Statistical Digital Signal Processing

Week 1 Overview of Random Process

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Course Description

- Covers key topics with major applications in:
  - ✓ Digital communication system
- Comprehensive analysis of signal processing algorithms for:
  - ✓ Modeling discrete-time deterministic and random signals
  - ✓ Optimum filter design
  - ✓ Power spectrum estimation of random signals
  - ✓ Implementation of adaptive filters
- Applications addressed::
  - ✓ Estimation of signals from noisy observations
  - ✓ Single-step and multi-step linear prediction
  - ✓ Deconvolution
  - ✓ Noise cancellation

Lecture Learning Outcomes

1. Explain the fundamental concepts of random signals and their importance in communication and signal processing systems.
2. Analyze and compute statistical properties of random variables, including probability distributions, expectation, variance, and correlation.
3. Describe random processes, including ensemble properties, time averages, and the concept of stationarity..
4. Evaluate the effect of linear filtering on stationary random processes, including the relationship between input and output autocorrelation and power spectral density.
5. Apply spectral factorization techniques to power spectral density functions

Week 1: Overview of Random Process

Outline

- Introduction: Random Signals
- Random Variables
- Random Process
- Filtering of Stationary Random Process
- Spectral Factorization

Introduction: Random Signals

- Signals can be broadly categorized as ***Deterministic*** and ***Random Signals***
- **Deterministic Signal:** A signal whose values are completely determined by a known mathematical function/formula
- **Random Signal:** signal whose values cannot be predicted exactly and are described using probability or by its statistical properties
- Random signals are often appeared in real life:

Example:

- ✓ The sequence of bit 1 and 0 in communication system
- ✓ Noise in communication system
- ✓ Packet arrival times at a node in a communication network

- Most of the signal associated with communication system are random signals which are the sequence of random variables indexed by time.

Random Variables

Definition: A random variable X is a function that assigns a real number to each possible outcome of a random experiment.

Classes of Random Variables:

I. **Discrete random variables:** X is a discrete random variable, if its range is countable or can take countable distinct values.

II. **Continuous random variables:** X is continuous random variable, if it has a range in the form of an interval or a union of non-overlapping intervals

- There are also **Mixed Random Variables** which are the mixture of both discrete and continuous random variables
- Instead of the exact value, a random variable is defined by its statistical characteristics—most notably by its **probability distribution.**

Random Variables

- Cumulative Distribution Function (CDF): The total probability accumulated up to the value x.

$$F_X(x) = \Pr\{X \le x\} \qquad (1)$$

- Probability Density Function: Given by the derivative of CDF

$$f_X(x) = \frac{d}{dx}\left(F_X(x)\right) \qquad (2)$$

- Mean: is the expected or average value that the random variable takes

$$m_X = E\{X\} = \int_{-\infty}^{\infty} x f_X(x) dx \qquad (3)$$

- Variance: Variance measures how much the values of a random variable deviate from the mean (average).

$$\sigma_X^2 = E\{(X - m_X)^2\} = \int_{-\infty}^{\infty} (x - m_X)^2 f_X(x) dx \qquad (4)$$

Random Variables

Second Moment: the average value of the square of the random variable.

$$E\{X^2\} = \int_{-\infty}^{\infty} x^2 f_X(x)dx = \sigma_X^2 + m_X \qquad (5)$$

- Joint Cumulative Distribution Function: The probability of two or more random variables occurring simultaneously.

$$F_{X,Y}(x,y) = \Pr\{X \le x, Y \le y\} \qquad (6)$$

- Joint Probability Density Function:

$$f_{X,Y}(x,y) = \frac{d^2}{\partial x \partial y} F_{X,Y}(x,y) \qquad (7)$$

- Correlation: how strongly random variables related to each other

$$r_{XY} = E\{XY^*\} \qquad (8)$$

Random Variables

- Covariance: Covariance measures how two random variables vary together

$$c_{X,Y}(x,y) = \mathrm{Cov}(x,y) = E\left\{(X - m_X)(Y - m_Y)^*\right\} = r_{XY} - m_X m_Y^* \qquad (9)$$

- Random variables X and Y are independent if the value of one does not affect the probability distribution of the other.:

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) \qquad (10)$$

- Random variables X and Y are uncorrelated if $r_{XY} = m_X m_Y^*$ OR $c_{XY} = 0$

- Random variables X and Y are Orthogonal if:

$$r_{XY} = 0 \qquad (11)$$

Discrete Random Variable Distribution

Bernoulli Distribution

- A Bernoulli random variable is a random variable that can only take two possible values, usually 0 and 1
- . Bernoulli random variables are used to model experiments having only two outcomes:

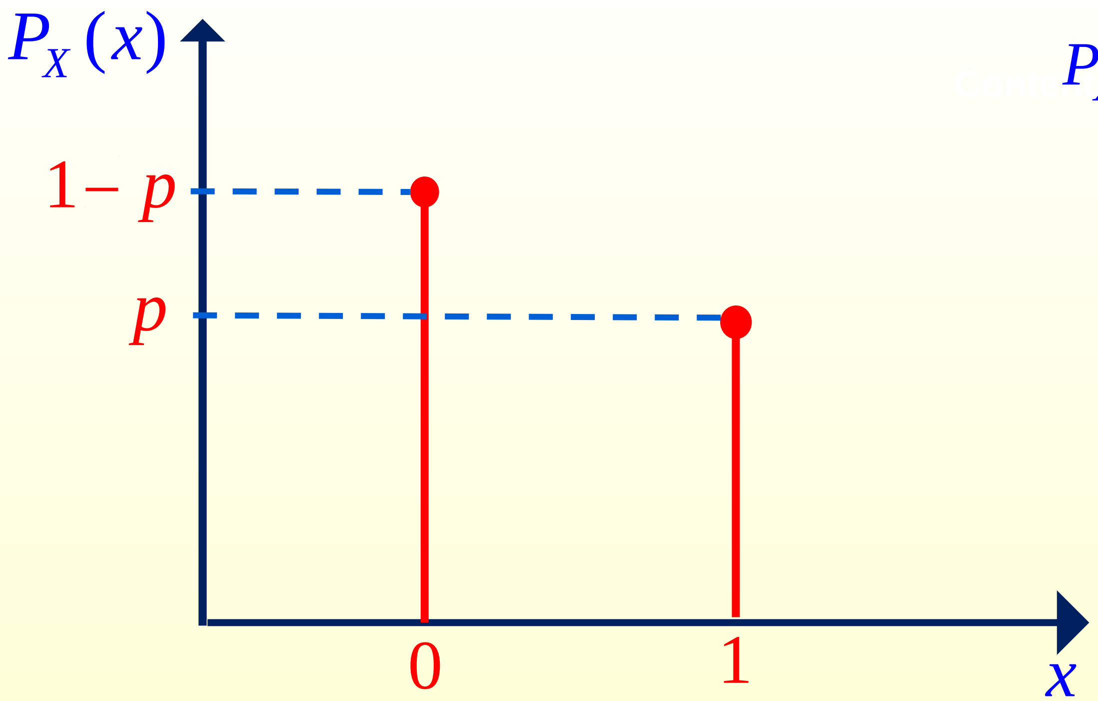
Example:

- Pass or fail in exams
- Lose or win in final cup soccer match
- Head or tail while tossing a coin
- A user is moving or not moving in cellular mobile service area

Discrete Random Variable Distribution

Bernoulli Distribution Cont.......

- A Bernoulli random variable has the following probability mass function:



$$P_X(x) = \begin{cases} p & for\ x = 1 \\ 1-p & for\ x = 0 \\ 0 & Otherwise \end{cases} \qquad (12)$$

Where:

$P_X(x)$ Is probability mass function (PMF)

Figure 1: Probability mass function of Bernoulli random variable

Discrete Random Variable Distribution

Poisson Distribution

- Poisson distribution used to model the number of occurrences of certain events in an interval of time or space.

  Example:

  - To count the number of packets received at the router port in 1 hour
  - To count the number of users interred in certain cellular network cell service area

- Poisson distribution has the following Probability mass function:

$$P_X(x) = \frac{e^{-\lambda}\lambda^x}{x!} \qquad (13)$$

Where: λ is the average occurrence per unit time

Discrete Random Variable Distribution

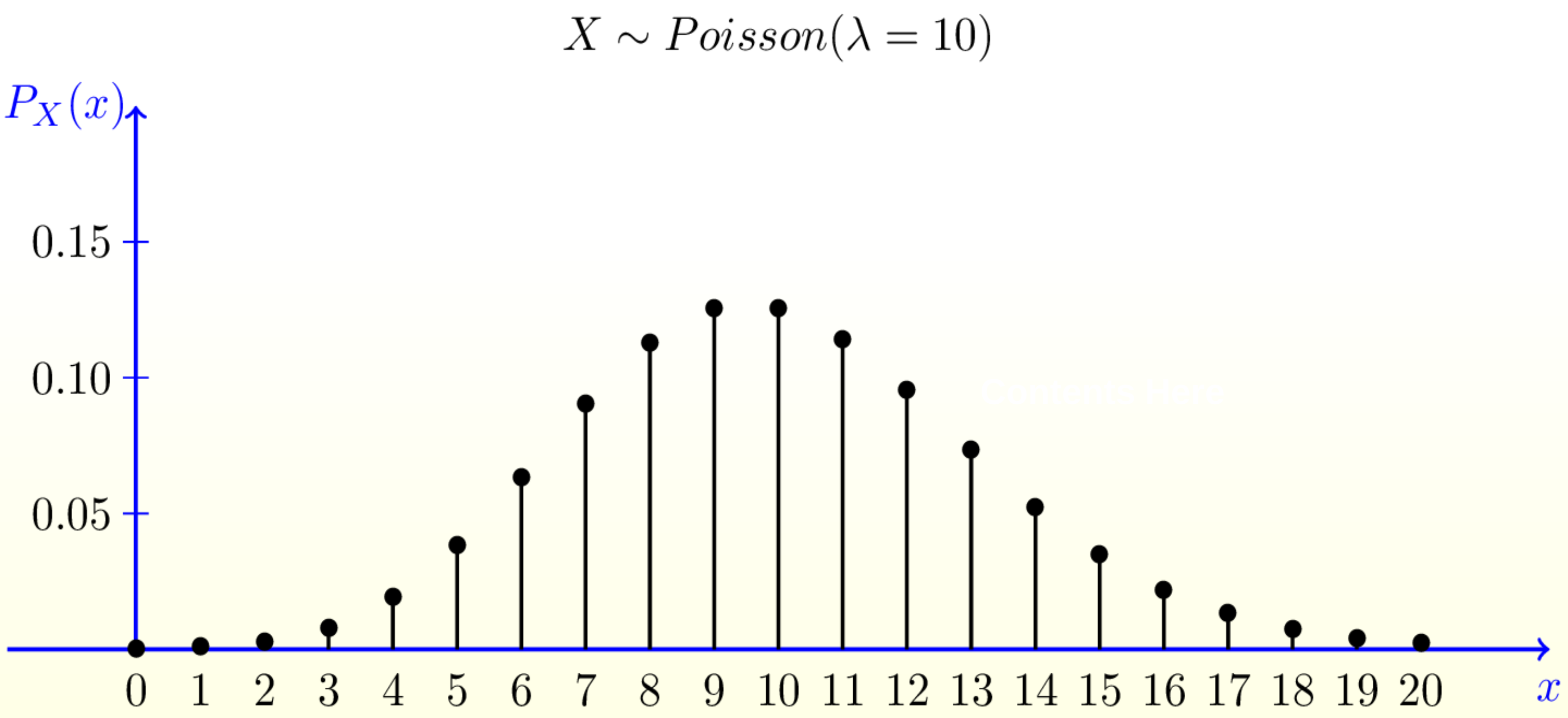


Figure 2: Probability mass function of Poisson random variable

Source: *"Explore Random and Stochastic Signals,"* Gray Sparrow. https://custom-images.strikinglycdn.com/res/hrscywv4p/image/upload/c_limit,fl_lossy,h_9000,w_1200,f_auto,q_auto/24525391/859028_261800.png

Continuous Random Variable Distribution

Uniform Distribution

- If a random variable X has uniform distribution in the interval $[a,b]$, its Probability density function is as follows:

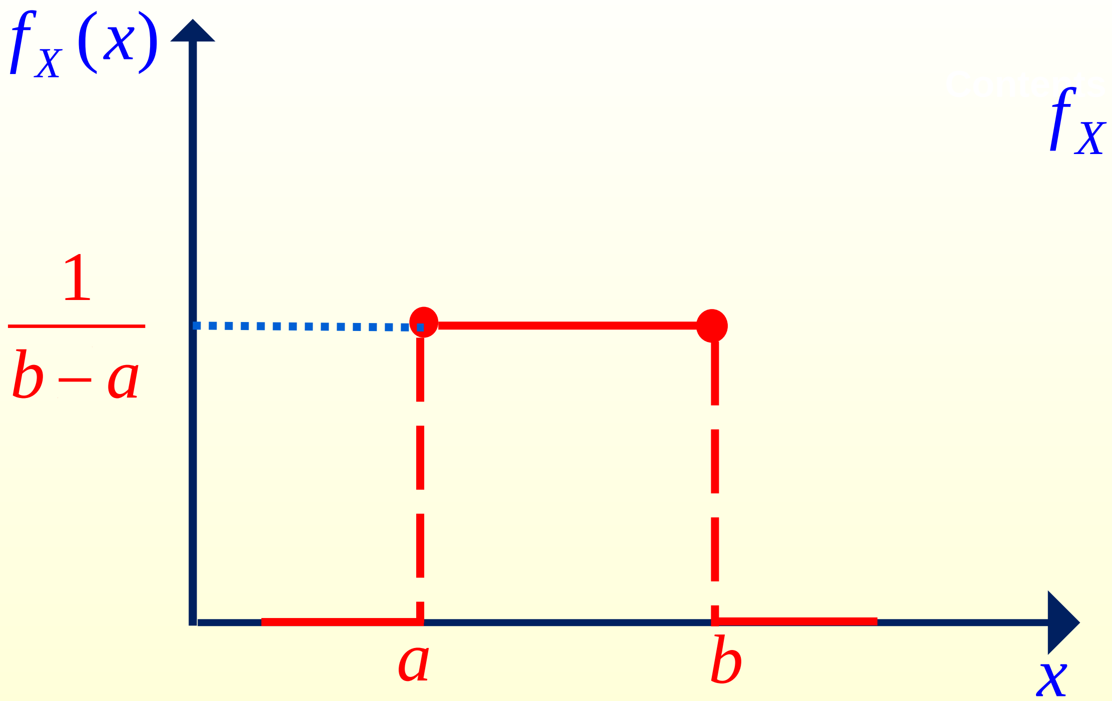


Figure 3: Probability density function of random variable having uniform Distribution

$$f_X(x) = \begin{cases} \frac{1}{b-a} & for\ a \le x \le b \\ 0 & Otherwise \end{cases} \qquad (14)$$

Continuous Random Variable Distribution

Normal (Gaussian) Distribution

- If X a Gaussian random variable with mean μ and variance σ^2

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2} \right) \qquad (15)$$

- Gaussian random variables are important to model random signals in real life as the sum of many random variables by applying the virtue of **Central Limit Theorem.**
- Noise in communication systems are widely modeled in terms of Gaussian random variables

Example:

- ✓ Additive White Gaussian Noise (AWGN) Channel
- ✓ Thermal Noise in Receivers
- ✓ Interference (sum of many independent interfering signals)

Continuous Random Variable Distribution

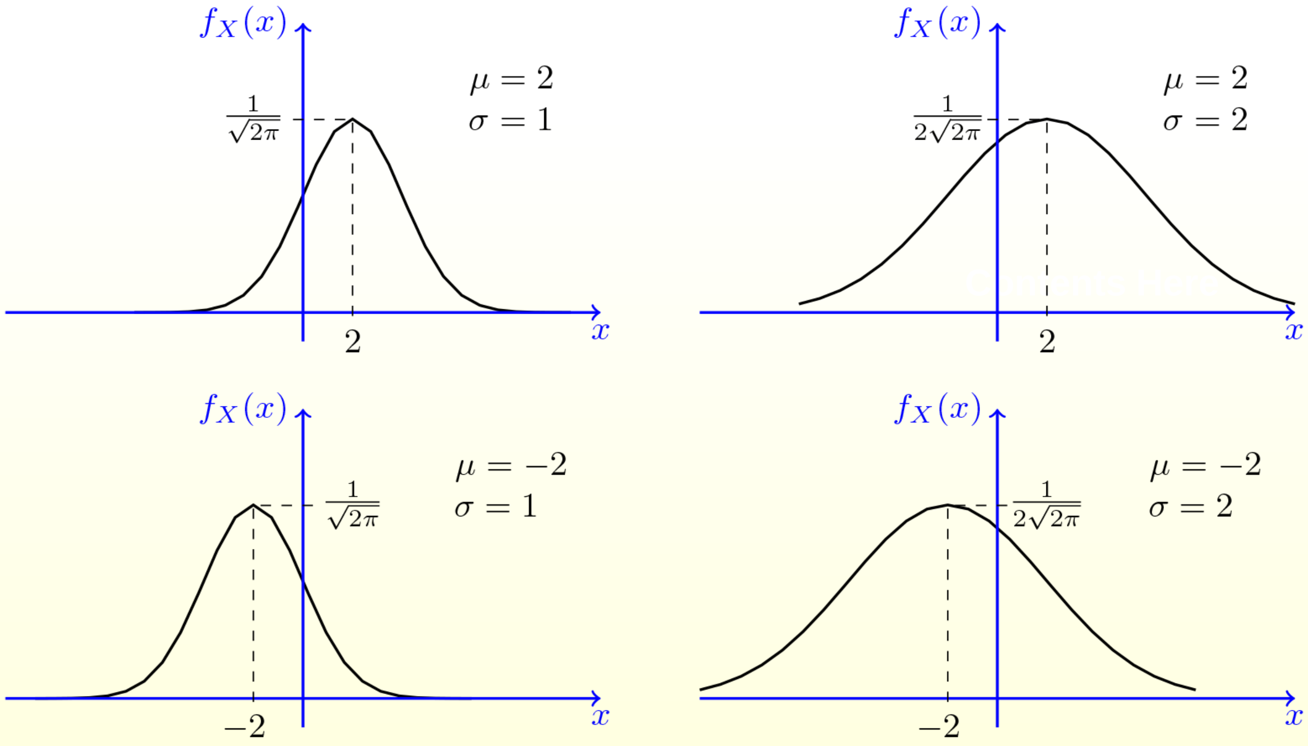


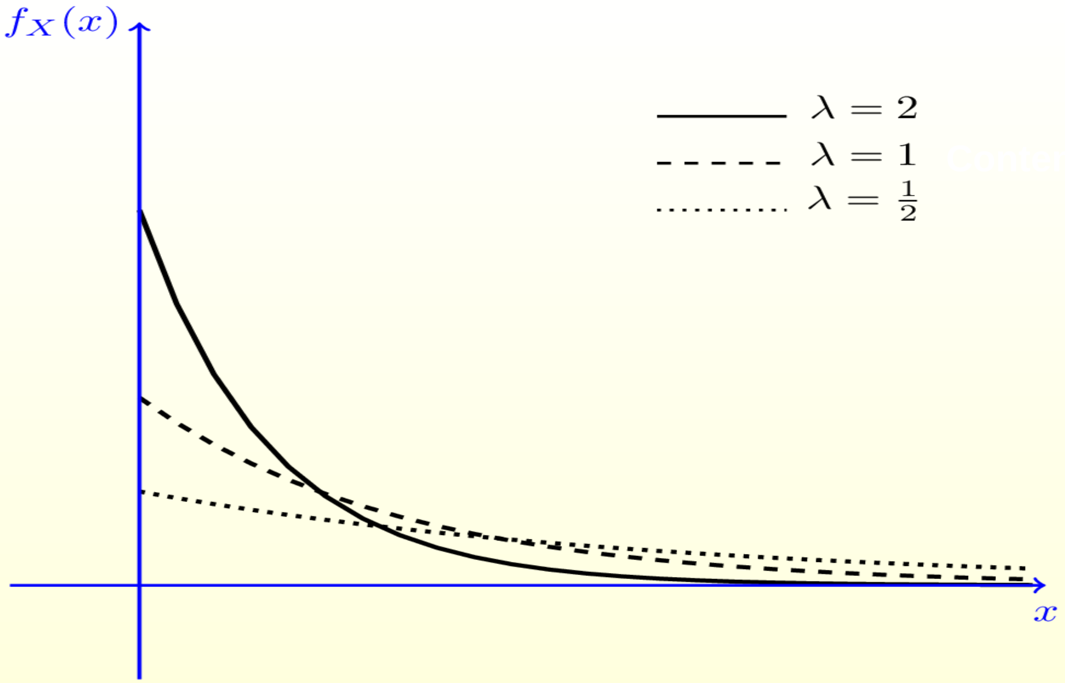
Figure 4: Probability density function of Gaussian distribution

Source: *"Explore Random and Stochastic Signals,"* Gray Sparrow. https://custom-images.strikinglycdn.com/res/hrscywv4p/image/upload/c_limit,fl_lossy,h_9000,w_1200,f_auto,q_auto/24525391/428986_449752.png

Continuous Random Variable Distribution

Exponential Distribution: describe the time between occurrence of events

- If random variable X has exponential distribution with parameter $\lambda > 0$



$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x > 0 \\ 0 & \text{Otherwise} \end{cases} \quad (16)$$

Figure 5: Probability density function of exponential distribution

Source: *"Explore Random and Stochastic Signals,"* Gray Sparrow. https://custom-images.strikinglycdn.com/res/hrscywv4p/image/upload/c_limit,fl_lossy,h_9000,w_1200,f_auto,q_auto/24525391/386277_493869.png

Random Process

Definition: A Random process X(n) is the sequence of indexed random variables

- Mean $(m_X(n))$ and Variance:

$$m_X(n) = E\{X(n)\} \quad (17)$$

$$\sigma_X^2(n) = E\{(X(n) - m_X(n))^2\} \quad (18)$$

- Autocorrelation and Cross-correlation:

$$r_X(l,m) = E\{X(l)X(m)^*\} \quad (19)$$

$$r_{X,Y}(l,m) = E\{X(l)\mathrm{Y}(m)^*\} \quad (20)$$

Random Process

Stationarity: It describes whether the statistical properties of a process change over time.

- First order stationarity:

$$f_{X(n)}(x) = f_{X(n+l)}(x) \quad (21)$$

$$m_X(n) = m_X(0) = m_X \quad (22)$$

- Second order stationarity:

$$r_X(l,m) = r_X(l-m) \quad (23)$$

- Wide Sense Staionarity (WSS) if:

$$m_X(n) = m_X(0) = m_X \quad (24)$$

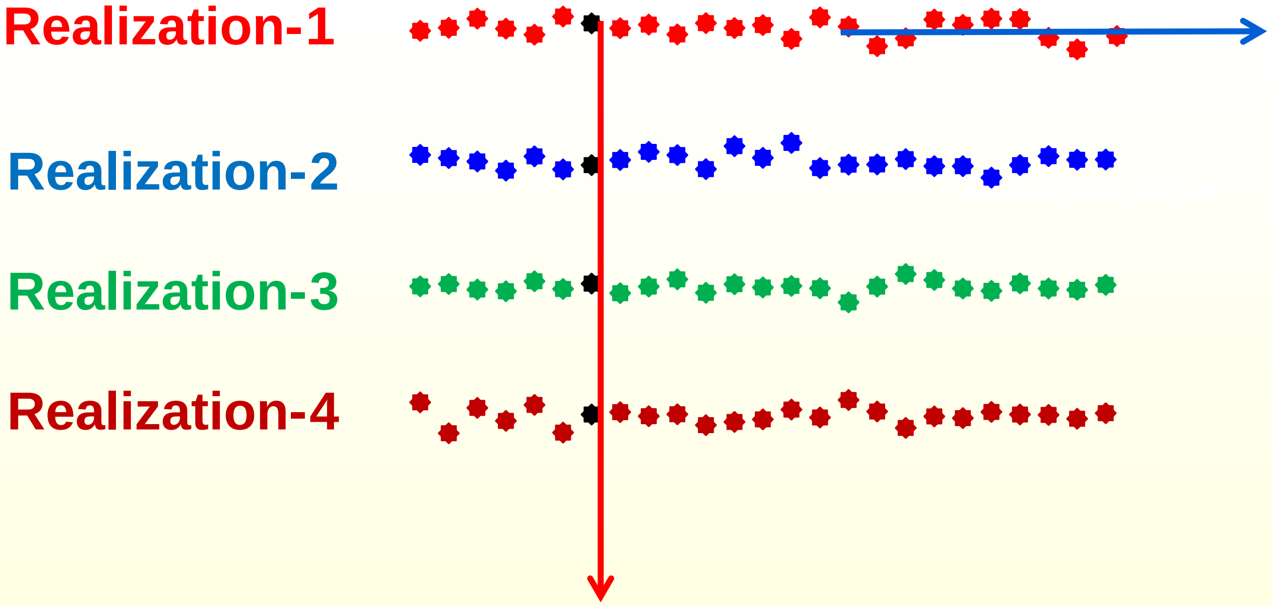
$$r_X(l,m) = r_X(l-m) \quad (25)$$

$$C_X(0) = r_X(0) - m_X m_X^* < \infty \quad (26)$$

Where: C_X is the covariance

Random Process

Sample Mean (Time Average) and Ensemble Mean (Average



Sample Mean:

$$\hat{m}_X(N) = \frac{1}{N}\sum_{n=1}^{N} X(n)$$

Ensemble Mean: $E\{X(n)\}$

Random Process

Ergodicity

- A WSS random process is ergodic if the sample mean and ensemble mean are equal considering long enough samples:

$$m_X = \lim_{N \to \infty} \hat{m}(N) \qquad (27)$$

Power spectrum

- double-sided z-transform of autocorrelation function:

$$P_X(z) = \sum_{k=-\infty}^{\infty} r_X(k) z^{-k} \qquad (28)$$

Random Process

Power spectrum…

- Assuming the autocorrelation is stable, we can write the Power spectrum as the DTFT of the autocorrelation function by substituting z by $e^{j\omega}$:

$$P_X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} r_X(k)e^{-j\omega k} \qquad (29)$$

- $P_X(e^{j\omega})$ is interpreted as the distribution of power of the signal X(n) over frequency
- From the inverse Z-transform, $r_X(k)$ can be recovered as:

$$r_X(k) = \oint_{u.c.} P_X(z)z^k \frac{dz}{2\pi jz}$$

Substituting

$$z = e^{j\omega} \rightarrow \frac{dz}{jz} = d\omega$$

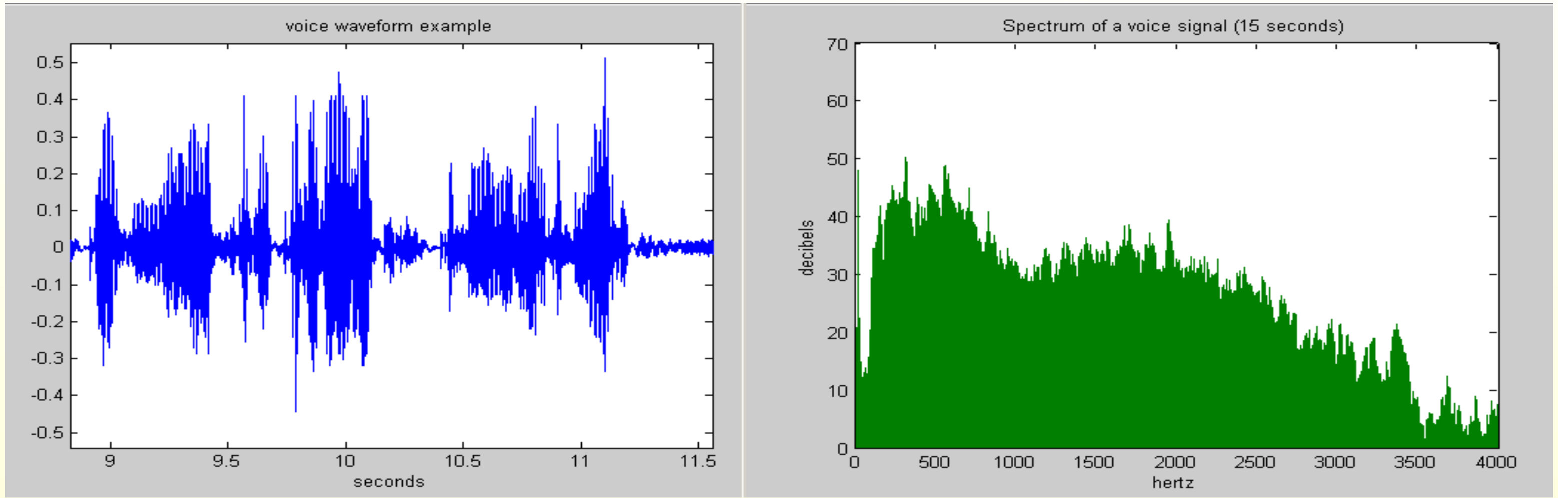
$$r_X(k) = \int_{-\pi}^{\pi} P_X(e^{j\omega})e^{jk\omega} \frac{d\omega}{2\pi} \qquad (30)$$

Random Process

Power spectrum…

- For Zero mean real valued WSS random process, the variance can be written as:

$$\sigma_X^2 = E\{X(n)^2\} = E\{X(n)X(n)\} = r_X(0) = \int_{-\pi}^{\pi} P_X(e^{j\omega})\frac{d\omega}{2\pi} \qquad (31)$$



(a) (b)

Figure 6: (a) The voice waveform over time (b) power spectrum of the voice

Source: *"Spectral density,"* Wikipedia, The Free Encyclopedia. https://en.wikipedia.org/wiki/Spectral_density#/media/File:Voice_waveform_and_spectrum.png

Random Process

Cross Power spectral Density: describes the frequency-domain relationship between two random signals.

The Cross power spectral density of two random signal X(n) and Y(n) is given by:

$$P_{XY}(z) = \sum_{k=-\infty}^{\infty} r_{XY}(k) z^{-k} \qquad (32)$$

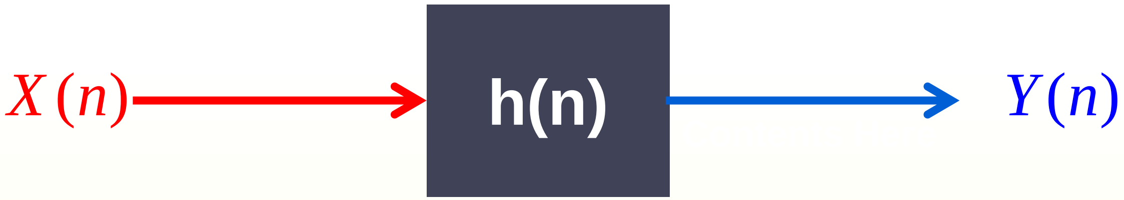
- Assuming the cross-correlation is stable, we can write the cross Power spectrum as the DTFT of the across-correlation function:

$$P_{XY}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} r_{XY}(k) e^{-j\omega k} \qquad (33)$$

- Cross power spectrum reveals at which frequencies two signals have high degree of correlations.

Filtering of Stationary Random Process

- Our objective is to drive the relationship of input and output signal in terms autocorrelation:



- The system function of the filter can be written:

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n} \qquad (34)$$

- Considering the filter is linear shift invariant (LSI):

$$Y(z) = H(z) X(z) \qquad (35)$$

Where:

$$Y(z) = \sum_{n=-\infty}^{\infty} Y(n) z^{-n} \quad \text{and} \quad X(z) = \sum_{n=-\infty}^{\infty} X(n) z^{-n}$$

Filtering of Stationary Random Process

- The Power spectrum of the output signal can be written [1]:

$$P_Y(z) = \sum_{k=-\infty}^{\infty} r_Y(k) z^{-k} = Y(z) Y^*(\frac{1}{z^*})$$
$$= H(z) X(z) H^*(\frac{1}{z^*}) X^*(\frac{1}{z^*})$$
$$= H(z) H^*(\frac{1}{z^*}) X(z) X^*(\frac{1}{z^*})$$
$$= P_H(z) P_X(z) \qquad (36)$$

- Setting $z = e^{j\omega}$, considering the autocorrelation is stable, we will have:

$$P_Y(e^{j\omega}) = H(e^{j\omega}) H^*(e^{j\omega}) P_X(e^{j\omega}) = |H(e^{j\omega})|^2 \; P_X(e^{j\omega}) \qquad (37)$$

- Applying Inverse Z transform both side on the above equation:

$$z^{-1}\{P_Y(z)\} = z^{-1}\{P_H(z) P_X(z)\} \qquad (38)$$

Filtering of Stationary Random Process

- The above equation can be written in terms of autocorrelation functions as:

$$r_Y(k) = r_h(k) * r_X(k)$$
$$= \sum_{m=-\infty}^{\infty} r_h(\mathrm{m}) r_X(k-m) \qquad (39)$$

- $r_h(k)$ is the autocorrelation of the filter impulse response h(n) which is deterministic:

$$r_h(k) = \sum_{n=-\infty}^{\infty} h(n)h(n+k)^* \qquad (40)$$

- The average power of the output signal y(n) can be written :

$$E\{y(n)y(n)^*\} = E\{|\,y(n)\,|^2\} = r_Y(0) \qquad (41)$$

Filtering of Stationary Random Process

Cross-Power Spectral density

- The cross-power spectrum of the two signals at the input and output of the LSI filter:

$$P_{YX}(z) = Y(z)X^*(\frac{1}{z^*}) = H(z)X(z)X^*(\frac{1}{z^*}) = H(z)P_X(z) \quad (42)$$

$$P_{XY}(z) = Y^*(\frac{1}{z^*})X(z) = H^*(\frac{1}{z^*})X^*(\frac{1}{z^*})X(z) = H^*(\frac{1}{z^*})P_X(z) \quad (43)$$

- Setting $z = e^{j\omega}$, we will have:

$$P_{YX}(e^{j\omega}) = H(e^{j\omega})P_X(e^{j\omega}) \quad (44)$$

$$P_{XY}(e^{j\omega}) = H^*(e^{j\omega})P_X(e^{j\omega}) \quad (45)$$

Filtering of Stationary Random Process

White Noise

- White noise x(n) is a random signal having the following autocorrelation function:

$$r_x(k) = \begin{cases} \sigma_x^2, & \text{for } k = 0 \\ 0, & Otherwise \end{cases} = \sigma_x^2 \delta(k) \qquad (46)$$

- When the input to the filter is white noise, the output signal power spectrum becomes:

$$P_Y(z) = P_H(z) P_x(z) = P_H(z) \sigma_x^2 \qquad (47)$$

$$P_{YX}(z) = H(z) P_X(z) = H(z) \sigma_X^2 \qquad (48)$$

Filtering of Stationary Random Process

White Noise

- In frequency domain:

$$P_Y(e^{j\omega}) = |H(e^{j\omega})|^2 \sigma_X^2 \quad (49)$$

$$P_{YX}(e^{j\omega}) = H(e^{j\omega})\sigma_X^2 \quad (50)$$

- The autocorrelation of the output signal is also:

$$r_Y(k) = \sigma_X^2 \sum_{n=-\infty}^{\infty} h(n)h(n+k)^* \quad (51)$$

$$r_{YX}(k) = h(k)\sigma_X^2 \quad (52)$$

Filtering of Stationary Random Process

Unknown System Identification by Cross-correlation Method

- Impulse response, h(k), of unknown system can be identified on the basis of input/output measurements

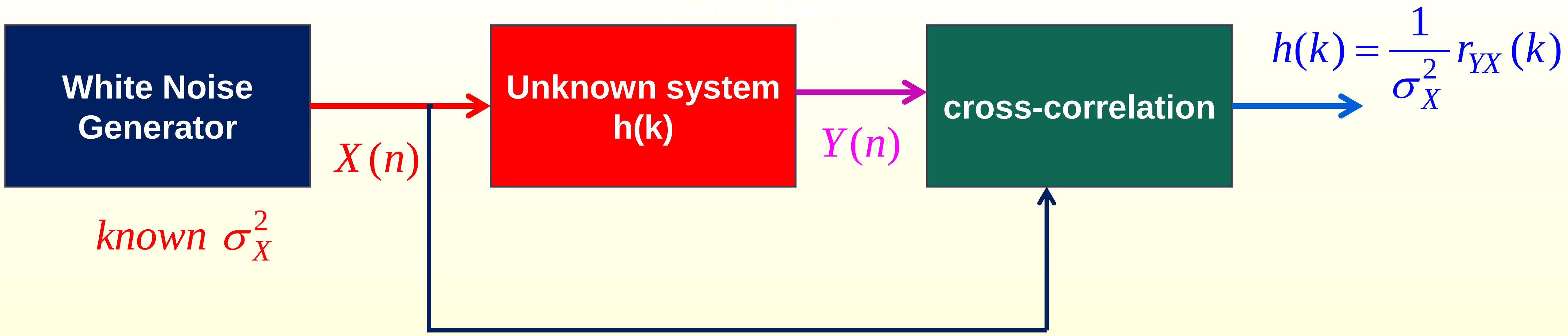


Figure 7: Unknown system identification

Spectral Factorization

- When the power spectrum $P_X(e^{j\omega})$ of a WSS process is continuous function of ω, $P_X(z)$ can be factorized in the following form [2]:

$$P_X(z) = \sum_{k=-\infty}^{\infty} r_X(k) z^{-k} = \sigma_0^2 Q(z) Q^*(1/z^*) \quad (53)$$

Known as spectral factorization of $P_X(z)$

Where: $\sigma_0^2 = \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln P_X(e^{j\omega}) d\omega \right\}$ and $Q(z) = \exp\left\{ \sum_{k=1}^{\infty} c(k) z^{-k} \right\}$

$$Q^*(1/z^*) = \exp\left\{ \sum_{k=-\infty}^{-1} c(k) z^{-k} \right\}$$

$$c(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln P_X(e^{j\omega}) e^{jk\omega} d\omega$$

Summary

- **Random Variables:**
  - ✓ Continuous Random Variables
  - ✓ Discrete Random Variables
- **Random Process:** Indexed random variables
  - ✓ **Non Stationary:** their statistics changing with time
  - ✓ **Stationary:** 1^{st} order**,** 2^{nd} order stationarity, and WSS
- **Power Spectrum:** Signal power distribution across frequency
  - ✓ Z-transform of autocorrelation function
- **Linear Filtering of Stationary Random Signal:**
  - ✓ Input-output relation in terms of power spectrum
  - ✓ **Application Example:** Unknown system identification
- **Spectral Factorization:** Representation of WSS signal power spectrum in Factorized form

References

[1] Sophocles J. Orfanidis,‘’ Optimum Signal Processing: An Introduction’’, McGraw-Hill Publishing Company, Pp.72-73, 2007.

[2] Monson H. Hayes, *“Statistical Digital Signal Processing and Modeling”,* John Wiley and sons, Pp.104-105, 1996.

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Thank You!