Quasiperiodic signals

Let y be a function of r independent variables:

$$y = y(t_1, t_2, \ldots, t_r).$$

y is *periodic*, of period 2π in *each* argument, if

$$y(t_1, t_2, \ldots, t_j + 2\pi, \ldots, t_r) = y(t_1, t_2, \ldots, t_j, \ldots, t_r), \quad j = 1, \ldots, r$$

y is called *quasiperiodic* if each t_j varies with time at a different rate (i.e., different "clocks"). We have then

$$t_j = \omega_j t, \qquad j = 1, \ldots, r.$$

The quasiperiodic function y has r fundamental frequencies:

$$f_j = \frac{\omega_j}{2\pi}$$

and r periods

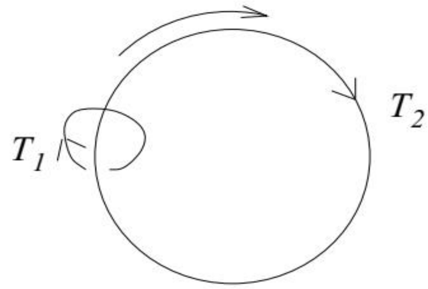
$$T_j = \frac{1}{f_j} = \frac{2\pi}{\omega_j}.$$

Example: The astronomical position of a point on Earth's surface changes due to

- rotation of Earth about axis ($T_1 = 24$ hours).
- revolution of Earth around sun ($T_2 \simeq 365$ days).

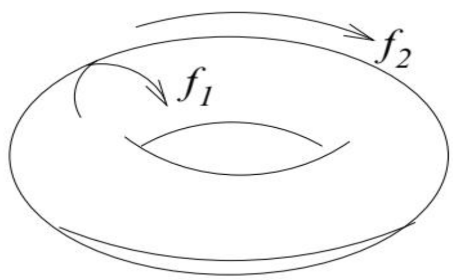
At long time scales, we also have changes in precession (26 Kyr), obliquity (41 Kyr), and eccentricity (∼100 Kyr).

Considering just two oscillations (e.g, rotation and revolution), we can conceive of such a function on a 2-D torus T^2, existing in a 3-D space.



Here we think of a disk spinning with period T_1 while it revolves along the circular path with period T_2.

Such behavior can be conceived as a trajectory on the *surface* of a doughnut or inner tube, or a torus T_2 in $\mathbb{R}^3$.



What is the power spectrum of a quasiperiodic signal $x(t)$? There are two possibilities:

1. The quasiperiodic signal is a *linear* combination of independent periodic functions. For example:

$$x(t) = \sum_{i=1}^{r} x_i(\omega_i t).$$

   Because the Fourier transform is a linear transformation, the power spectrum of $x(t)$ is a set of peaks at frequencies

$$f_1 = \omega_1/2\pi, \ f_2 = \omega_2/2\pi, \ldots$$

   and their harmonics

$$m_1 f_1, \ m_2 f_2, \ldots \qquad (m_1, m_2, \ldots \text{ positive integers}).$$

2. The quasiperiodic signal $x(t)$ depends nonlinearly on periodic functions. For example,

$$x(t) = \sin(2\pi f_1 t)\sin(2\pi f_2 t) = \frac{1}{2}\cos(|f_1 - f_2|2\pi t) - \frac{1}{2}\cos(|f_1 + f_2|2\pi t).$$

The fundamental frequencies are

$$|f_1 - f_2| \qquad \text{and} \qquad |f_1 + f_2|.$$

The harmonics are

$$m_1|f_1 - f_2| \qquad \text{and} \qquad m_2|f_1 + f_2|, \qquad m_1, m_2 \text{ positive integers.}$$

The nonlinear case requires more attention. In general, if $x(t)$ depends nonlinearly on r periodic functions, then the harmonics are

$$|m_1 f_1 + m_2 f_2 + \ldots + m_r f_r|, \qquad m_i \text{ arbitrary integers.}$$

In what follows, we specialize to $r = 2$ frequencies, and forget about finite Δf.

Each nonzero component of the spectrum of $x(\omega_1 t, \omega_2 t)$ is a peak at

$$f = |m_1 f_1 + m_2 f_2|, \qquad m_1, m_2 \text{ integers .}$$

There are two cases:

1. f_1/f_2 rational $\Rightarrow$ *sparse spectrum.*
2. f_1/f_2 irrational $\Rightarrow$ *dense spectrum.*

To understand this, rewrite f as

$$f = f_2 \left| m_1 \frac{f_1}{f_2} + m_2 \right|.$$

In the rational case,

$$\frac{f_1}{f_2} = \frac{\text{integer}}{\text{integer}}.$$

Then

$$\left| m_1 \frac{f_1}{f_2} + m_2 \right| = \left| \frac{\text{integer}}{f_2} + \text{integer} \right| = \text{ integer multiple of } \frac{1}{f_2}.$$

Thus the peaks of the spectrum must be separated (i.e., sparse).

Alternatively, if f_1/f_2 is irrational, then m_1 and m_2 may always be chosen so that

$$\left| m_1 \frac{f_1}{f_2} + m_2 \right| \text{ is not similarly restricted.}$$

These distinctions have further implications.

In the rational case,

$$\frac{f_1}{f_2} = \frac{n_1}{n_2}, \qquad n_1, n_2 \text{ integers.}$$

Since

$$\frac{n_1}{f_1} = \frac{n_2}{f_2}$$

the quasiperiodic function is *periodic* with period

$$T = n_1 T_1 = n_2 T_2.$$

All spectral peaks must then be harmonics of the fundamental frequency

$$f_0 = \frac{1}{T} = \frac{f_1}{n_1} = \frac{f_2}{n_2}.$$

Thus the rational quasiperiodic case is in fact periodic, and some writers restrict quasiperiodicity to the irrational case.

Note further that, in the irrational case, the signal never exactly repeats itself.

One may consider, as an example, the case of a child walking on a sidewalk, attempting with uniform steps to never step on a crack (and breaking his mother's back...).

Then if $x(t)$ were the distance from the closest crack at each step, it would only be possible to avoid stepping on a crack if the ratio

$$\frac{\text{step size}}{\text{crack width}}$$

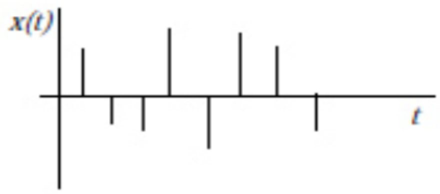
were rational.

Aperiodic signals

Aperiodic signals are neither periodic nor quasiperiodic.

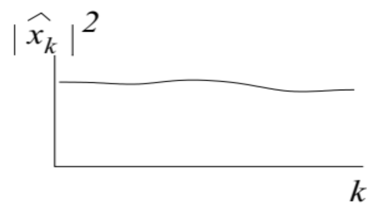
Aperiodic signals appear random, though they may have a deterministic foundation.

An example is white noise, which is a signal that is "new" and unpredictable at each instant, e.g.,



Statistically, each sample of a white-noise signal is independent of the others, and therefore uncorrelated to them.

The power spectrum of white noise is, on average, flat:



The flat spectrum of white noise is a consequence of its lack of harmonic structure (i.e., one cannot recognize any particular tone, or dominant frequency).

We proceed to derive the spectrum of a white noise signal $x(t)$.

Rather than considering only one white-noise signal, we consider an *ensemble* of such signals, i.e.,

$$x^{(1)}(t), x^{(2)}(t), \ldots$$

where the superscipt denotes the particular realization within the ensemble. Each realization is independent of the others.

Now discretize each signal so that

$$x_j = x(j\Delta t), \qquad j = 0, \ldots, n-1$$

We take the signal to have finite length n but consider the ensemble to contain an infinite number of realizations.

We use angle brackets to denote *ensemble averages*.

The ensemble-averaged mean of the jth sample is then

$$\langle x_j \rangle = \lim_{p\to\infty} \frac{1}{p} \sum_{i=1}^{p} x_j^{(i)}$$

Similarly, the mean-square value of the jth sample is

$$\left\langle x_j^2 \right\rangle = \lim_{p\to\infty} \frac{1}{p} \sum_{i=1}^{p} \left(x_j^{(i)} \right)^2$$

Now assume *stationarity*: $\langle x_j \rangle$ and $\left\langle x_j^2 \right\rangle$ are independent of j. We take these mean values to be $\langle x \rangle$ and $\left\langle x^2 \right\rangle$, respectively, and assume $\langle x \rangle = 0$.

Recall the autocorrelation ψ_m:

$$\psi_m = \sum_{j=0}^{n-1} x_j x_{j+m}.$$

By definition, each sample of white noise is uncorrelated with its past and future. Therefore

$$\begin{aligned} \langle \psi_m \rangle &= \left\langle \sum_j x_j x_{j+m} \right\rangle \\ &= n \left\langle x^2 \right\rangle \delta_m \end{aligned}$$

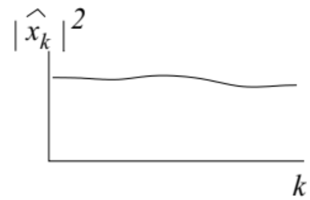
where

$$\delta_m = \begin{cases} 1 & m = 0 \\ 0 & \text{else} \end{cases}$$

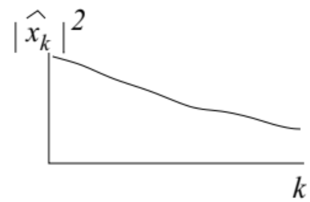
We obtain the power spectrum from the autocorrelation function by the Wiener-Khintchine theorem:

$$\begin{aligned} \left\langle |\hat{x}_k|^2 \right\rangle &= \sum_{m=0}^{n-1} \langle \psi_m \rangle \exp\left(-i \frac{2\pi mk}{n} \right) \\ &= \sum_{m=0}^{n-1} n \left\langle x^2 \right\rangle \delta_m \exp\left(-i \frac{2\pi mk}{n} \right) \\ &= n \left\langle x^2 \right\rangle \\ &= \text{constant.} \end{aligned}$$

Thus for white noise, the spectrum is indeed flat, as previously indicated:

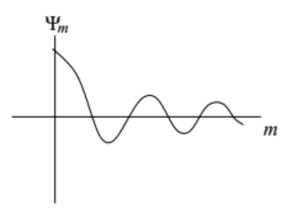


A more common case is "colored" noise: a continuous spectrum, but not constant:



In such (red) colored spectra, there is a relative lack of high frequencies. The signal is still apparently random, but only beyond some interval Δt.

The autocorrelation of colored noise is broader, e.g.,



1.7 Power spectrum of a random walk

Colored noise often has a power spectrum that decays like

$$\left\langle |\hat{x}_k|^2 \right\rangle \propto k^{-\beta}, \qquad \beta \simeq 2.$$

Here we show how the case $\beta = 2$ derives from a random walk.

Suppose that $\{x_j\}$ is a random walk. Then the increments

$$\eta_j = x_j - x_{j-1}$$

are all independent, so that the the autocorrelation

$$\begin{aligned}\langle \psi_m \rangle &= \sum_j \langle \eta_j \eta_{j+m} \rangle \\ &= n \left\langle \eta^2 \right\rangle \delta_m.\end{aligned}$$

Therefore the power spectrum of the increments is flat:

$$\left\langle |\hat{\eta}_k|^2 \right\rangle = \text{const.}$$

We can also calculate $\hat{\eta}_k$ from the Fourier transform of x_j:

$$\begin{aligned}\hat{\eta}_k &= \sum_j (x_j - x_{j-1}) \exp\left(-i\frac{2\pi jk}{n}\right) \\ &= \hat{x}_k - \sum_j x_j \exp\left(-i\frac{2\pi(j+1)k}{n}\right) \\ &= \hat{x}_k \left[1 - \exp\left(-i\frac{2\pi k}{n}\right)\right].\end{aligned}$$

where we have used the periodicity of x_j, i.e., $x_{j+n} = x_j$, in writing the summation in the second relation.

Squaring both sides above, we obtain the power spectrum

$$|\hat{\eta}_k|^2 = 2|\hat{x}_k|^2 \left[1 - \cos\left(\frac{2\pi k}{n}\right)\right].$$

Since

$$\cos x = 1 - \frac{x^2}{2} + \mathcal{O}(x^4)$$

we have, for $2\pi k \ll n$,

$$|\hat{\eta}_k|^2 \propto k^2 |\hat{x}_k|^2.$$

But we know that $\left\langle |\hat{\eta}_k|^2 \right\rangle = \text{const.}$ Therefore

$$\left\langle |\hat{x}_k|^2 \right\rangle \propto k^{-2}.$$

Thus when spectra decay like $1/k^2$, the underlying time series could be a random walk.

But beware: many other processes also give spectra that decay like $1/k^2$.

Identification of spectral peaks

Suppose you compute the DFT of a particular signal and identify a spectral peak. Is the peak real?

To answer this question, we need a null hypothesis and ask a specific question:

> *If a time series is composed of uncorrelated (white) noise, what is the probability of observing a spectral peak with a power greater than the power observed?*

Suppose that x_j is Gaussian white noise with zero mean and variance

$$\sigma^2 = \langle x_j^2 \rangle = \frac{1}{n} \sum_j x_j^2.$$

The DFT of x_j is

$$\begin{aligned} \hat{x}_k &= \sum_j x_j \exp\left(-i \frac{2\pi j k}{n}\right) \\ &\equiv a_k + i b_k. \end{aligned}$$

where

$$a_k = \sum_j x_j \cos\left(2\pi j k / n\right)$$

and

$$b_k = -\sum_j x_j \sin\left(2\pi j k / n\right)$$

Because the x_j are independent Gaussian random variables with zero mean, so too are a_k and b_k. The mean power spectrum S_k is

$$\begin{aligned} S_k = \langle |\hat{x}_k|^2 \rangle &= \langle a_k^2 + b_k^2 \rangle \\ &= \langle a_k^2 \rangle + \langle b_k^2 \rangle \\ &= n\sigma^2 \end{aligned}$$

For evenly sampled data the variance is equally shared so that

$$\langle a_k^2 \rangle = \langle b_k^2 \rangle = \frac{n\sigma^2}{2} \equiv s_0^2$$

Consequently the probability density functions of a_k and b_k are Gaussian with zero mean and variance s_0^2:

$$p(a) = \frac{1}{\sqrt{2\pi}s_0} e^{-a^2/2s_0^2}$$

$$p(b) = \frac{1}{\sqrt{2\pi}s_0} e^{-b^2/2s_0^2},$$

where we have dropped the index k since all k-components are identically distributed.

Because the random variables a and b are independent, the joint probability density function

$$\begin{aligned} p_{ab}(a,b) &= p(a)p(b) \\ &= \frac{1}{2\pi s_0^2} e^{-(a^2+b^2)/2s_0^2}. \end{aligned}$$

Now define the spectral power

$$\phi = a^2 + b^2.$$

The probability of observing a power $\leq \phi$ at any particular spectral index is given by the cumulative density function

$$P(\phi) = \int_{a^2+b^2 \leq \phi} p_{ab}(a,b)\mathrm{d}a\mathrm{d}b.$$

Set

$$a = r\cos\theta \quad \text{and} \quad b = r\sin\theta.$$

Then, integrating over r, from $r^2 = 0$ to $r^2 = \phi$,

$$\begin{aligned} P(\phi) &= \frac{1}{2\pi s_0^2} \int_0^{\sqrt{\phi}} 2\pi r \mathrm{d}r\, e^{-r^2/2s_0^2} \\ &= -\, e^{-r^2/2s_0^2}\Big|_0^{\sqrt{\phi}} \\ &= 1 - e^{-\phi/2s_0^2}. \end{aligned}$$

Recalling that $2s_0^2 = n\sigma^2$, we have

$$P(\phi; n, \sigma^2) = 1 - e^{-\phi/n\sigma^2}.$$

To see what this means, suppose you observe a spectral peak with power ϕ_0 in a time series of length n with a mean-square fluctuation of σ^2.

If the time series were Gaussian white noise with the same mean square fluctuation, the probability of observing a peak with power *greater* than ϕ_0 would be

$$\begin{aligned}\text{Prob}(\phi > \phi_0) &= 1 - P(\phi_0; n, \sigma^2) \\ &= e^{-\phi_0/n\sigma^2}.\end{aligned}$$

This quantity, often called the *p-value*, gives the *statistical significance* of the peak (lower p-values mean greater statistical significance).

As expected, observing a greater power ϕ_0 implies greater statistical significance.

But as the length n of the time series increases, we require a proportionately greater power to maintain the same statistical significance!

Moreover the probability of fluctuations drops off only exponentially with their size.

Thus large fluctuations are common in power spectra, and one must carefully interpret any spectral peak to be confident that it corresponds to a true periodic signal.

References

[1] Bracewell, R. N. *The Fourier Transform and its Applications* (McGraw Hill, New York, 1978).

[2] Muller, R. A. & Macdonald, G. J. *Ice Ages and Astronomical Causes* (Springer, New York, 2000).

[3] Bergé, P., Pomeau, Y. & Vidal, C. *Order within Chaos: Towards a Deterministic Approach to Turbulence* (John Wiley and Sons, New York, 1984).

[4] Emerson, S. R. & Hedges, J. I. *Chemical Oceanography and the Marine Carbon Cycle* (Cambridge, New York, 2008).

[5] Petit, J. *et al.* Climate and atmospheric history of the past 420,000 years from the Vostok ice core, Antarctica. *Nature* **399**, 429–436 (1999).

[6] Shannon, C. E. Communiction in the presence of noise. *Proceedings of the IRE* **37**, 10–21 (1949).

[7] Scargle, J. D. Studies in astronomical time series analysis. II - Statistical aspects of spectral analysis of unevenly spaced data. *Astrophysical Journal* **263**, 835–853 (1982).