Statistical Digital Signal Processing

Week 4 Stochastic Signal Modeling

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Previous Topic (Week-3)

Deterministic Signal Modeling (Part-2)

- Prony's Method: Introduction
- Prony's Method: Pole-Zero Modeling
- Prony's Method Modeling Error
- Shank's Method

Week-4: Lecture Learning Outcomes

1. Explain the fundamentals of stochastic signal modeling, including the definition and key properties of random processes and their applications in signal processing.
2. Describe the structure and characteristics of Autoregressive Moving Average (ARMA) models.
3. Formulate and analyze Autoregressive (AR) models, including determining model parameters.
4. Compare AR and ARMA models, and appropriate use cases in practical signal modeling scenarios.
5. Apply stochastic modeling techniques, including selecting suitable models and evaluating their performance using statistical measures.

Week 4: Stochastic Signal Modeling

Outline

- Stochastic Signal Modeling : Introduction
- Autoregressive Moving Average Model (ARMA)
- Autoregressive Model (AR)

Stochastic Signal Modeling : Introduction

- In the previous sessions, we have seen different methods for deterministic signal modeling
- In the case of deterministic signal modeling, the value of the signal $(x(n))$ is known at least for values of n over fixed finite interval
- However, Some applications require modeling random processes or signals whose behavior is defined in probabilistic terms
- Random signals whose time evolution is influenced by unknown or random factors include:
  - Network traffic variations in data networks
  - Modulated digital signals in communication systems
  - Electrocardiograms
  - Seismograms
  - Sonar Data
  - Population Statistics

Stochastic Signal Modeling : Introduction

- The errors that are minimized in the case of deterministic signal modeling are no longer feasible for random signal modeling since $(x(n))$ is known in probabilistic sense
- For example in Prony's method $a_p(k)$ is found by minimizing the following deterministic squared error ε_p:

$$\varepsilon_p = \sum_{n=q+1}^{\infty} |e(n)|^2 = \sum_{n=q+1}^{\infty} \left| x(n) + \sum_{k=1}^{p} a_p(k) x(n-k) \right|^2 \qquad (1)$$

- Eq(1) shows that the error ε_p is written based on the value of the signal $(x(n))$
- However, the exact vale of the signal $(x(n))$ is not known in the case of random signal and such deterministic signal modeling methods are no longer appropriate

Stochastic Signal Modeling : Introduction

- In addition, the input signal to the model is random signal for the case random signal modeling
- Let's consider linear modeling of stationary random signal $(x(n))$ using rational filter $H(z)$ [1]

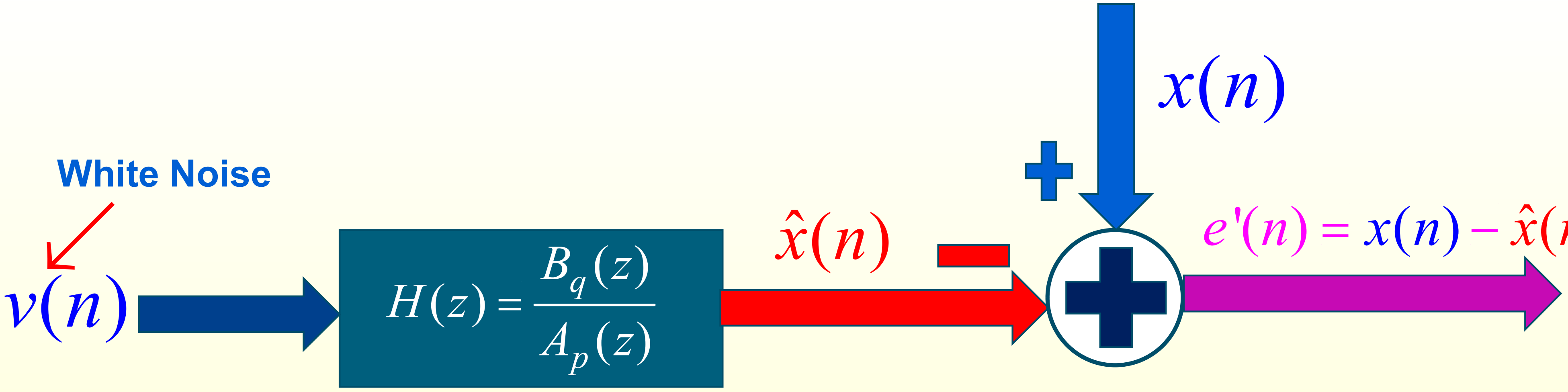


Figure 1: Illustration of Linear Modeling of Stationary Random Signal

Stochastic Signal Modeling : Introduction

- In the case of stochastic signal modeling, Instead of minimizing $\sum |e'(n)|^2$, it make sense to minimize :

$$E\{|e'(n)|^2\} = E\{|x(n) - \hat{x}(n)|^2\} \qquad (2)$$

- However, minimizing the above equation, eq(2), is not mathematically tractable
- Therefore, we will proceed by matching the correlation sequence $r_x(k)$ of $x(n)$ with the correlation sequence $r_{\hat{x}}(k)$ of $\hat{x}(n)$ as:

$$r_{\hat{x}}(k) \approx r_x(k) \qquad (3)$$

Autoregressive Moving Average Model (ARMA)

- A wide sense stationary ARMA (p,q) process, $x(n)$, having p poles and q zeros can be modeled using linear shift invariant filter with a white noise input $v(n)$

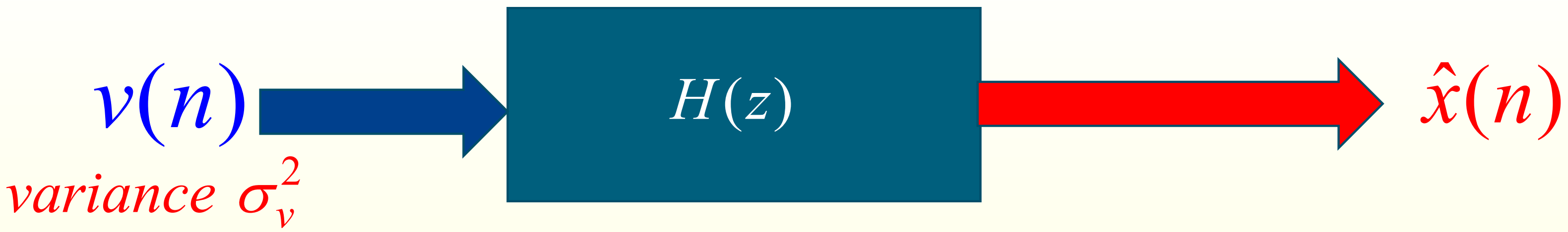


Figure 2: Stochastic Signal Modeling as an output of LSI Filter with white Noise Input

Where:

$$H(z) = \frac{B_q(z)}{A_p(z)} = \frac{\sum_{k=0}^{q} b_q(k) z^{-k}}{1 + \sum_{k=1}^{p} a_p(k) z^{-k}} \qquad (4)$$

Autoregressive Moving Average Model (ARMA)

- The power spectrum of the output signal of the filter $\hat{x}(n)$ is given by:

$$P_X(z) \approx P_{\hat{X}}(z) = \mathbf{z}\{r_{\hat{x}}(k) \approx r_x(k)\} = P_V H(z) H^*(1/z^*) = \sigma_v^2 \frac{B_q(z) B_q^*(1/z^*)}{A_p(z) A_p^*(1/z^*)} \tag{5}$$

$\uparrow$

Z-Transform

Or

$$P_X(e^{j\omega}) \approx P_{\hat{X}}(e^{j\omega}) = \mathbf{DTFT}\{r_{\hat{x}}(k) \approx r_x(k)\} = \sigma_v^2 \frac{B_q(e^{j\omega}) B_q^*(e^{j\omega})}{A_p(e^{j\omega}) A_P^*(e^{j\omega})} = \sigma_v^2 \frac{|B_q(e^{j\omega})|^2}{|A_p(e^{j\omega})|^2} \tag{6}$$

$\uparrow$

Discrete Time
Fourier Transform

Autoregressive Moving Average Model (ARMA)

- The output signal of the filter $\hat{x}(n)$ is given by:

$$\hat{x}(n) = h(n) * v(n) \qquad (7)$$

- In z-domain:

$$\hat{X}(z) = H(z)V(z) = \frac{B_q(z)}{A_p(z)}V(z) \qquad (8)$$

- From eq(8), we have:

$$A_p(z)\hat{X}(z) = B_q(z)V(z) \qquad (9)$$

- Writing eq(9) in time domain:

$$a_p(n) * \hat{x}(n) = b_q(n) * v(n) \qquad (10)$$

- From eq(9), we have :

$$\hat{x}(n) + \sum_{l=1}^{p} a_p(l)\hat{x}(n-l) = \sum_{l=0}^{q} b_q(l)v(n-l) \qquad (11)$$

Autoregressive Moving Average Model (ARMA)

- Multiplying eq(11) by $\hat{x}^*(n-k)$ both sides:

$$\hat{x}(n)\hat{x}^*(n-k)+\sum_{l=1}^{p}a_p(l)\hat{x}(n-l)\hat{x}^*(n-k)=\sum_{l=0}^{q}b_q(l)v(n-l)\hat{x}^*(n-k) \qquad (12)$$

- Taking the expectation of eq(12) both sides:

$$E\{\hat{x}(n)\hat{x}^*(n-k)\}+\sum_{l=1}^{p}a_p(l)E\{\hat{x}(n-l)\hat{x}^*(n-k)\}=\sum_{l=0}^{q}b_q(l)E\{v(n-l)\hat{x}^*(n-k)\} \qquad (13)$$

- Writing eq(13) using autocorrelation and cross-correlation functions :

$$r_{\hat{x}}(k)+\sum_{l=1}^{p}a_p(l)\, r_{\hat{x}}(k-l)=\sum_{l=0}^{q}b_q(l)\, r_{v\hat{x}}(k-l) \qquad (14)$$

- Since we have matched the autocorrelations of $x(n)$ and $\hat{x}(n)$:

$$r_{\hat{x}}(k)\approx r_x(k), \quad r_{\hat{x}}(k-l)\approx r_x(k-l), \text{ and } r_{v\hat{x}}(k-l)\approx r_{vx}(k-l) \qquad (15)$$

Autoregressive Moving Average Model (ARMA)

- From eq(14) and eq(15), we have:

$$r_x(k) + \sum_{l=1}^{p} a_p(l)\, r_x(k-l) = \sum_{l=0}^{q} b_q(l)\, r_{vx}(k-l) \qquad (16)$$

- The correlation between $v(n)$ and $x(n)$ $\rightarrow r_{vx}(k-l)$ can be written :

$$r_{vx}(k-l) = E\{v(k)x^*(l)\} = E\{v(k)[h(l) * v(l)]^*\}$$

$$= \sum_{m=-\infty}^{\infty} E\{v(k)v^*(l-m)\}h^*(m)$$

$$= \sum_{m=-\infty}^{\infty} r_v(m+k-l)h^*(m) = \sigma_v^2 h^*(l-k) \qquad (17)$$

Where:

$$r_v(m+k-l) = \begin{cases} \sigma_v^2 \text{ , for } m = l-k \\ 0, \text{ Otherwise} \end{cases}$$

Autoregressive Moving Average Model (ARMA)

- Using eq(16) and eq(17), we have:

$$r_x(k) + \sum_{l=1}^{p} a_p(l)\, r_x(k-l) = \sigma_v^2 \sum_{l=0}^{q} b_q(l)\, h^*(l-k) \qquad (18)$$

- For $k \geq 0$, we will obtain the following **Yule-Walker equations**

$$r_x(k) + \sum_{l=1}^{p} a_p(l)\, r_x(k-l) = \begin{cases} \sigma_v^2 \sum_{l=0}^{q} b_q(l)\, h^*(l-k) = \sigma_v^2 c_q(k)\ , \text{ for } 0 \leq k \leq q \\ 0\ , \text{ for } k > q \end{cases} \qquad (19)$$

- Assuming the input signal $v(n)$ has unity variance $(\sigma_v^2 = 1)$:

$$r_x(k) + \sum_{l=1}^{p} a_p(l)\, r_x(k-l) = c_q(k) \qquad (20)$$

Where: $c_q(k) = \begin{cases} \sum_{l=0}^{q} b_q(l)\, h^*(l-k)\ , \text{ for } 0 \leq k \leq q \\ 0\ , \text{ for } k > q \end{cases}$

Autoregressive Moving Average Model (ARMA)

- From eq(20), Yule-Walker equations in matrix form for :

$$\begin{bmatrix} r_x(0) & r_x(-1) & r_x(-2)\cdots & r_x(-p) \\ r_x(1) & r_x(0) & r_x(-1)\cdots & r_x(-p+1) \\ \vdots & \vdots & \vdots \;\; \cdots & \vdots \\ r_x(q) & r_x(q-1) & r_x(q-2)\cdots & r_x(q-p) \\ \hline r_x(q+1) & r_x(q) & r_x(q-1)\cdots & r_x(q-p+1) \\ \vdots & \vdots & \vdots & \vdots \\ r_x(q+p) & r_x(q+p-1) & r_x(q+p-2)\cdots & r_x(q) \end{bmatrix} \begin{bmatrix} 1 \\ a_p(1) \\ a_p(2) \\ \vdots \\ a_p(p) \end{bmatrix} = \begin{bmatrix} c_q(0) \\ c_q(1) \\ \vdots \\ c_q(q) \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (21)$$

- Using the relation $r_x(-k) = r_x^*(k)$, eq(21) can be written in the following form:

Autoregressive Moving Average Model (ARMA)

$$\begin{bmatrix} r_x(0) & r_x^*(1) & r_x^*(2)\cdots & r_x^*(p) \\ r_x(1) & r_x(0) & r_x^*(1)\cdots & r_x^*(p-1) \\ \vdots & \vdots & \vdots \quad \cdots & \vdots \\ r_x(q) & r_x(q-1) & r_x(q-2)\cdots & r_x(q-p) \\ \hline r_x(q+1) & r_x(q) & r_x(q-1)\cdots & r_x(q-p+1) \\ \vdots & \vdots & \vdots & \vdots \\ r_x(q+p) & r_x(q+p-1) & r_x(q+p-2)\cdots & r_x(q) \end{bmatrix} \begin{bmatrix} 1 \\ a_p(1) \\ a_p(2) \\ \vdots \\ a_p(p) \end{bmatrix} = \begin{bmatrix} c_q(0) \\ c_q(1) \\ \vdots \\ c_q(q) \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (22)$$

- For $k = q+1, q+2, \ldots, q+p$, we have :

$$r_x(k) + \sum_{l=1}^{p} a_p(l)\, r_x(k-l) = 0 \qquad (23)$$

Autoregressive Moving Average Model (ARMA)

- Writing eq(23) in matrix form, we will obtain the lower **(green colored)** matrix of eq(21 & 22):

$$\begin{bmatrix} r_x(q+1) & r_x(q) & r_x(q-1)\cdots & r_x(q-p+1) \\ r_x(q+2) & r_x(q+1) & r_x(q)\cdots & r_x(q-p+2) \\ \vdots & \vdots & \vdots & \vdots \\ r_x(q+p) & r_x(q+p-1) & r_x(q+p-2)\cdots & r_x(q) \end{bmatrix} \begin{bmatrix} 1 \\ a_p(1) \\ a_p(2) \\ \vdots \\ a_p(p) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (24)$$

- From eq(24):

$$\begin{pmatrix} r_x(q) & r_x(q-1) & \cdots & r_x(q-p+1) \\ r_x(q+1) & r_x(q) & \cdots & r_x(q-p+2) \\ \vdots & \vdots & \cdots & \vdots \\ r_x(q+p-1) & r_x(q+p-2)\cdots & & r_x(q) \end{pmatrix} \begin{bmatrix} a_p(1) \\ a_p(2) \\ a_p(3) \\ \vdots \\ a_p(p) \end{bmatrix} = -\begin{bmatrix} r_x(q+1) \\ r_x(q+2) \\ \vdots \\ r_x(q+p) \end{bmatrix} \quad (25)$$

Autoregressive Moving Average Model (ARMA)

- Eq(25) is the **Modified Yule-Walker equations**
- Given the autocorrelations of $x(n)$, the coefficient $a_p(k)$ can be calculated using eq(25)
- After $a_p(k)$ is found, the next step is finding $b_q(k)$
- From eq(20), for $k = 0,1,\ldots,q$, we have:

$$\begin{pmatrix} r_x(0) & r_x(-1) & \cdots & r_x(-p) \\ r_x(1) & r_x(0) & \cdots & r_x(-p+1) \\ \vdots & \vdots & \cdots & \vdots \\ r_x(q) & r_x(q-1) & \cdots & r_x(q-p) \end{pmatrix} \begin{bmatrix} 1 \\ a_p(1) \\ a_p(2) \\ \vdots \\ a_p(p) \end{bmatrix} = \begin{bmatrix} c_q(0) \\ c_q(1) \\ \vdots \\ c_q(q) \end{bmatrix} \qquad (26)$$

- From relation $r_x(-k) = r_x^*(k)$, eq(26) can be written in alternative form as:

Autoregressive Moving Average Model (ARMA)

$$\begin{pmatrix} r_x(0) & r_x^*(1) & \cdots & r_x^*(p) \\ r_x(1) & r_x(0) & \cdots & r_x^*(p-1) \\ \vdots & \vdots & \cdots & \vdots \\ r_x(q) & r_x(q-1) & \cdots & r_x(q-p) \end{pmatrix} \begin{bmatrix} 1 \\ a_p(1) \\ a_p(2) \\ \vdots \\ a_p(p) \end{bmatrix} = \begin{bmatrix} c_q(0) \\ c_q(1) \\ \vdots \\ c_q(q) \end{bmatrix} \quad (27)$$

- For $k = 0,1,2,\ldots,q$, recalling $c_q(k)$ is:

$$c_q(k) = \sum_{l=0}^{q} b_q(l)\, h^*(l-k) = b_q(k) * h^*(-k) = h^*(-k) * b_q(k) = \sum_{l=0}^{q-k} b_q(l+k)\, h^*(l) \quad (28)$$

- In z-domain, we have:

$$C_q(z) = \mathbf{z}\left\{ b_q(k) * h^*(-k) \right\} = B_q(z) H^*(1/z^*) = B_q(z) \frac{B_q^*(1/z^*)}{A_p^*(1/z^*)} \quad (29)$$

Autoregressive Moving Average Model (ARMA)

- From eq(29), we have:

$$C_q(z)A_p^*(1/z^*) = B_q(z)B_q^*(1/z^*) \qquad (30)$$

- Recalling $c_q(k)$ is known for $k \geq 0$:

$$c_q(k) = \begin{cases} \sum_{l=0}^{q} b_q(l)\, h^*(l-k) \text{ , for } 0 \leq k \leq q \\ 0 \text{ , for } k > q \end{cases} \qquad (31)$$

- We know only the **casual part** of $C_q(z)$ denoted by $[C_q(z)]_+$:

$$[C_q(z)]_+ = \sum_{k=0}^{\infty} c_q(k) z^{-k} \qquad (32)$$

Autoregressive Moving Average Model (ARMA)

- Denoting the anti-causal part by $[C_q(z)]_-$:

$$[C_q(z)]_- = \sum_{k=-\infty}^{-1} c_q(k) z^{-k} = \sum_{k=1}^{\infty} c_q(-k) z^{k} \qquad (33)$$

- From eq(32) and eq(33):

$$C_q(z) = [C_q(z)]_- + [C_q(z)]_+ = \sum_{k=1}^{\infty} c_q(-k) z^{k} + \sum_{k=0}^{\infty} c_q(k) z^{-k} \qquad (34)$$

Unknown Term

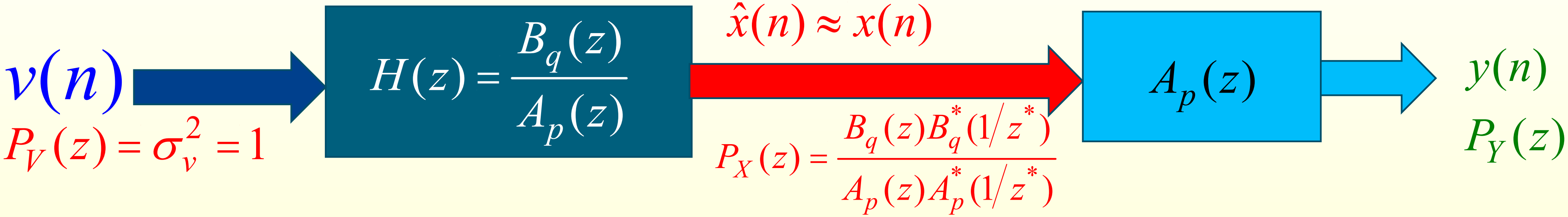
Known Term

Autoregressive Moving Average Model (ARMA)

- From eq(5), if we set $\sigma_v^2 = 1$, the power spectrum of the filter output signal becomes:

$$P_X(z) = \frac{B_q(z)B_q^*(1/z^*)}{A_p(z)A_p^*(1/z^*)} \qquad (35)$$

- Filtering the output signal $x(n)$ using linear shift invariant filter having system function $A_p(z)$



- The output signal power spectrum $P_Y(z)$ is given by

$$P_Y(z) = P_X(z)A_p(z)A_p^*(1/z^*) = B_q(z)B_q^*(1/z^*) \qquad (36)$$

Autoregressive Moving Average Model (ARMA)

- From eq(30) and eq(36), we have :

$$P_Y(z) = C_q(z)A_p^*(1/z^*) = B_q(z)B_q^*(1/z^*) \quad (37)$$

- Substituting eq(34) for $C_q(z)$ in eq(37), we have :

$$P_Y(z) = C_q(z)A_p^*(1/z^*) = \left([C_q(z)]_- + [C_q(z)]_+\right)A_p^*(1/z^*)$$

$$= [C_q(z)]_- A_p^*(1/z^*) + [C_q(z)]_+ A_p^*(1/z^*) \quad (38)$$

- Since $a_p(k) = 0$ for $k < 0$, Hence $A_p^*(1/z^*)$ only positive power of k

$$P_Y(z) = [P_Y(z)]_- + [P_Y(z)]_+ = [C_q(z)]_- A_p^*(1/z^*) + [C_q(z)]_+ A_p^*(1/z^*) \quad (39)$$

Only Anti-Causal **Causal + Anti-Causal**

Autoregressive Moving Average Model (ARMA)

- Then the causal part of $P_Y(z)$ which is $[P_Y(z)]_+$ can be obtained :

$$[P_Y(z)]_+ = \Big[[C_q(z)]_+ A_p^*(1/z^*) \Big]_+ \qquad (40)$$

- Then using the conjugate symmetry property, the entire $P_Y(z)$ can be obtained from $[P_Y(z)]_+$

- Finally by performing the following spectral factorization:

$$P_Y(z) = B_q(z) B_q^*(1/z^*) \qquad (41)$$

$B_q(z)$ Can be found

ARMA: Example

- Let us consider the problem of modeling a real-valued random process $x(n)$ having the following autocorrelation values using an ARMA(p,q)=ARMA(1,1) model [2]:

$$r_x(0) = 26\ ;\ r_x(1) = 7\ ;\ r_x(2) = \frac{7}{2}$$

Solution:

- From eq(22), the Yule-Walker equations become:

$$\begin{pmatrix} r_x(0) & r_x^*(1) \\ r_x(1) & r_x(0) \\ r_x(2) & r_x(1) \end{pmatrix} \begin{pmatrix} 1 \\ a_1(1) \end{pmatrix} = \begin{pmatrix} c_1(0) \\ c_1(1) \\ 0 \end{pmatrix}$$

- The modified Yule-Walker equations become:

$$r_x(2) + a_1(1)\ r_x(1) = 0$$

ARMA: Example

- Then:

$$a_1(1) = -\frac{r_x(1)}{r_x(2)} = -\frac{7}{7/2} = -\frac{1}{2}$$

- For the nominator coefficient, we begin by solving for $c_1(0)$ and $c_1(1)$ using the Yule-Walker equations:

$$\begin{pmatrix} r_x(0) & r_x^*(1) \\ r_x(1) & r_x(0) \end{pmatrix} \begin{pmatrix} 1 \\ a_1(1) \end{pmatrix} = \begin{pmatrix} c_1(0) \\ c_1(1) \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} c_1(0) \\ c_1(1) \end{pmatrix} = \begin{pmatrix} 26 & 7 \\ 7 & 26 \end{pmatrix} \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} = \begin{pmatrix} 45/2 \\ -6 \end{pmatrix}$$

ARMA: Example

- Therefore, the causal part of $C_1(z) \rightarrow [C_1(z)]_+$ becomes:

$$[C_1(z)]_+ = \frac{45}{2} - 6z^{-1}$$

- And also, we have:

$$A_1(z) = 1 - 0.5z^{-1} \quad \Rightarrow \quad A_1^*(1/z^*) = 1 - 0.5z$$

- Multiplying $[C_1(z)]_+$ by $A_1^*(1/z^*)$ we have:

$$[C_1(z)]_+ A_1^*(1/z^*) = \left(\frac{45}{2} - 6z^{-1}\right)\left(1 - 0.5z\right) = -\frac{45}{4}z + \frac{51}{2} - 6z^{-1}$$

- Therefore, the causal part of $P_Y(z) \rightarrow [P_Y(z)]_+$ becomes:

$$[P_Y(z)]_+ = \left[[C_1(z)]_+ A_1^*(1/z^*)\right]_+ = \frac{51}{2} - 6z^{-1}$$

ARMA: Example

- Using the symmetry property of $P_Y(z)$, we obtain:

$$P_Y(z) = C_1(z)A_1^*(1/z^*) = B_1(z)B_1^*(1/z^*) = -6z + \frac{51}{2} - 6z^{-1}$$

- Using spectral factorization:

$$B_1(z)B_1^*(1/z^*) = 24\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{4}z\right) = \left[\sqrt{24}\left(1 - \frac{1}{4}z^{-1}\right)\right]\left[\sqrt{24}\left(1 - \frac{1}{4}z\right)\right]$$

- Hence:

$$B_1(z) = \sqrt{24}\left(1 - \frac{1}{4}z^{-1}\right) = 2\sqrt{6}\left(1 - \frac{1}{4}z^{-1}\right)$$

- Then, $H(z)$ becomes:

$$H(z) = \frac{B_1(z)}{A_1(z)} = 2\sqrt{6}\;\frac{1 - 0.25z^{-1}}{1 - 0.5z^{-1}}$$

Autoregressive Model (AR)

- An order p wide-sense stationary autoregressive process is a special case of ARMA(p,q) when q=0.

$$H(z) = \frac{B_0(z)}{A_p(z)} = \frac{b_0(0)}{1 + \sum_{k=1}^{p} a_p(k) z^{-k}} \tag{42}$$

- From the Yule-Walker equations we have had:

$$r_x(k) + \sum_{l=1}^{p} a_p(l)\, r_x(k-l) = \begin{cases} c_q(k)\ , \text{ for } 0 \le k \le q \\ 0\ , \text{ for } k > q \end{cases} \tag{43}$$

Where:

$$c_q(k) = \sum_{l=0}^{q} b_q(l)\, h^*(l-k) \tag{44}$$

Autoregressive Model (AR)

- When $q = 0$, eq(43) boils down to:

$$r_x(k) + \sum_{l=1}^{p} a_p(l)\, r_x(k-l) = \begin{cases} c_0(0)\ , \text{ for } k = q = 0 \\ 0\ , \text{ for } k > 0 \end{cases} \qquad (45)$$

- From eq (44), $c_0(0)$ can be found as:

$$c_0(0) = b_0(0)\, h^*(0) \qquad (46)$$

- From eq (42), we have:

$$A_p(z)H(z) = B_0(z) \qquad (47)$$

- writing eq (47) in time domain:

$$h(n) + \sum_{k=1}^{p} a_p(k)h(n-k) = \begin{cases} b_0(0), & n \le q, \ \textbf{or } n = 0 \\ 0, & n > q \quad \textbf{or } n > 0 \end{cases} \qquad (48)$$

Autoregressive Model (AR)

- we can obtain $h(n)$ for $n \le q$ from eq(48) as:

$$h(n) = b_0(0) - \sum_{k=1}^{p} a_p(k) h(n-k) \qquad (49)$$

- From eq (49), for $n = 0$, we will have:

$$h(0) = b_0(0) \qquad (50)$$

Conjugating

$$h^*(0) = b_0^*(0) \qquad (51)$$

- From eq (46 & 51), we have:

$$c_0(0) = b_0(0)\, h^*(0) = b_0(0) b_0^*(0)$$

$$= | b_0(0) |^2 \qquad (52)$$

Autoregressive Model (AR)

- Then using eq(45 & 52) we will have a:

$$r_x(k) + \sum_{l=1}^{p} a_p(l)\, r_x(k-l) = \begin{cases} |\,b_0(0)\,|^2 \,, \text{ for } k = 0 \\ 0 \,, \text{ for } k > 0 \end{cases} \tag{53}$$

- Alternatively eq(53) can be written as:

$$r_x(k) + \sum_{l=1}^{p} a_p(l)\, r_x(k-l) = \left|b_0(0)\right|^2 \delta(k) \;; \text{ for } k \geq 0 \tag{54}$$

Unit Sample Function

Autoregressive Model (AR)

- For $k = 1, 2, \ldots, p$, writing eq(54) in matrix form:

$$\begin{bmatrix} r_x(0) & r_x^*(1) & r_x^*(2) \cdots & r_x^*(p-1) \\ r_x(1) & r_x(0) & r_x^*(1) \cdots & r_x^*(p-2) \\ r_x(2) & r_x(1) & r_x(0) \;\; \cdots & r_x^*(p-3) \\ \vdots & \vdots & \vdots \;\;\; \cdots & \vdots \\ r_x(p-1) & r_x(p-2) & r_x(p-3) \cdots & r_x(0) \end{bmatrix} \begin{bmatrix} a_p(1) \\ a_p(2) \\ a_p(3) \\ \vdots \\ a_p(p) \end{bmatrix} = -\begin{bmatrix} r_x(1) \\ r_x(2) \\ r_x(3) \\ \vdots \\ r_x(p) \end{bmatrix} \tag{55}$$

- Once $a_p(k)$ is found form eq(55), $b_0(0)$ can be found by setting $k = 0$ in eq(54):

$$\left|b_0(0)\right|^2 = r_x(0) + \sum_{l=1}^{p} a_p(l)\, r_x(-l) = r_x(0) + \sum_{l=1}^{p} a_p(l)\, r_x^*(l) \tag{56}$$

ARMA Model Vs. AR Model

- **ARMA Model:**
  - Can model a signal with considerably fewer parameters compared to AR model
  - Its design and analysis is more complex than AR model
  - It has higher modeling accuracy
  - Captures both signal dynamics and noise effects, therefore, it is suitable for modeling signal which depends on both past values and past noise effects
  - **Example:** Network traffic
- **AR Model:**
  - Its design and analysis is Simpler than ARMA model
  - Best for modeling a signal which has strong correlation with past values
  - **Example:** Temperature change

Summary

- **Stochastic Signal Modeling:**
  - Using LSI filter with white noise input
  - The signal is modeled as an out put of the LSI filter
- **Autoregressive Moving Average Model (ARMA (p,q))**
  - With p order of the autoregressive part and;
  - q order of the moving-average part
- **Autoregressive Model (AR (p))**
  - With p order or number of lags;

References

[1] Charles W. Therrien,‘’ *Discrete Random Signals and Statistical Signal Processing”*, Prentice Hall, Pp.503-505, 1992.

[2] Monson H. Hayes, *“Statistical Digital Signal Processing and Modeling”,* John Wiley and sons, Pp.192-193, 1996.

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Thank You!