Week 14

Impulse and Momentum for Rigid Bodies

Lecturer : Daniel Hambissa Datti

Contents

By the end of this lecture, you are able to:

1. Define and explain impulse momentum for rigid body
2. Define and explain Linear momentum for rigid body
3. Define and explain impulse for rigid body
4. Define Equations for impulse momentum for rigid body
5. Understand Impact and conservation of momentum

Understand the impulse momentum For rigid bodies

- We now apply the principle of impulse and momentum to the plane motion of rigid bodies and of systems of rigid bodies to solve problems involving force, velocity, and time as related to the planar motion of a rigi d body[1].
- This method is an alternative to the force–acceleration approach for solving rigid-body planar motion probl ems [1].
- It is especially useful when:
  - ✓ Forces act over a short time (e.g., impact, collisions).
  - ✓ You know initial and final velocities, but not the time history of forces
  - ✓ You are interested in velocity change, not acceleration.

Understand the impulse momentum

- Mathematically, for a particle:

$$mv_1 + \sum \int_{t_1}^{t_2} F dt = mv_2$$

- For a rigid body, we extend this and the impulse–momentum relationship can be written for both translation and rotation and can be combined to handle general plane motion.
- Before doing this, we will first formalize the methods for obtaining a body's linear and angular momentum, assuming the body is symmetric with respect to an inertial x-y reference plane

Linear Momentums

- The linear momentum of a rigid body is determined by summing vectorially the linear momenta of all the particles of the body.

$$L = \sum m_i v_i$$

- Since: $L = \sum m_i v_i = m v_G$,we can also write:

$$L = m v_G$$

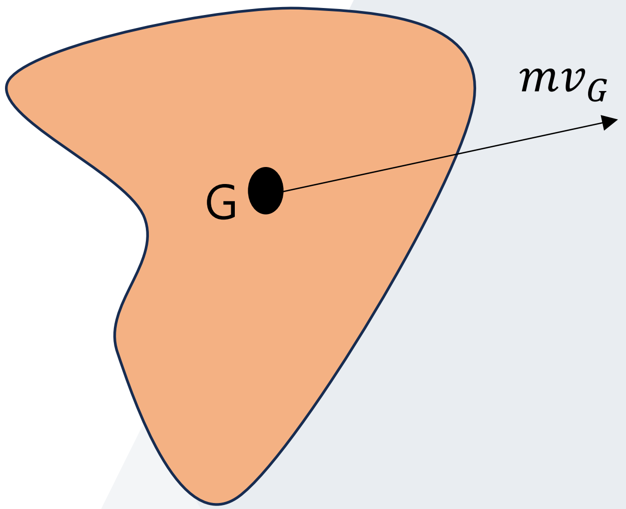


Figure 1. linear momentum

- This means the body's linear momentum can be represented by a single particle of mass *m* moving with velocity v_G .
- The vector $m_i v_i$ is called the sum of linear momentum, denoted as L. This vector has the same direction as v. The linear momentum vector has units of (kg·m)/s or (slug·ft)/s. mass is constant!

Angular Momentum

- The angular momentum of the body about G is equal to the product of the moment of inertia of the body about an axis passing through G

$$H_G = I_G \omega$$

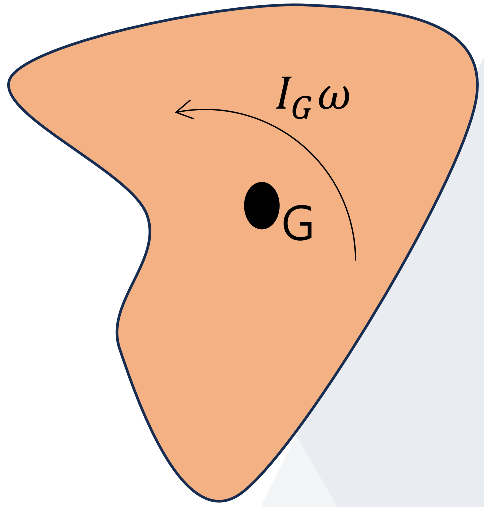


Figure 2. angular momentum

- ✓ H_G is a vector quantity having a magnitude $I_G \omega$, which is commonly measured in units of kg $\cdot$ m^2/s, and a direction defined by ω, which is always perpendicular to the plane of motion.
- ✓ Angular momentum represents the rotational "quantity of motion" and indicates the body's resistance to rotational speed change

- Using these results, we will now consider three types of motion.

Translation

- When a rigid body is subjected to either rectilinear or curvilinear translation, Fig. 3, then, $\omega = 0$ and its mass center has a velocity of v_G.
- Hence, the linear momentum, and the angular momentum about G, become:

$$H_G = 0$$
$$L = mv_G$$

- If the angular momentum is computed about some other point A, the " moment" of the linear momentum L must be found about the point.
- Since d is the "moment arm" as shown in Fig. 3,then;

$$H_A = d(mv_G)$$

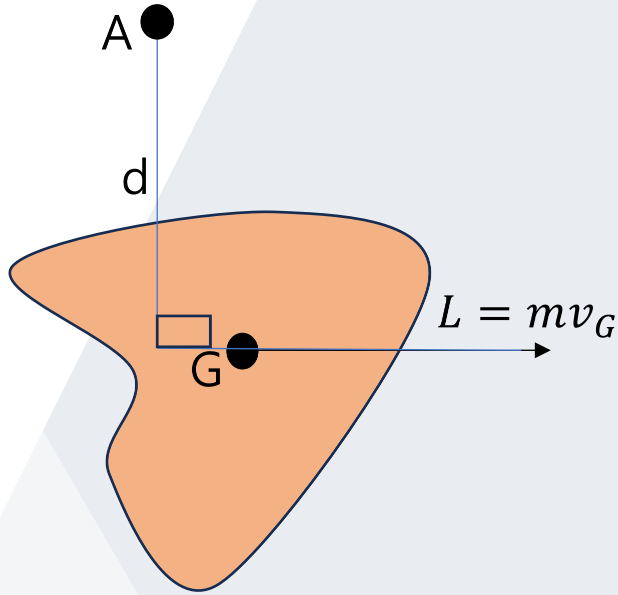


Figure 3. Translation

Rotation About a Fixed Axis

- When a rigid body is rotating about a fixed axis, Fig.4, the linear momentum, and the angular momentum about G, are:

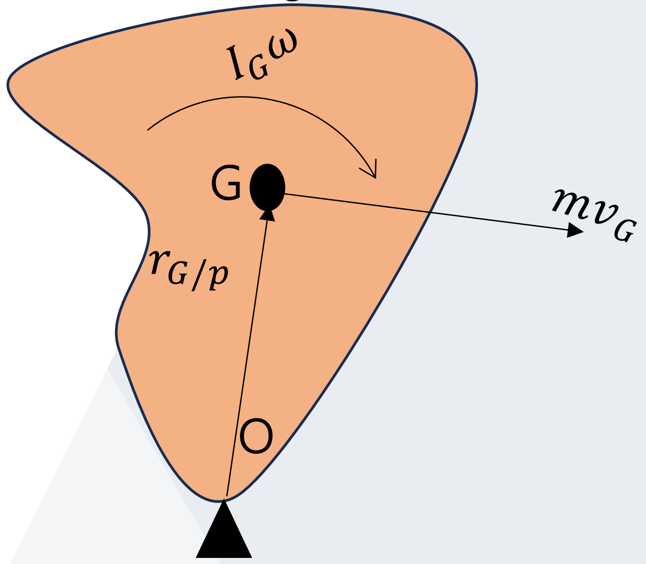


Figure 4. Rotation

$$L = mv_G$$
$$H_G = I_G\omega$$

It is sometimes convenient to compute the angular momentum about point O. Noting that L $or\ v_G$ is always perpendicular to $r_{G/p}$, we have:

$$H_O = I_G\omega + r_{G/p}(mv_G)$$

Since $v_G = (r_{G/p})\omega$, this equation can be written as: $H_O = \omega\ (I_G + (mr_{G/p}{}^2))$

Using the parallel-axis theorem,

$$H_O = I_o\omega$$

For the calculation, then, either the above equation can be used.

General Plane Motion

- Then a rigid body is subjected to general plane motion, Fig. 5, the lin ear momentum, and the angular momentum about G, become:

$$L = mv_G$$
$$H_G = I_G\omega$$

- If the angular momentum is computed about point A, Fig. 5, it is nec essary to include the moment of L and He about this point. In this ca se,

$$H_O = I_G\omega + d(mv_G)$$

Here d is the moment arm, as shown in the figure $H_O = \omega\,(I_G + (mr_{G/p}{}^2))$

As a special case, if point A is the instantaneous center of zero velocity then, we can write the above equation as

$$H_{IC} = I_{IC}\omega$$

where I_{IC} is the moment of inertia of the body about the I_C

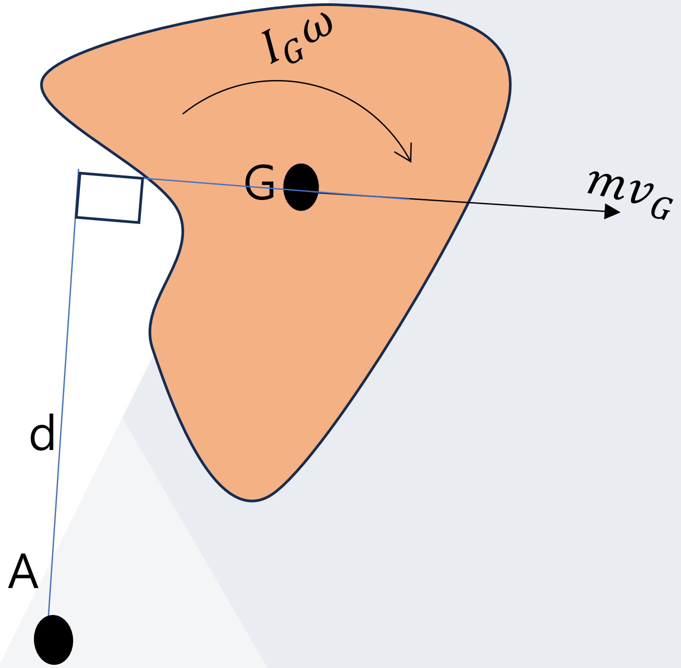


Figure 5. general plane motion

Impulse

- Impulse represents the total effect of a force acting over time.

Linear Impulse

$$I = \int_{t_1}^{t_2} f\,dt$$

It changes the linear momentum of the body.

Angular Impulse

$$I_M = \int_{t_1}^{t_2} M\,dt$$

It changes the angular momentum of the body.

Principle of Linear Impulse and Momentum for Rigid Bodies

- Like the case for particle motion, the principle of impulse and momentum for a rigid body can be developed by combining the equation of motion with kinematics.

- This principle relates external impulses to changes in linear and angular momentum.
- The resulting equation will yield a direct solution to problems involving force, velocity, and time.

Principle of Linear Impulse and Momentum

- Combining Newton's 2nd law ($\sum F = ma$) with integration with respect to time we get,

$$mv_{(G)1} + \sum \int_{t_1}^{t_2} F dt = mv_{(G)2}$$

- This equation is referred to as the principle of linear impulse and momentum.
- It states that the sum of all the impulses created by the external force system which acts on the body during the time interval t_1 to t_2 is equal to the change in the linear momentum of the body during this time interval

Cont'd….

$$mv_{(G)1} + \sum \int_{t_1}^{t_2} F dt = mv_{(G)2}$$

- This equation is referred to as the principle of linear impulse and momentum.

- It states that the sum of all the impulses created by the external force system which acts on the body during the time interval t_1 to t_2 is equal to the change in the linear momentum of the body during this time interval

Principle of Angular Impulse and Momentum

- In a similar manner, for rotation about a fixed axis passing through point G, Combining rotational equation ($\sum m_G = I_G \alpha$) with integration with respect to time we get,:

$$I_G \omega_1 + \sum \int_{t_1}^{t_2} M_G \, dt = I_G \omega_2$$

In a similar manner, for rotation about a fixed axis passing through point O,

$$I_O \omega_1 + \sum \int_{t_1}^{t_2} M_o \, dt = I_O \omega_2$$

- This Equations are referred to as the principle of angular impulse and momentum.

- Both equations state that the sum of the angular impulses acting on the body during the time interval $t_1 \; to \; t_2$ is equal to the change in the body's angular momentum during this time interval.

These three terms are illustrated graphically on the impulse and momentum diagrams shown in Fig. .6

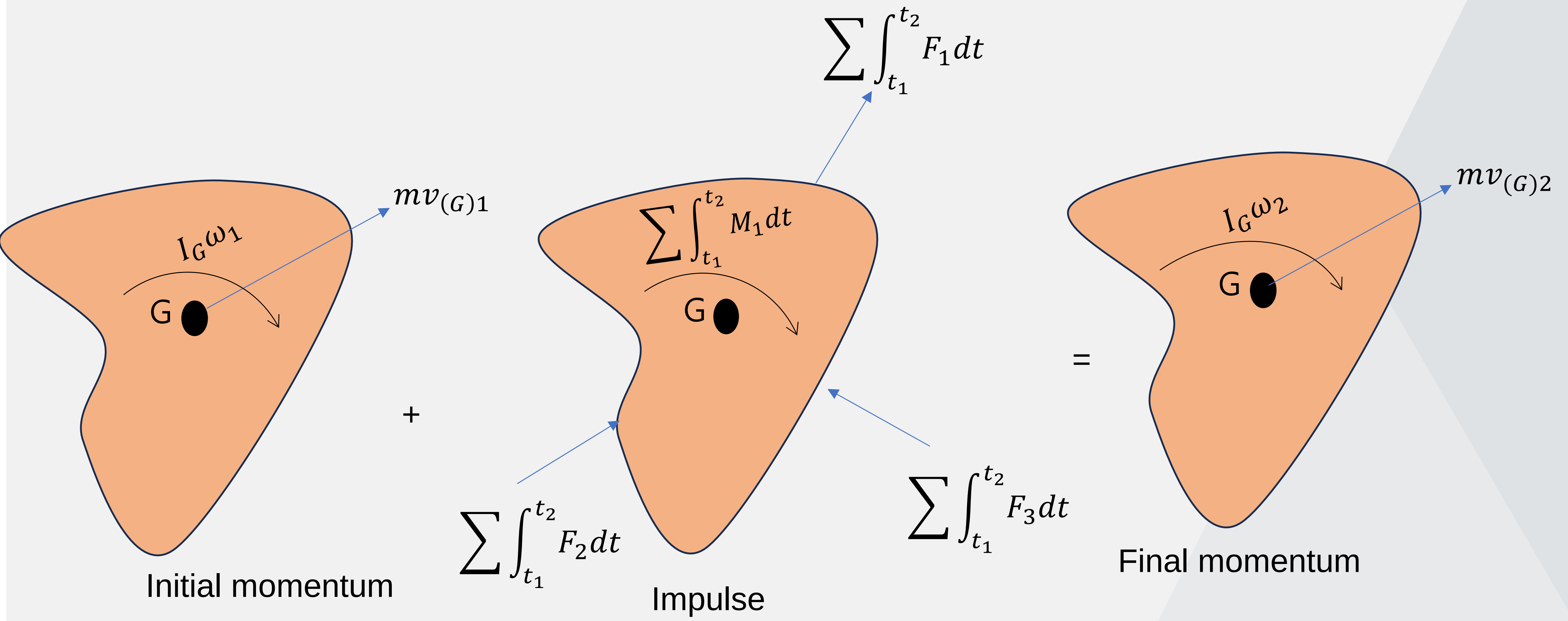


Figure 6. Impulse and momentum diagrams

- To summarize these concepts, if motion occurs in the x-y plane, the following three scalar equations can be written to describe the planar motion of the body.

$$X - direction \qquad m v_{(Gx)_1} + \sum \int_{t_1}^{t_2} F_x dt = m v_{Gx(2)}$$

$$\text{y}-direction \qquad m v_{(Gy)_1} + \sum \int_{t_1}^{t_2} F_y dt = m v_{Gy(2)}$$

$$I_G \omega_1 + \sum \int_{t_1}^{t_2} M_G dt = I_G \omega_2$$

Note that the linear momentum $mv_{(G)}$ is applied at the body's mass center, whereas the angular momentum $I_G\omega$ is a free vector, and therefore, like a couple moment, it can be applied at any point on the body.

If F and M ,are constant, such is the case for the body's weight ,integration of the impulses yields F(t_2 - t_1) and M (t_2 - t_1), respectively.

Procedure for Analysis

Step 1. Free-Body Diagram (FBD)

- Establish Inertial Frame: Define the x, y, z inertial coordinate system.
- Draw FBD: Account for all forces and couple moments that produce impulses.
- Establish Velocities: Define the initial and final velocity of the mass center V_G and ω the angular velocity
- Moment of Inertia: Compute the moment of inertia, I_G or I_O .
- This principle relates external impulses to changes in linear and angular momentum.

Procedure for Analysis

Step 2. Principle of Impulse and Momentum

- Apply Scalar Equations: Use the three scalar equations derived from the principle..

- Angular Momentum (H_O for a Fixed Axis): For a rigid body rotating about a fixed axis, the angular momentum is given by:

$$H_O = I_O\omega$$

- **Impulsive Forces:**

✓ All forces on the FBD create an impulse, but some do no work.

✓ Forces that are functions of time must be integrated to obtain the impulse.

Kinematics

- When to Use: Kinematics must be used if more than three equations are needed for a complete solution.

- Purpose: Relate the velocity of the mass center V_G to the body's angular velocity ω.

- Aid: Kinematic (velocity) diagrams can be helpful in obtaining the necessary relationships.

Procedure for Analysis

Kinematics

- When to Use: Kinematics must be used if more than three equations are needed for a complete solution.
- Purpose: Relate the velocity of the mass center V_G to the body's angular velocity ω.
- Aid: Kinematic (velocity) diagrams can be helpful in obtaining the necessary relationships.

Conservation of linear momentum

conservation of linear momentum

- When the sum of the external impulses acting on a body is zero in a specific direction, the impulse momentum equation for body reduces to a simplified form, namely,

$$\sum mv_{(G)_1} = \sum mv_{(G)_2} \qquad i.e \sum \int_{t_1}^{t_2} F dt = 0$$

- This equation is referred to as the conservation of linear momentum.

Impact and conservation of linear momentum

Angular Momentum Conservation

- The angular momentum of a system of connected rigid bodies is conserved about the system's center of mass G, or a fixed point O, when the sum of all the angular impulses about these points is zero or appreciably small (non-impulsive)

$$H_{(G)1} = H_{(G)2}$$

$$\sum \int_{t_1}^{t_2} M_G \, dt = 0$$

Impact of rigid bodes

- When two bodies collide, very large forces act for an extremely short time. This phenomenon is known as impact.
- During impact, these impulsive contact forces dominate the motion, while non-impulsive forces such as weight or small friction can be neglected.
- For rigid bodies collision results in an abrupt change in velocity and often produces both translational and rotational motion.
- In particle mechanics, impact is analyzed using the coefficient of restitution (e) and conservation of linear momentum. Each particle is treated as a point mass; therefore, only translational motion is considered.
- When the bodies are rigid rather than point particles, each has mass, geometry, and rotational inertia. The same physical laws apply, but now the analysis must include rotational motion as well as translation.

Impact of rigid bodes

- Thus, in rigid-body impact problems, we extend the particle approach by applying:
  - Linear impulse–momentum equations (as in the particle case), and
  - Angular impulse–momentum equations about a convenient point (mass center or contact point).
- The impact of rigid bodies is essentially the particle impact theory plus rotation.
- The rigid body impact is studied considering two type of impact:
  - Central Impact
  - Eccentric Impact

Central Impact

The line of impact passes through both mass centers. Only translation occurs, similar to the particle case.

The same two equations for particle used (momentum and restitution), is applied.

$$m_A v_{(A)_1} + m_B v_{(B)_1} = m_A v_{(A)_2} + m_B v_{(B)_2}$$

$$e = \frac{v_{(B)_2} - v_{(A)_2}}{v_{(A)_1} - v_{(B)_1}}$$

Figure 6. Line of contact and plane of contact

Eccentric Impact

- The line of impact does not pass through the centers of mass then eccentric impact will occur
- Each body experiences both translational and angular velocity changes.
- The analysis now requires both linear and angular impulse–momentum equations.
- This type of impact often occurs when one or both of the bodies are constrained to rotate about a fixed axis.

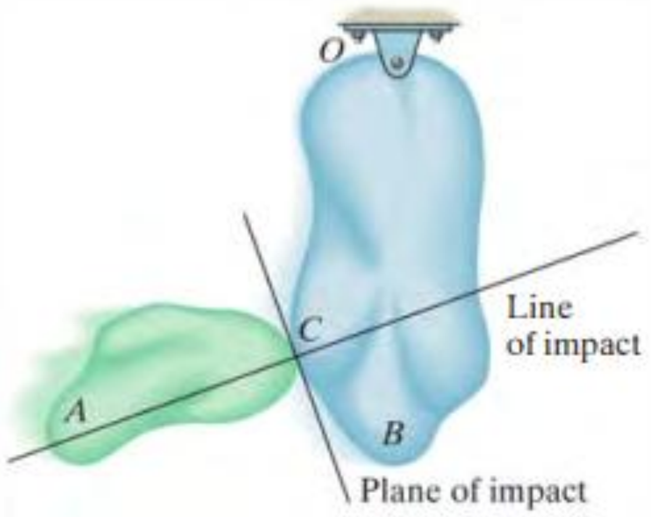


Figure 7. eccentric impact

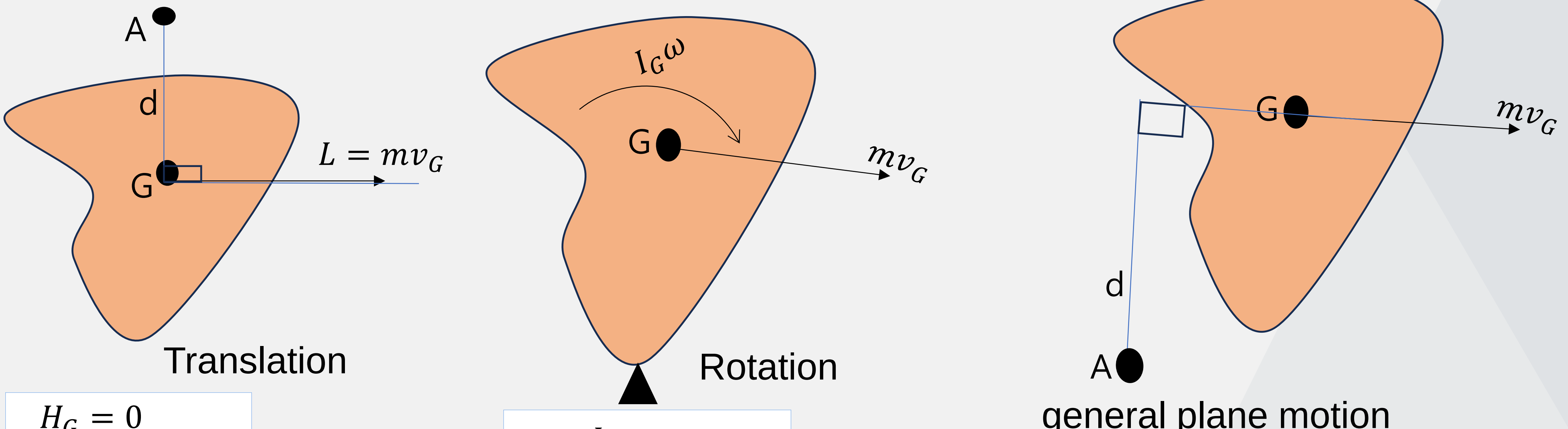
Summary Impact analysis for rigid bodies

- Real impacts involve deformation, vibration, and energy dissipation, which are not captured by rigid-body assumptions [3].
- The coefficient of restitution is an empirical factor, not a constant physical property.
- Detailed modeling of contact behavior is extremely complex and usually requires advanced computational methods.
- The impulse–momentum and momentum conservation principles provide the most practical and educationally meaningful way to study impacts.
- Therefore, detailed study of rigid-body impact mechanics is intentionally omitted in elementary dynamics courses.

Summary on Impulse momentum

Linear and Angular Momentum

- The linear and angular momentum of a rigid body can be referenced to its mass center G.
- If the angular momentum is to be determined about an axis other than the one passing through the mass center, then the angular momentum is determined by summing vector He and the moment of vector L about this axis.



$$H_G = 0$$
$$L = mv_G$$

$$L = mv_G$$
$$H_G = I_G\omega$$
$$H_O = I_o\omega$$

$$L = mv_G$$
$$H_G = I_G\omega$$
$$H_O = I_G\omega + d(mv_G)$$

Figure 8. summary

Summary on Impulse momentum

- The principles of linear and angular impulse and momentum are used to solve problems that involve force, velocity, and time.
- Before applying these equations, it is important to establish the x, y, z inertial coordinate system
- The free-body diagram for the body should also be drawn in order to account for all of the forces and couple moments that produce impulses on the body

$$mv_{(Gx)_1} + \sum \int_{t_1}^{t_2} F_x dt = mv_{Gx(2)} \qquad mv_{(Gy)_1} + \sum \int_{t_1}^{t_2} F_y dt = mv_{Gy(2)} \qquad I_G\omega_1 + \sum \int_{t_1}^{t_2} M_G dt = I_G\omega_2$$

Summary Conservation of Momentum

- Provided the sum of the linear impulses acting on a system of connected rigid bodies is zero in a particular direction, then the linear momentum for the system is conserved in this direction.

- Conservation of angular momentum occurs if the impulses pass through an axis or are parallel to it. Momentum is also conserved if the external forces are small and thereby create non-impulsive forces on the system.

$$\sum mv_{(G)_1} = \sum mv_{(G)_2} \qquad H_{(G)1} = H_{(G)2}$$

Summary

In This Lecture We Covered:

1. Introduction to Kinetics of Rigid bodies → Impulse - momentum method
2. Define and explain momentum for rigid bodies and its types
3. Define and explain Impulse for rigid bodies and its types
4. Equation Impulse - momentum method
5. Impact and conservation of energy for rigid bodies→ description

References

[1] Dynamics, Hibbeler, Rusel M.,Prentice Hall,10th ed., 2003

[2] Cengel, Yunus, and John Cimbala. *Ebook: Fluid mechanics fundamentals and applications (si units)*. McGraw Hill, 2013.

[3] Engineering Mechanics - Dynamics, Meriam J.L.,John Wiley & Sons, 9th ed., 2020.,

[4] Vector Mechanics for Engineers: Dynamics, Johnston, E. R., & Clausen, W. E., McGraw-Hill, 11th ed., 2015