Statistical Digital Signal Processing

Week 3 Deterministic Signal Modeling (Part-2)

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Previous Topic (Week-2)

Deterministic Signal Modeling (Part-1)

- Basics of Signal modeling
- Parametric and Non-Parametric Signal Modeling
- Motivations for Parametric Signal Modeling
- Parametric Signal Modeling Steps
- Deterministic Signal Modeling using Linear Shift Invariant Filter
- Least Square or Direct Method
- Padé Approximation

Lecture Learning Outcomes

1. Explain the fundamental principles of Prony's Method and its role in parametric signal modeling.
2. Apply Prony's Method to develop pole-zero models for representing deterministic discrete-time signals and systems..
3. Analyze and interpret modeling errors associated with Prony's Method and evaluate their impact on model accuracy.
4. Implement Shank's Method for improved rational approximation of signals and compare it with Prony's approach.
5. Assess and select appropriate modeling techniques (Prony vs. Shank) based on signal characteristics and application requirements.

Week 3: Deterministic Signal Modeling (Part-2)

Outline

- Prony's Method: Introduction
- Prony's Method: Pole-Zero Modeling
- Prony's Method Modeling Error
- Shank's Method

Prony’s Method: Introduction

- **Limitation Of Pade Approximation:** it can not give guaranty for the minimization of modeling error for data values out of [0, p+q] interval
- This limitation comes because the data values out of [0, p+q] interval are not included in the modeling process
- Prony’s model relaxes the requirement for an exact fit over the interval [0, p+q], allowing a more accurate approximation of the signal
- Similar to the Pade approximation, Prony’s method involves solving of linear equations
- We will begin by general problem of Pole-Zero modeling using Prony's method
- Then, We will proceed to Shank’s model which is the modification of Prony's method

Prony's Method: Pole-Zero Modeling

- The signal $x(n)$ will be model using linear shift invariant filter having p poles q zeros:

$$H(z) = \frac{B_q(z)}{A_p(z)} \qquad (1)$$

- Modeling $x(n)$ as impulse response of the filter, the modeling error $E'(z)$ becomes :

$$E'(z) = X(z) - \frac{B_q(z)}{A_p(z)} \qquad (2)$$

- Multiplying both side of the above equation by

$$A_p(z)E'(z) = A_p(z)X(z) - B_q(z) \qquad (3)$$

- Denoting the left side of the above equation by new error $E(z)$

$$E(z) = A_p(z)X(z) - B_q(z) \qquad (4)$$

Prony’s Method: Pole-Zero Modeling

- In time domain, the previous equation can be written as [1]:

$$e(n) = a_p(n) * x(n) - b_q(n) = \hat{b}_q(n) - b_q(n) \qquad (5)$$

Filtered signal using a filter having unit sample response of $a_p(n)$

- Eq(5) can be written in the form:

$$e(n) = x(n) + \sum_{l=1}^{p} a_p(l)x(n-l) - b_q(n) \qquad (6)$$

Prony's Method: Pole-Zero Modeling

- Since $b_q(n) = 0$ for $n > q$, we will have:

$$e(n) = \begin{cases} x(n) + \sum_{l=1}^{p} a_p(l)x(n-l) - b_q(n), & for\ n = 0,1,2,\ldots,q \\ x(n) + \sum_{l=1}^{p} a_p(l)x(n-l), & for\ n > q \end{cases} \quad (7)$$

- Prony's method begin by finding $a_p(k)$ which minimizes the squared error $\varepsilon_{p,q}$ as follows:

$$\varepsilon_{p,q} = \sum_{n=q+1}^{\infty} |e(n)|^2 = \sum_{n=q+1}^{\infty} \left| x(n) + \sum_{l=1}^{p} a_p(l)x(n-l) \right|^2 \quad (8)$$

Prony's Method: Pole-Zero Modeling

- Since $\varepsilon_{p,q}$ depends only on $a_p(k)$, the coefficient that minimizes $\varepsilon_{p,q}$ can be found from the following partial derivative:

$$\frac{\partial \varepsilon_{p,q}}{\partial a_p^*(k)} = \sum_{n=q+1}^{\infty} \frac{\partial |e(n)|^2}{\partial a_p^*(k)} = \sum_{n=q+1}^{\infty} \frac{\partial [e(n)e^*(n)]}{\partial a_p^*(k)} = \sum_{n=q+1}^{\infty} e(n)\frac{\partial e^*(n)}{\partial a_p^*(k)} = 0; \quad k = 1, 2, \ldots, p \qquad (9)$$

- From eq(7), for particular value of $l = k$, the term $e^*(n)$ can be written as:

$$e^*(n) = x^*(n) + a_p^*(k)x^*(n-k) \qquad (10)$$

- Taking the partial derivative of eq(10):

$$\frac{\partial e^*(n)}{\partial a_p^*(k)} = \frac{\partial [x^*(n) + a_p^*(k)x^*(n-k)]}{\partial a_p^*(k)} = x^*(n-k) \qquad (11)$$

Prony's Method: Pole-Zero Modeling

- From eq(9) and eq(11), we have

$$\sum_{n=q+1}^{\infty} e(n)x^*(n-k) = 0; \qquad (12)$$

- Writing $e(n)$ in the form given in eq(7) for $\rightarrow n > q$:

$$\sum_{n=q+1}^{\infty} [x(n) + \sum_{l=1}^{p} a_p(l)x(n-l)]x^*(n-k) = 0 \qquad (13)$$

- Equivalently, eq(13) can be written in the form:

$$\sum_{n=q+1}^{\infty} x(n)x^*(n-k) + \sum_{l=1}^{p} a_p(l) \sum_{n=q+1}^{\infty} x(n-l)x^*(n-k) = 0 \qquad (14)$$

Prony's Method: Pole-Zero Modeling

- From eq(14), we have

$$\sum_{l=1}^{p} a_p(l) \sum_{n=q+1}^{\infty} x(n-l)x^*(n-k) = -\sum_{n=q+1}^{\infty} x(n)x^*(n-k) \qquad (15)$$

- In order to simplify eq(15), lets define:

$$r_x(k,l) = \sum_{n=q+1}^{\infty} x(n-l)x^*(n-k) \qquad (16)$$

Where: $r_x(k,l) = r_x^*(l,k)$

- Therefore, eq(15) turns out to be:

$$\sum_{l=1}^{p} a_p(l) r_x(k,l) = -r_x(k,0) \; ; \quad k = 1,2,\ldots,p \qquad (17)$$

Prony's Method: Pole-Zero Modeling

- Eq(17) can be written in matrix form as:

$$\underbrace{\begin{bmatrix} r_x(1,1) & r_x(1,2) & r_x(1,3)\cdots & r_x(1,p) \\ r_x(2,1) & r_x(2,2) & r_x(2,3)\cdots & r_x(2,p) \\ r_x(3,1) & r_x(3,2) & r_x(3,3)\cdots & r_x(3,p) \\ \vdots & \vdots & \vdots & \vdots \\ r_x(p,1) & r_x(p,2) & r_x(p,3)\cdots & r_x(p,p) \end{bmatrix}}_{\mathbf{R}_x} \underbrace{\begin{bmatrix} a_p(1) \\ a_p(2) \\ a_p(3) \\ \vdots \\ a_p(p) \end{bmatrix}}_{\overline{\mathbf{a}}_p} = - \underbrace{\begin{bmatrix} r_x(1,0) \\ r_x(2,0) \\ r_x(3,0) \\ \vdots \\ r_x(p,0) \end{bmatrix}}_{-\mathbf{r}_x} \qquad (18)$$

- Eq(18) can be written concisely as:

$$\mathbf{R}_x \overline{\mathbf{a}}_p = -\mathbf{r}_x \qquad (19)$$

Prony's Method: Modeling Error

- The minimum value of the modeling error $(\varepsilon_{p,q})$ is given by:

$$\varepsilon_{p,q} = \sum_{n=q+1}^{\infty} |e(n)|^2 = \sum_{n=q+1}^{\infty} e(n)e^*(n) = \sum_{n=q+1}^{\infty} e(n)[x(n) + \sum_{k=1}^{p} a_p(k)x(n-k)]^*$$

$$= \sum_{n=q+1}^{\infty} e(n)x^*(n) + \sum_{n=q+1}^{\infty} e(n)\left[\sum_{k=1}^{p} a_p(k)x(n-k)\right]^* \qquad (20)$$

- From Orthogonality principle, the second term of eq(20) becomes zero:

$$\sum_{n=q+1}^{\infty} e(n)\left[\sum_{k=1}^{p} a_p(k)x(n-k)\right]^* = \sum_{k=1}^{p} a_p^*(k)\left[\sum_{n=q+1}^{\infty} e(n)x^*(n-k)\right] = 0 \qquad (21)$$

- The minimum value of modeling error can be written by using eq (20) and eq(21) as:

Prony's Method: Modeling Error

$$\varepsilon_{p,q} = \sum_{n=q+1}^{\infty} e(n)x^*(n) = \sum_{n=q+1}^{\infty}\left[x(n) + \sum_{k=1}^{p} a_p(k)x(n-k)\right]x^*(n) \qquad (22)$$

- Eq(22) can be written in terms of autocorrelation functions as:

$$\varepsilon_{p,q} = \sum_{n=q+1}^{\infty} x(n)x^*(n) + \sum_{k=1}^{p} a_p(k) \sum_{n=q+1}^{\infty} x(n-k)x^*(n)$$

$$= r_x(0,0) + \sum_{k=1}^{p} a_p(k) r_x(0,k) \qquad (23)$$

- However, the actual mean square modeling error can be given by

$$\varepsilon_{MSE} = \frac{1}{N}\sum_{n=0}^{N-1}\left[e'(n)\right]^2 \qquad (24)$$

Total actual squared error

Prony's Method: Modeling Error

- Eq(23) can be casted or augmented in eq(18)
- First let's rewrite eq(18) in the following form:

$$\begin{bmatrix} r_x(1,0) \\ r_x(2,0) \\ r_x(3,0) \\ \vdots \\ r_x(p,0) \end{bmatrix} + \begin{bmatrix} r_x(1,1) & r_x(1,2) & r_x(1,3)\cdots & r_x(1,p) \\ r_x(2,1) & r_x(2,2) & r_x(2,3)\cdots & r_x(2,p) \\ r_x(3,1) & r_x(3,2) & r_x(3,3)\cdots & r_x(3,p) \\ \vdots & \vdots & \vdots & \vdots \\ r_x(p,1) & r_x(p,2) & r_x(p,3)\cdots & r_x(p,p) \end{bmatrix} \begin{bmatrix} a_p(1) \\ a_p(2) \\ a_p(3) \\ \vdots \\ a_p(p) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (25)$$

Prony's Method: Modeling Error

- The previous equation, eq(24) can also be written as:

$$\begin{bmatrix} r_x(1,0) & r_x(1,1) & r_x(1,2)\cdots & r_x(1,p) \\ r_x(2,0) & r_x(2,1) & r_x(2,2)\cdots & r_x(2,p) \\ r_x(3,0) & r_x(3,1) & r_x(3,2)\cdots & r_x(3,p) \\ \vdots & \vdots & \vdots & \vdots \\ r_x(p,0) & r_x(p,1) & r_x(p,2)\cdots & r_x(p,p) \end{bmatrix} \begin{bmatrix} 1 \\ a_p(1) \\ a_p(2) \\ \vdots \\ a_p(p) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (26)$$

- Then augmenting eq(23) in eq (26), we will have the following equation

Prony's Method: Modeling Error

$$\underbrace{\begin{bmatrix} r_x(0,0) & r_x(0,1) & r_x(0,2)\cdots & r_x(0,p) \\ r_x(1,0) & r_x(1,1) & r_x(1,2)\cdots & r_x(1,p) \\ r_x(2,0) & r_x(2,1) & r_x(2,2)\cdots & r_x(2,p) \\ \vdots & \vdots & \vdots & \vdots \\ r_x(p,0) & r_x(p,1) & r_x(p,2)\cdots & r_x(p,p) \end{bmatrix}}_{\bar{\mathbf{R}}_x} \underbrace{\begin{bmatrix} 1 \\ a_p(1) \\ a_p(2) \\ \vdots \\ a_p(p) \end{bmatrix}}_{\mathbf{a}_p} = \underbrace{\begin{bmatrix} \varepsilon_{p,q} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\varepsilon_{p,q}\mathbf{u}_1} \qquad (27)$$

- Concisely writing eq(27)

$$\bar{\mathbf{R}}_x \mathbf{a}_p = \varepsilon_{p,q} \mathbf{u}_1 \qquad (28)$$

↑

Unit Vector, $\mathbf{u}_1 = [1,0,\ldots,0]^T$

Prony's Method: Modeling Error

- In Prony's method, $a_p(k)$ can be found from eq (19) or eq(28)
- Once $a_p(k)$ found, $b_q(k)$ can be calculated by setting the following error to zero

$$e(n) = a_p(n) * x(n) - b_q(n), \quad for \quad n = 0,1,\ldots,q \qquad (29)$$

- Setting $e(n) = 0$, we will have;

$$b_q(n) = a_p(n) * x(n) = x(n) + \sum_{k=1}^{p} a_p(k)x(n-k), \quad for \quad n = 0,1,\ldots,q \qquad (30)$$

- In matrix form, eq(30) becomes

$$\begin{pmatrix} x(0) & 0 & & \cdots & 0 \\ x(1) & x(0) & 0 & \cdots & 0 \\ x(2) & x(1) & x(0) & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ x(q) & x(q-1) & x(q-2) & \cdots & x(0) \end{pmatrix} \begin{pmatrix} 1 \\ a_p(1) \\ a_p(2) \\ \vdots \\ a_p(p) \end{pmatrix} = \begin{pmatrix} b_q(0) \\ b_q(1) \\ b_q(2) \\ \vdots \\ b_q(q) \end{pmatrix} \qquad (31)$$

Prony's Method: Example

- Let a signal $x(n)$ is given as [2]:

$$x(n)=\begin{cases}1 & ; \quad n=0,1,\ldots,N-1\\ 0 & ; \quad else\end{cases}$$

Objective: our objective is to model $x(n)$ using LSI filter having **one pole** and **one zero** applying Prony's method

Solution: from the information given, we have $p=1, q=1$ and $H(z)$ will have the form:

$$H(z)=\frac{b(0)+b(1)z^{-1}}{1+a(1)z^{-1}}$$

- Since $p=1$, we have the following from eq (18):

$$a(1)r_x(1,1)=-r_x(1,0) \quad \Rightarrow \quad a(1)=\frac{-r_x(1,0)}{r_x(1,1)}$$

Prony's Method: Example

- From the definition given at eq (16), we have:

$$r_x(k,l) = \sum_{n=q+1}^{\infty} x(n-l)x^*(n-k)$$

- Therefore:

$$r_x(1,1) = \sum_{n=2}^{\infty} x(n-1)x^*(n-1) = \sum_{n=2}^{\infty} x^2(n-1) = N-1$$

$$r_x(0,1) = \sum_{n=2}^{\infty} x(n)x^*(n-1) = \sum_{n=2}^{\infty} x(n)x(n-1) = N-2$$

- Then:

$$a(1) = -\frac{r_x(1,0)}{r_x(1,1)} = -\frac{N-2}{N-1}$$

Prony's Method: Example

- Similarly from eq(31), we have two normal equation for the numerator coefficient:

$$b(0) = x(0) = 1$$

$$b(1) = x(1) + a(1)x(0) = 1 - \frac{N-2}{N-1} = \frac{1}{N-1}$$

- Now $H(z)$ can be calculated as:

$$H(z) = \frac{b(0) + b(1)z^{-1}}{1 + a(1)z^{-1}} = \frac{1 + \frac{1}{N-1} z^{-1}}{1 - \frac{N-2}{N-1} z^{-1}}$$

- The minimum squared modeling error can also be calculated from eq(23) as:

$$\varepsilon_{p,q} = r_x(0,0) + \sum_{k=1}^{p} a(k) r_x(0,k)$$

Prony's Method: Example

- Then, we will have:

$$\varepsilon_{1,1} = r_x(0,0) + a(1) r_x(0,1)$$

- Then:

$$r_x(0,0) = \sum_{n=2}^{\infty} x(n)x^*(n) = \sum_{n=2}^{\infty} x^2(n) = N-2$$

$$r_x(0,1) = \sum_{n=2}^{\infty} x^*(n)x(n-1) = \sum_{n=2}^{\infty} x(n)x(n-1) = N-2$$

- Then:

$$\varepsilon_{1,1} = (N-2) - \left(\frac{N-2}{N-1}\right) N - 2 = \frac{N-2}{N-1}$$

Prony's Method: Example

- If we consider $N = 21$, we will have:

$$H(z) = \frac{1 + 0.05z^{-1}}{1 - 0.95z^{-1}} \quad \text{and} \quad \varepsilon_{1,1} = \frac{N-2}{N-1} = \frac{19}{20} = 0.95$$

- The impulse response of the filter, $h(n)$, becomes:

$$h(n) = \delta(n) + (0.95)^{n-1}u(n-1)$$

- The difference between the actual signal and the model, $e'(n)$, is:

$$e'(n) = x(n) - h(n)$$

- The actual total squared-error is :

$$\varepsilon_{\mathbf{Pro,min}} = \sum_{n=0}^{\infty} \left[e'(n)\right]^2 \approx 4.5954$$

- Hence, there is a huge discrepancy between the actual squared error and the minimum squared modeling error

Shank's Method

- In Prony's method the modeling error is set to zero, $e(n) = 0$, for $\rightarrow n = 0, 1, 2, \ldots, q$
- Prony's method forces the model to be exact over the data value interval $[0, q]$
- However, It does not take in to account the data for $\rightarrow n > q$
- To tackle the gaps of Prony's method, Shank has proposed a better approach
- Shank performs a least square minimization of the following modeling error $(e'(n))$ over the entire data values:

$$e'(n) = x(n) - \hat{x}(n) \qquad (32)$$

Output of the model/Modeled signal

- In Shank method, the filter $H(z)$ is taken as cascaded of two filters having system functions of $\frac{1}{A_p(z)}$ and $B_q(z)$

Shank's Method

- Hence,

$$H(z) = \left[\frac{1}{A_p(z)}\right] B_q(z) \qquad (33)$$

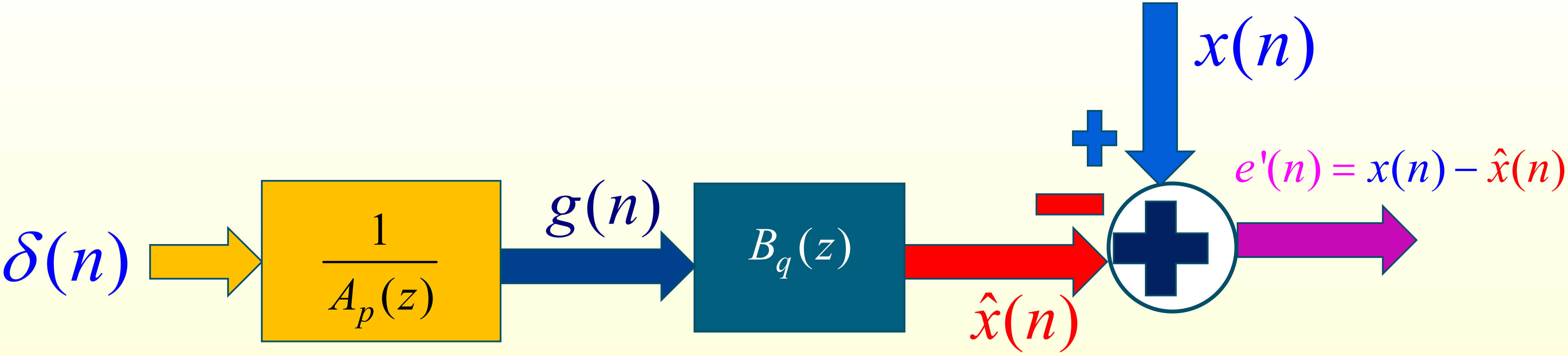

Figure 1: Shank's Method Illustration

Shank's Method

- In Shank method, $a_p(k)$ will be found first using Prony's method
- Once $a_p(k)$ is found, it is possible to calculate the out put of the first filter, $g(n)$ in figure(1)
- In z-domain, we have:

$$G(z) = \frac{\mathbf{z}\{\delta(n)\}}{A_p(z)} \qquad (34)$$

- From eq(34), we have:

$$A_p(z)G(z) = \mathrm{z}\{\delta(n)\} \qquad (35)$$

- Writing eq(35) in time domain:

$$\delta(n) = a_p(n) * g(n) = g(n) + \sum_{k=1}^{p} a_p(k) g(n-k) \qquad (36)$$

- From eq(36), we have:

$$g(n) = \delta(n) - \sum_{k=1}^{p} a_p(k) g(n-k) \qquad (37)$$

Shank's Method

- Shank methods minimizes the following squared modeling error $\rightarrow \varepsilon_S$:

$$\varepsilon_S = \sum_{n=0}^{\infty} |e'(n)|^2 \tag{38}$$

Where:

$$e'(n) = x(n) - \hat{x}(n) = x(n) - \sum_{l=0}^{q} b_q(l) g(n-l) \tag{39}$$

- Minimizing ε_S with respect to $b_q(k)$:

$$\frac{\partial \varepsilon_S}{\partial b_q^*(k)} = \sum_{n=0}^{\infty} e'(n) \frac{\partial [e'(n)]^*}{\partial b_q^*(k)} = -\sum_{n=0}^{\infty} e'(n) g^*(n-k) = 0 \tag{40}$$

Shank's Method

- Substituting eq(39) in eq(40):

$$-\sum_{n=0}^{\infty}\left[x(n)-\sum_{l=0}^{q}b_q(n-l)g(n-l)\right]g^*(n-k)=0 \qquad (41)$$

- Rewriting eq(41) in the form:

$$\sum_{l=0}^{q}b_q(l)\left[\sum_{n=0}^{\infty}g(n-l)g^*(n-k)\right]=\sum_{n=0}^{\infty}x(n)g^*(n-k) \qquad (42)$$

- Writing eq(42) in terms of deterministic autocorrelations and cross correlations::

$$\sum_{l=0}^{q}b_q(l)r_g(k,l)=r_{xg}(k);\ \text{for } k=0,1,\ldots,q \qquad (43)$$

Shank's Method

- Writing eq(43) in matrix form

$$\begin{bmatrix} r_g(0,0) & r_g(0,1) & r_g(0,2)\cdots & r_g(0,q) \\ r_g(1,0) & r_g(1,1) & r_g(1,2)\cdots & r_g(1,q) \\ r_g(2,0) & r_g(2,1) & r_g(2,2)\cdots & r_g(2,q) \\ \vdots & \vdots & \vdots & \vdots \\ r_g(q,0) & r_g(q,1) & r_g(q,2)\cdots & r_g(q,q) \end{bmatrix} \begin{bmatrix} b_q(0) \\ b_q(1) \\ b_q(2) \\ \vdots \\ b_q(q) \end{bmatrix} = \begin{bmatrix} r_{xg}(0) \\ r_{xg}(1) \\ r_{xg}(2) \\ \vdots \\ r_{xg}(q) \end{bmatrix} \qquad (44)$$

- Replacing

$$r_g(k,l) \rightarrow r_g(k-l) \rightarrow r_g^*(l-k)$$

Shank's Method

- Eq(44) becomes:

$$\underbrace{\begin{bmatrix} r_g(0) & r_g^*(1) & r_g^*(2)\cdots & r_g^*(q) \\ r_g(1) & r_g(0) & r_g^*(1)\cdots & r_g^*(q-1) \\ r_g(2) & r_g(1) & r_g(0)\cdots & r_g^*(q-2) \\ \vdots & \vdots & \vdots & \vdots \\ r_g(q) & r_g(q-1) & r_g(q-2)\cdots & r_g(0) \end{bmatrix}}_{\mathbf{R}_g} \underbrace{\begin{bmatrix} b_q(0) \\ b_q(1) \\ b_q(2) \\ \vdots \\ b_q(q) \end{bmatrix}}_{\mathbf{b}_q} = \underbrace{\begin{bmatrix} r_{xg}(0) \\ r_{xg}(1) \\ r_{xg}(2) \\ \vdots \\ r_{xg}(q) \end{bmatrix}}_{\mathbf{r}_{xg}} \qquad (45)$$

- Concisely:

$$\mathbf{R}_g \mathbf{b}_q = \mathbf{r}_{xg} \qquad (46)$$

Shank's Method: Modeling Error

- The squared error can be found as:

$$\varepsilon_S = \sum_{n=0}^{\infty} |e'(n)|^2 = \sum_{n=0}^{\infty} e'(n)[e'(n)]^* = \sum_{n=0}^{\infty} e'(n)\left[x(n) - \sum_{k=0}^{q} b_q(k) g(n-k)\right]^*$$

$$= \sum_{n=0}^{\infty} e'(n) x^*(n) - \sum_{k=0}^{q} b_q^*(k) \left[\sum_{n=0}^{\infty} e'(n) g^*(n-k)\right] \qquad (47)$$

This term is zero due to orthogonality property

- Therefore, we have now the minimum squared error $\{\varepsilon_S\}_{\min}$:

$$\{\varepsilon_S\}_{\min} = \sum_{n=0}^{\infty} e'(n) x^*(n) = \sum_{n=0}^{\infty} \left[x(n) - \sum_{k=0}^{q} b_q(k) g(n-k)\right] x^*(n) \qquad (48)$$

Shank's Method: Modeling Error

- Eq (48) can be further simplified as:

$$\{\varepsilon_S\}_{\min} = \sum_{n=0}^{\infty} x(n)x^*(n) - \sum_{q=0}^{q} b_q(k)\left[\sum_{n=0}^{\infty} g(n-k)x^*(n)\right] \qquad (49)$$

- In terms of autocorrelation and cross-correlation functions:

$$\{\varepsilon_S\}_{\min} = r_x(0) - \sum_{k=0}^{q} b_q(k) r_{gx}(-k) \qquad (50)$$

- Since $r_{gx}(-k) = r_{xg}^*(k)$, we will have

$$\{\varepsilon_S\}_{\min} = r_x(0) - \sum_{k=0}^{q} b_q(k) r_{xg}^*(k) \qquad (51)$$

Shank's Method: Example

- Lets consider the previous problem [2]:

$$x(n) = \begin{cases} 1 \;\; ; \;\; n = 0,1,\dots,N-1 \\ 0 \;\; ; \;\; else \end{cases}$$

- From Prony's method, for the case of $p = q = 1$, we have had:

$$A(z) = 1 - \frac{N-2}{N-1} z^{-1}$$

- Then using shank method, let's begin by evaluating $g(n)$ to find $B(z)$:

$$G(z) = \frac{1}{A(z)} = \frac{1}{1 - \frac{N-2}{N-1} z^{-1}} \quad \text{and} \quad g(n) = \left(\frac{N-2}{N-1} \right)^n u(n)$$

- Next we need to find the autocorrelations of $g(n)$ and the cross correlation between $x(n)$ and $g(n)$

Shank's Method: Example

- Then, we have :

$$r_g(0) = \frac{1}{1-\left(\frac{N-2}{N-1}\right)^2} \qquad r_g(1) = \frac{N-2}{N-1} r_g(0)$$

$$r_{xg}(0) = (N-1)\left[1-\left(\frac{N-2}{N-1}\right)^N\right] \qquad r_{xg}(1) = (N-1)\left[1-\left(\frac{N-2}{N-1}\right)^{N-1}\right]$$

- Then solving the following equation:

$$\begin{pmatrix} r_g(0) & r_g^*(1) \\ r_g(1) & r_g(0) \end{pmatrix} \begin{pmatrix} b(0) \\ b(1) \end{pmatrix} = \begin{pmatrix} r_{xg}(0) \\ r_{xg}(1) \end{pmatrix}$$

Shank's Method: Example

- Then, we will have :

$$b(0)=1 \quad \text{and} \quad b(1)=(N-1)\left[\frac{1}{N-1}-\left(\frac{N-2}{N-1}\right)^{N-1}+\left(\frac{N-2}{N-1}\right)^{N+1}\right]$$

- Considering $N=21$, we will have

$$H(z)=\frac{1+0.301z^{-1}}{1-0.95z^{-1}}$$

$$\varepsilon_{\mathbf{Shank,min}}=r_x(0)-b(0)r_{xg}(0)-b(1)r_{xg}(1)=3.95$$

- Comparing with Prony's method, Shank is better in minimizing the modeling error

$$\varepsilon_{\mathbf{Pro,min}}=4.5954>\varepsilon_{\mathbf{Shank,min}}=3.95$$

Shank's Vs Prony's Method

- As we have seen from the previous examples, It is revealed that:
  - Shank's method more efficient than Prony's method in minimizing the modeling error
  - However, Shank's method introduce Extra computation of sequence $g(n)$, autocorrection of $g(n)$, cross correlation of $x(n)$ and $g(n)$

Summary

- **Difference between Padé Approximation and Prony method:**
  - ✓ In terms of model development
  - ✓ In terms modeling error minimization
- **Prony methods Vs Shank Method**
  - ✓ **In terms of model development**
  - ✓ **In terms of Performance**

References

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Thank You!