FINAL EXAMINATION

TIME: THREE HOURS

ATTEMPT ALL THE QUESTIONS

QUESTION 1. (20 Marks)

If you ever have a chance to visit Yellowstone in the winter you should do it. The geysers not only shoot out jets of water hundreds of feet in the air, but release giant towers of frozen steam. The favored mechanism for these marvels is as follows. An empty tube deep in the earth is surrounded by super-heated rock. Water from the surface fills the tube at a roughly constant rate where it is then heated by the surrounding rock. When the water at the surface of the column reaches the boiling point it triggers a cascade of boiling from the surface to the bottom creating a jet of water and steam.

(a) Based on the above mechanism what would you expect the wait time distribution between eruptions to look like?

(b) How would you expect random variability to express itself?

QUESTION 2. (20 Marks)

Borehole Paleothermometry: One way we know about past climate variability is by drilling deep holes (boreholes) into bedrock or glaciers, and measuring the temperature at different depths. Surface temperature perturbations have diffused downwards below the surface, and we can solve for past surface temperatures by integrating the diffusion equation backwards in time, and solving for the surface boundary condition as a function of time. Here we will tackle a slightly easier example of how surface temperature variations can leave a direct temperature record below the ground or ice.

Assume that we have a large, homogeneous body of rock or glacial ice, infinite in horizontal extent, so that we can consider the time-dependent diffusion equation only in the vertical direction. Consider z to be the vertical coordinate, positive downwards. We will solve an equation for the temperature perturbation from the long-term mean, $T'(z,t)$. Let the surface boundary condition be:

$$T'(0,t) = \Delta\cos(\omega t), \tag{1}$$

i.e., oscillation about zero with an angular frequency ω and an amplitude Δ. Assuming a thermal diffusivity D, show that

$$T'(z,t) = \Delta \cos(\omega t - kz) e^{-kz} \tag{2}$$

represents a solution to the diffusion equation, if an appropriate value of k is used. What is the appropriate value of k?

QUESTION 3. (30 Marks)

Precession of the Major Axis: Without any perturbations from the outside, a planet orbiting in an elliptical orbit will remain in the same orbit forever. It turns out, however, that the stability of these fixed ellipses is tied very closely to the $1/r$ potential in which the planets sit. In this problem, we will explore what happens if we add a

small perturbing force,

$$f \sim \frac{1}{r^4}, \tag{3}$$

to our system. This higher order term comes about both from interactions with other planets and through relativistic considerations – here we will quote the relativistic results alone for the strength of the perturbing force, since perturbations from other planets are complicated to calculate, and introduce forces that scale with other powers of r (e.g. $f \sim r, f \sim 1/r^3$) – it is not immediately clear what terms dominate on long time scales. We will start with a conversion of our orbital equation from one in terms of the potential, to one in terms of the force. Based on the equation for the orbital energy derived in class, it can be shown that an expression can be written in terms of the force field, $F(r)$:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{l^2} F(r). \tag{4}$$

Showing this is a little involved, so we're not asking that you do it, but eqn. (4) will be the basis for what follows.

(a) By enforcing the gravitational potential and including the following definitions:

$$\frac{1}{\alpha} \equiv \frac{G m_1^2 m_2}{l^2}, \qquad u = \frac{1}{r} \tag{5}$$

show that equation (4) reduces to

$$\frac{d^2 u}{d\theta^2} + u = \frac{1}{\alpha}. \tag{6}$$

(b) Show that the conic section solution for the gravitational potential,

$$u = \frac{1}{\alpha}\left(1 + \epsilon \cos(\theta)\right), \tag{7}$$

is a solution to equation (6).

(c) At this point, we are ready to explore the consequences of including a small term which depends on r^{-4} to $F(r)$. This allows us to alter equation (6) to get

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{\alpha} + \delta u^2 \tag{8}$$

where δ is a very small constant. Specifically, δ must be small relative to the orbital radius; relativity predicts $\delta = 3Gm_2/c^2 \approx 4400$ m, which is indeed miniscule compared to Earth's orbital radius of 1.5×10^{11} m. Because the $u^2\delta$ term is so small it can be solved through a series of successive guesses where we start by guessing the solution of equation (6) and plug it into equation (8). By this we mean that we guess that $u \approx u_0 + u_1$, where $u_1 << u_0$, and then determine u_0 as

the solution to the unperturbed equation:

$$\frac{d^2u_0}{d\theta^2} + u_0 = \frac{1}{\alpha} \tag{9}$$

The perturbation u_1 is solved for by using $u^2 \approx u_0^2$ and then solving for the LHS that matches δu_0^2:

$$\frac{d^2u_1}{d\theta^2} + u_1 = \delta u_0^2. \tag{10}$$

Since δ is very small, we expect the correction u_1 to be very small, and thus terms like u_0u_1 in the square of u are further small corrections to the solution, which we will not consider. For this problem, use u_0 given by (7), and show that:

$$u_1 = \frac{\delta}{\alpha^2}\left(1 + \frac{\epsilon^2}{2} + \epsilon\theta\sin\theta - \frac{\epsilon^2}{6}\cos 2\theta\right) \tag{11}$$

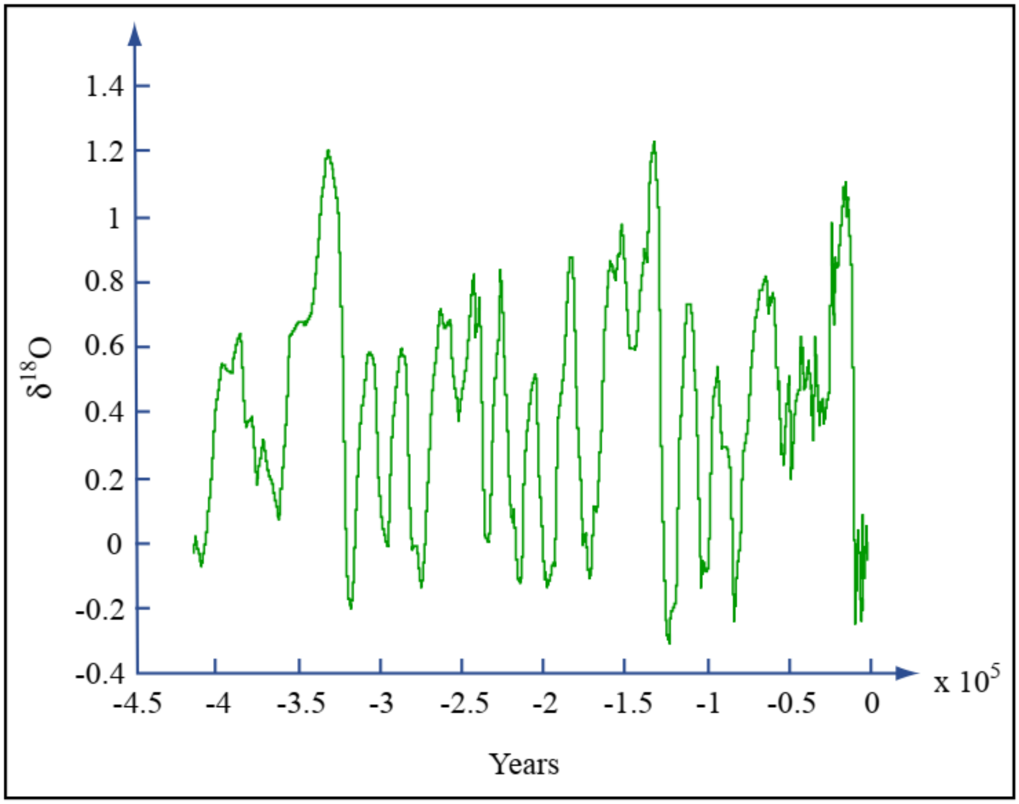
is a solution consistent with (10).

(d) Show, remembering the small angle approximation, that our solution for $u_0 + u_1$ can be simplified to

$$u \cong \frac{1}{\alpha}\left[1 + \epsilon\cos\left(\theta - \frac{\delta}{\alpha}\theta\right)\right] + \frac{\delta}{\alpha^2}\left(1 + \frac{\epsilon^2}{2} - \frac{\epsilon^2}{6}\cos 2\theta\right). \tag{12}$$

QUESTION 4. (10 Marks)

Ice Volume: In class we have looked at various records found in the ice cores which appear to exhibit frequencies consistent with orbital forcing. Specifically, we looked in depth at the record of $\delta^{18}O$ as a proxy for the ice volume:



Explain why and in what direction we would expect $\delta^{18}O$ in the atmosphere to change as a result of glaciation and deglaciation.

QUESTION 5. (20 Marks)

Equilibrium of CO_2 with the Oceans & Timescale of CO_2 Uptake: In class we discussed how carbon dioxide (CO_2) leaves a volcano and ends up both in the atmosphere and in the ocean. If CO_2 was not reactive in water then we could predict the concentration of CO_2 in the water based solely on its concentration in the air through our knowledge of Henry's Law (cough, 5.111), assuming that we waited long enough for the system to reach equilibrium. Henry's law normally takes the following form:

$$[CO_2] = k_H p_{CO_2}. \tag{1}$$

Here, $[CO_2]$ is the concentration of carbon dioxide in the solution (units: M), k_H is the Henry's law coefficient (a function of the chemical, solvent and temperature, units:

M atm^{-1}) and p_{CO_2} is the partial pressure of carbon dioxide in the gas phase (units: atm; 1 atm=101325 Pa). This works quite well for finding the amount of dissolved gas in a solution. Unfortunately for the purposes of this pset (though fortunately for the habitability of Earth), carbon dioxide is not inert in water and can take on multiple forms. Two reactions in the 'carbonate system' and their respective equilibrium constants are:

$$\mathrm{CO_2 + H_2O \rightleftharpoons HCO_3^- + H^+}, \qquad K_1 = \frac{[\mathrm{HCO_3^-}][\mathrm{H^+}]}{[\mathrm{CO_2}][\mathrm{H_2O}]} \tag{2}$$

and

$$\mathrm{HCO_3^- \rightleftharpoons CO_3^{2-} + H^+}, \qquad K_2 = \frac{[\mathrm{CO_3^{2-}}][\mathrm{H^+}]}{[\mathrm{HCO_3^-}]}. \tag{3}$$

In the following, we will assume the oceans have a known pH, where you should recall that:

$$\mathrm{pH} = -\log_{10}[\mathrm{H^+}] \tag{4}$$

Because carbon dioxide takes part in these reversible reactions the concentration of carbon dioxide stored in the ocean is not $[CO_2]$, but rather:

$$[\mathrm{DIC}] = [\mathrm{CO_2}] + [\mathrm{HCO_3^-}] + [\mathrm{CO_3^{2-}}], \tag{5}$$

where [DIC] signifies "dissolved inorganic carbon," (units: M).

(a) Show that the expression for [DIC] as a function of the carbon dioxide concentration, the various equilibrium constants and the hydrogen ion concentration $[H^+]$

(i.e. pH), is given by:

$$\begin{aligned} [\mathrm{DIC}] &= k_H^* p_{\mathrm{CO_2}} \\ k_H^* &\equiv k_H \left(1 + \frac{K_1}{[\mathrm{H^+}]} + \frac{K_1 K_2}{[\mathrm{H^+}]^2}\right) \end{aligned} \tag{6}$$

(b) Now we will use equation (6) to estimate the partitioning of carbon between the atmosphere and ocean. Assume that the atmosphere has mass M_A (units: kg), mean molecular weight $\bar{m}$ (units: kg/mol), a surface pressure of 1 atm, and that CO_2 is a trace species (has minimal impact on overall atmospheric mass, pressure, etc.). Also assume that the ocean has volume V_O (units: L). Now, derive an expression for the mass fractions of carbon in the atmosphere and ocean – i.e. how much carbon of a total amount C_T is found in the ocean as [DIC], and how much is found in the atmosphere as gaseous CO_2? Explain qualitatively how the ocean pH affects these mass fractions.

(c) At 275 K (i.e. close to the temperature of most of the world's ocean water) $k_H = 6.7\times10^{-2}$ M atm^{-1}, $K_1 = 3.3\times10^{-7}$ M, and $K_2 = 2.9\times10^{-11}$ M (k_H, K_1, and K_2 are all fairly temperature-dependent). Look up the mass of Earth's atmosphere, the volume of the ocean, and the average ocean pH, and estimate the partitioning of carbon between the atmosphere and ocean. For a surface partial pressure p_{CO_2}=280 μ-atm, what is the mass of carbon in the atmosphere and ocean?

(d) The molecular diffusivity D_m of all of the components of [DIC] in water is on the order of 10^{-9} m^2 s^{-1}. Using this, and a characteristic ocean depth, derive a time scale for molecular diffusion of carbon from the atmosphere to the bottom of the ocean.

End of exam