CONTINUOUS ASSESSMENT TEST

QUESTION 1.

Kelvin's age of the Earth. In this problem, we'll repeat the initial wrongheaded calculation by Lord Kelvin of Earth's age. The value here will differ somewhat from the 20 Myr quoted in the course notes, which was a later revision from his initial estimate.

Assume the Earth started out as a homogeneous molten sphere, which uniformly solidified at temperature T_M, at a time $\tau_\oplus$ before present. Also assume a known present near-surface geothermal gradient $-\partial T/\partial z|_0$, a constant thermal diffusivity D, and constant surface temperature boundary condition T_0.

Kelvin's values for these parameters are $T_M - T_0 = 7000°\text{F}$, $-\partial T/\partial z|_0 = 1°\text{F}/50\text{ft}$, and $D = 400\text{ft}^2/\text{yr}$.

(a) Justify the assumption that the spherical geometry of Earth, and the fact that it is not semi-infinite in spatial extent, can be ignored for the purposes of this problem.

(b) Derive a general expression for $\tau_\oplus$ and put in parameters to arrive at Kelvin's estimate of $\tau_\oplus^{\text{Kelvin}}$.

QUESTION 2.

Growth of a Cloud Droplet. How rapidly can a cloud droplet grow by condensation from supersaturated air? This question presents a highly idealized model of the problem, based again on the steady-state diffusion equation. This time we will work in spherical coordinates, and we will assume that the water vapor concentration q (units: kg m^{-3}) depends only on the radius from the center of the drop, r.

(a) Show that at steady state the diffusion equation for q becomes:

$$D_v \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dq}{dr}\right) = 0, \tag{4}$$

where D_v is the molecular diffusivity of water vapor in air.

(b) If we impose the boundary conditions that $q = Hq^*$ at $r = \infty$ and $q = q^*$ at $r = a$, where a is the radius of the droplet, then what is the solution to equation (4)? Here, q^* represents the water vapor mass concentration of a parcel of air that is saturated, and H is the relative humidity (humidity relative to saturation). $H > 1$

(greater than 100 %) indicates supersaturation of the air around the droplet and allows for growth with time, while $H < 1$ implies subsaturated air and allows for evaporation of the droplet. Show that the diffusive flux of water vapor into the sphere (J_v), in mass per unit time, is given by:

$$J_v = 4\pi D_v a q^* (H - 1). \tag{5}$$

(c) Now, we want to consider how the droplet grows in time, due to the diffusive flux of water vapor into the droplet. Note that if we want to use equation (5), we need

to justify our assumption that the atmospheric water vapor field is quasi-steady. In other words, we need to justify our neglect of the time-derivative of the water vapor concentration in our derivation up to this point. How might we do this?

(d) Assuming that we can use equation (5) to model the growth rate of a cloud droplet, write down the relevant differential equation for the time rate of change of a droplet radius a, given condensation rate J_v, and assuming that the droplet density is that of water, ρ_w.

(e) Solve the differential equation for droplet growth rate, subject to the initial condition that $a = a_0$ at $t = 0$. How does the radius depend on time? Is this surprising?

(f) Using $D_v = 2 \times 10^{-5}$ m^2 s^{-1}, $q^* = 5 \times 10^{-3}$ kg m^{-3}, and $H = 1.005$ (clouds are normally only very weakly supersaturated), make a log-log plot of $a(t)$ for droplets with a_0 =1, 10, and 100 μm. How long does it take a cloud droplet to grow to 1 mm under these conditions? If these conditions are relatively favorable for droplet growth, and a cloud droplet needs to grow to ~1 mm before it becomes a raindrop, what does that tell you about mechanisms for raindrop formation?