle located at its initial and final points along the path and choose a datum.
- Calculate the kinetic energy at initial T_1 and Final point T_2,considering the magnitude of both translational and rotational motion
- Compute the spring potential energy and gravitational potential energy at each end of the path.
- Substitute all your values to the general conservation of energy equation and solve for the known.

Summary

Work –energy equation

- useful for solving problems that involve force, velocity, and displacement
- Equation $T_1 + \sum U_{1-2} = T_2$
- Kinetic energy $T = \frac{1}{2} m\, V_G{}^2 + \frac{1}{2} I_G \omega^2$
- Work is due to forces and couple

$$U_{1\rightarrow 2} = \int_1^2 ((F\cos\theta))ds$$

$$U = M(\theta_2 - \theta_1)$$

- Applied in all cases.

Conservation of energy

- A conservative force does work that is independent of its path. Two examples are the weight of a particle and the spring force.
- The work done by a conservative force depends upon its position relative to a datum.
- When this work is referenced from a datum, it is called potential energy..
- The potential energy is the sum of the body's gravitational and elastic potential energies.
- Equation $T_1 + V_1 = T_2 + V_2$

Summary

In This Lecture We Covered:

1. Introduction to Kinetics of Rigid bodies→ work energy method
2. Work and its types
3. Kinetic energy for rigid bodies
4. Work energy equation for rigid bodies
5. conservation of energy rigid bodies

References

[1] Dynamics, Hibbeler, Rusel M.,Prentice Hall,10th ed., 2003

[2] Engineering Mechanics - Dynamics, Meriam J.L.,John Wiley & Sons, 9th ed., 2020.,

[3] Vector Mechanics for Engineers: Dynamics, Johnston, E. R., & Clausen, W. E., McGraw-Hill, 11th ed., 2015

[4] Cengel, Yunus, and John Cimbala. *Ebook: mechanics fundamentals and applications (si units)*. Mc Graw Hill, 2013.